What is...a matrix?

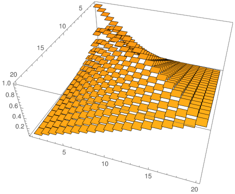
Or: Incarnations of the same beast.

Seriously, what is a matrix (visually)?

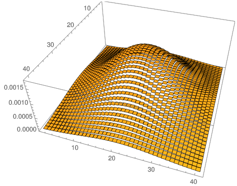
Answer 1. A rectangle of numbers *e.g.*

$$\begin{pmatrix} 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix}$$

Answer 2. A staircase 3d function, *e.g.*



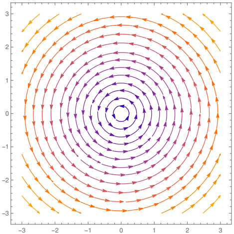
$$\leftrightsquigarrow \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1 & 2/3 & 1/2 & 2/5 \\ 1/3 & 2/3 & 1 & 3/4 & 3/5 \\ 1/4 & 1/2 & 3/4 & 1 & 4/5 \\ 1/5 & 2/5 & 3/5 & 4/5 & 1 \end{pmatrix}$$



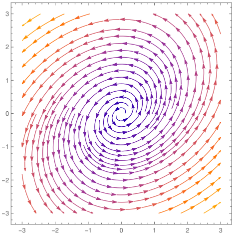
$$\leftrightsquigarrow \begin{pmatrix} 0.002589 & 0.0107788 & 0.0241466 & 0.0107788 & 0.002589 \\ 0.0107788 & 0.0448755 & 0.10053 & 0.0448755 & 0.0107788 \\ 0.0241466 & 0.10053 & 0.225206 & 0.10053 & 0.0241466 \\ 0.0107788 & 0.0448755 & 0.10053 & 0.0448755 & 0.0107788 \\ 0.002589 & 0.0107788 & 0.0241466 & 0.0107788 & 0.002589 \end{pmatrix}$$

Seriously, what is a matrix (via actions)?

Answer 3. A transformation of space, *e.g.*



$$\leftrightsquigarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$



$$\leftrightsquigarrow \begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}$$

Answer 4. A transformation of shapes, *e.g.*

$$\leftrightsquigarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\leftrightsquigarrow \begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}$$

Seriously, what is a matrix (via actions)?

Answer 5. An algebraic object allowing certain operations, *e.g.*...

- ...multiplication by scalars.

$$2 \cdot \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix} = \begin{pmatrix} 20 & 22 & 24 \\ 26 & 28 & 30 \\ 32 & 34 & 36 \end{pmatrix}$$

- ...addition.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix} = \begin{pmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 27 \end{pmatrix}$$

- ...transposition (mirroring).

$$\mathrm{Transpose}\left(\begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix}\right) = \begin{pmatrix} 10 & 13 & 16 \\ 11 & 14 & 17 \\ 12 & 15 & 18 \end{pmatrix}$$

- ...multiplication.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix} = \begin{pmatrix} 84 & 90 & 96 \\ 201 & 216 & 231 \\ 318 & 342 & 366 \end{pmatrix}$$

For completeness: A formal definition.

A matrix M is a rectangular array of numbers, or other mathematical objects for which operations such as addition and multiplication are defined.

The individual entries of $M = (m_{ij})_{i=1,\dots,m}^{j=1,\dots,n}$ are arranged by

$$\begin{array}{ccccc} & 1 & 2 & \cdots & n \\ 1 & m_{11} & m_{12} & \cdots & m_{1n} \\ 2 & m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m & m_{m1} & m_{m2} & \cdots & m_{mn} \end{array}$$

Matrices can be multiplied by scalars and added (if they are of the same size) together componentwise, transposed and their is a rule for multiplication. (And many more cool things!)

What is this strange matrix multiplication?

Matrix multiplication is constructed such that shape-action compose:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thank you for your attention!

I hope that was of some help.