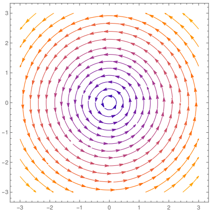
What is...a change of basis?

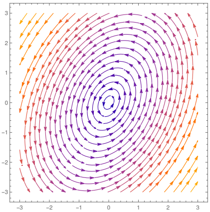
Or: Moving, rotating and scaling axes.

Twice the same beast?

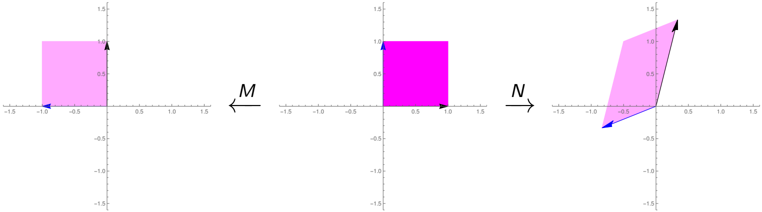
A rotation! $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Another rotation? $N = \begin{pmatrix} 1/3 & -5/6 \\ 4/3 & -1/3 \end{pmatrix}$





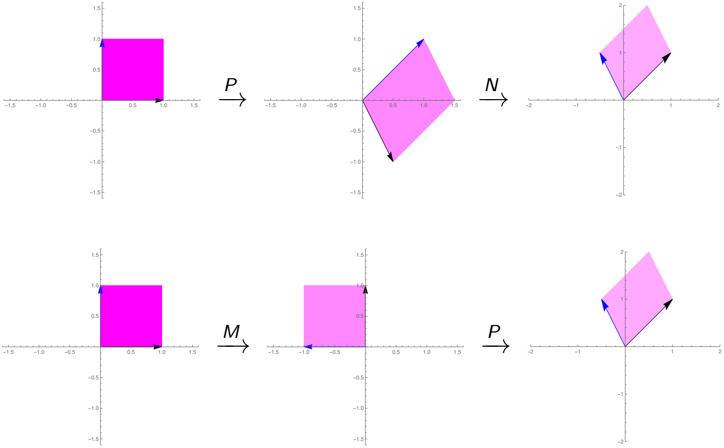
The action is similar:



Question. In what sense are they equal?

Maps in coordinates

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1/2 & 1 \\ -1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 2/3 & -2/3 \\ 2/3 & 1/3 \end{pmatrix}, \quad N = \begin{pmatrix} 1/3 & -5/6 \\ 4/3 & -1/3 \end{pmatrix}$$



The matrix P is the base-change, a.k.a. change of coordinates.

Lets think about functions

$$\mathbb{R}\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\} = \mathbb{R}^2 \xrightarrow{M} \mathbb{R}^2 = \mathbb{R}\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$$

$$\mathbb{R}\{\begin{pmatrix}1/2\\-1\end{pmatrix},\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\} = \mathbb{R}^2 \xrightarrow{N} \mathbb{R}^2 = \mathbb{R}\{\begin{pmatrix}1/2\\-1\end{pmatrix},\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$$

These are the same functions:

$$M\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \boxed{0}\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + \boxed{1}\cdot\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \quad N\begin{pmatrix}1/2\\-1\end{pmatrix} = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = \boxed{0}\cdot\left(\begin{smallmatrix}1/2\\1\end{smallmatrix}\right) + \boxed{1}\cdot\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$$

$$M\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right) = \boxed{-1}\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + \boxed{0}\cdot\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \quad N\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = \begin{pmatrix}-1/2\\-1\end{pmatrix} = \boxed{-1}\cdot\left(\begin{smallmatrix}1/2\\1\end{smallmatrix}\right) + \boxed{0}\cdot\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$$

Up to change of coordinates, M and N describe the same linear map!

For completeness: A formal definition.

Let $B^{\text{old}} = \{v_1, ..., v_n\}$ and $B^{\text{new}} = \{w_1, ..., w_n\}$ be two bases of $\mathbb{K}^n$.

- We can define a vector $a \in \mathbb{K}^n$ by its coordinates $(a_1, ..., a_n) = a_1 v_1 + ... + a_n v_n$ in B^{old}
- We can define a vector $b \in \mathbb{K}^n$ by its coordinates $(b_1, ..., b_n) = b_1 w_1 + ... + b_n w_n$ in B^{new}
- Even if $a = b$ abstractly, $(a_1, ..., a_n) \neq (b_1, ..., b_n)$

A change-of-basis matrix P is a nxn matrix such that $(b_1, ..., b_n) = P(a_1, ..., a_n)$ holds for all coordinate vectors that represent the same vector.

In other words, P changes B^{old} into B^{new}

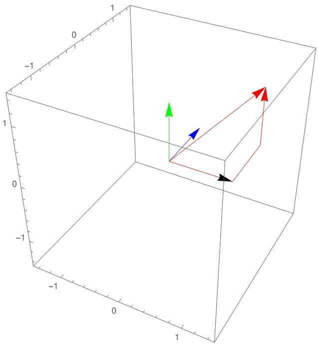
If $N = PMP^{-1}$, then M and N are the same linear map up to choice of coordinates.

Different axes for the same vector

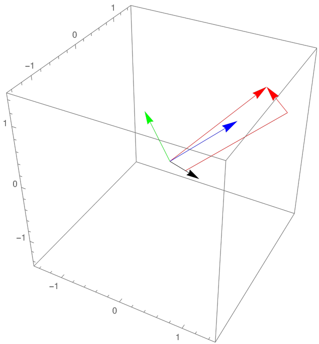
$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1/2 \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 3/2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1/2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

The vector is the same, but coordinates are different:

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \quad \textit{vs.} \quad \begin{pmatrix} 1/2 & 3/2 & 1/2 \end{pmatrix}$$



vs.



Thank you for your attention!

I hope that was of some help.