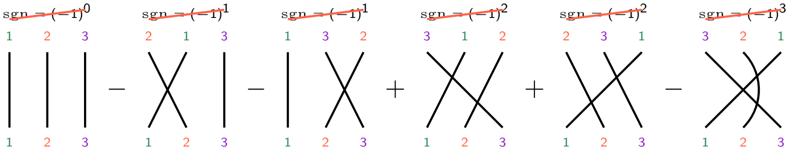
What is...the permanent?

Or: The trivial representation.

The determinant without signs

$$\mathrm{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma_i} \qquad \det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

Example. For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$:

perm(A)= 

$$= a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}$$

Determinant-geometry, permanent-combinatorics

Riddle. How many ways are there to choose a distinct element from each subset of

$$X = \{\{3,5,6,7\},\{3,7\},\{1,2,4,5,7\},\{3\},\{1,3,6\},\{1,5,7\},\{1,2,3,6\}\}?$$

Answer. Calculate the permanent of

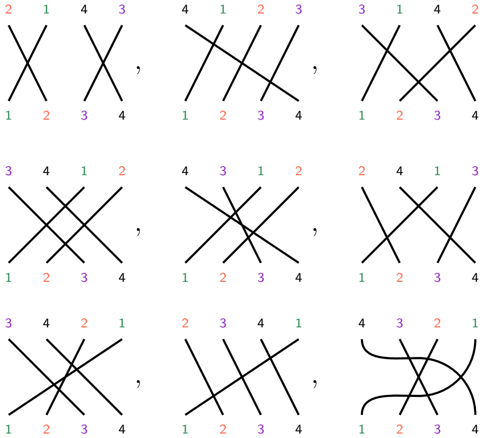
$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

You get $\mathrm{perm}(M) = 2$.

The permanent always counts something

$$\operatorname{perm}(\,0\,) = 0, \quad \operatorname{perm}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1, \quad \operatorname{perm}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2, \quad \operatorname{perm}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 9, \quad \ldots$$

This counts the number of fixpoint-free permutations, *e.g.*



For completeness: A formal definition.

The determinant perm is the unique function (non-trivial: it exists!) from *n*x*n* matrices to the ground field such that:

- $\mathrm{perm}(id) = 1$
- perm is multilinear on columns
- perm is symmetric

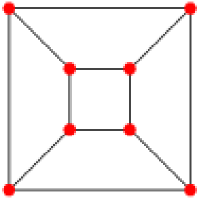
The second bullet point is *e.g.*

$$\mathrm{perm}\begin{pmatrix} a & b+e \\ c & d+f \end{pmatrix} = \mathrm{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \mathrm{perm}\begin{pmatrix} a & e \\ c & f \end{pmatrix}$$

The third bullet point is *e.g.*

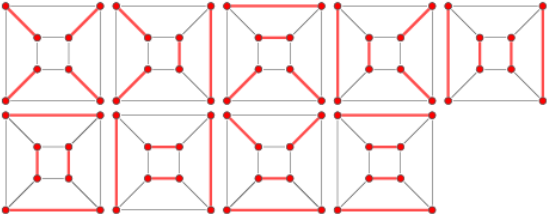
$$\mathrm{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{perm}\begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

More counting



$$\rightsquigarrow \operatorname{perm}\begin{pmatrix} 0&1&1&0&1&0&0&0 \\ 1&0&0&1&0&1&0&0 \\ 1&0&0&1&0&0&1&0 \\ 0&1&1&0&0&0&0&1 \\ 1&0&0&0&0&1&1&0 \\ 0&1&0&0&1&0&0&1 \\ 0&0&1&0&1&0&0&1 \\ 0&0&0&1&0&1&1&0 \end{pmatrix} = 81 = 9^2$$

The permanent counts matchings:



Thank you for your attention!

I hope that was of some help.