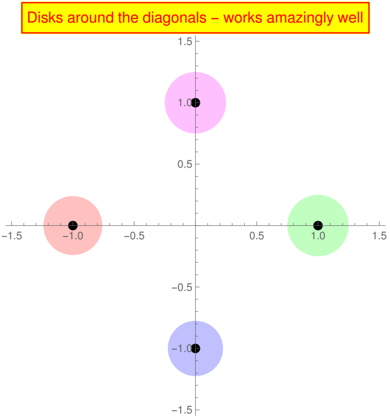
What is...the Gershgorin circle theorem?

Or: Circling the eigenvalues.

Find the eigenvalues without computing them: diagonal entries $\gg$ rest

$$\begin{pmatrix} i & 0.1i & 0.25 & 0 \\ 0 & -1 & -0.1+0.1i & 0.1 \\ 0 & -0.15 & 1 & 0.1 \\ 0.1 & 0 & -0.125 & -i \end{pmatrix}$$

Let us put circles around the diagonals:



Eigenvalues:

$-0.000 + 1.000i$

$-1.007 + 0.008i$

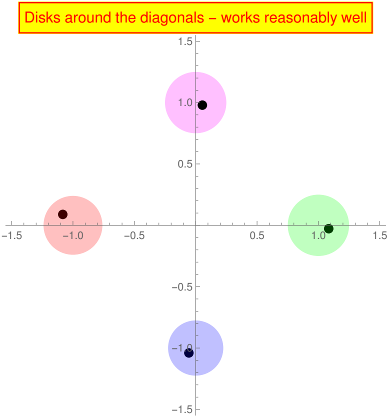
$1.003 - 0.002i$

$0.004 - 1.007i$

Find the eigenvalues without computing them: diagonal entries ≈ rest

$$\begin{pmatrix} i & 1.1i & 0.25 & 0 \\ 0 & -1 & -1.1+0.1i & 0.1 \\ 0 & -0.15 & 1 & 0.1 \\ 1.1 & 0 & -0.125 & -i \end{pmatrix}$$

Let us put circles around the diagonals:



Eigenvalues:

$0.055 + 0.980i$

$-1.083 + 0.089i$

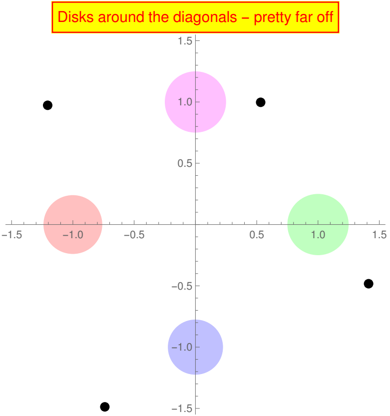
$1.084 - 0.028i$

$-0.056 - 1.040i$

Find the eigenvalues without computing them: diagonal entries < rest

$$\begin{pmatrix} i & 1.8i & 1.25 & 0 \\ 0 & -1 & -1.1+1.5i & 0.1 \\ 0 & -0.15 & 1 & 1.75 \\ 1.1 & 0 & -0.125 & -i \end{pmatrix}$$

Let us put circles around the diagonals:



Eigenvalues:

$0.532+0.997i$

$-1.205+0.972i$

$1.412-0.482i$

$-0.739-1.487i$

Enter, the theorem!

Let $M = (a_{ij})$ be a matrix with entries in $\mathbb{C}$. Then:

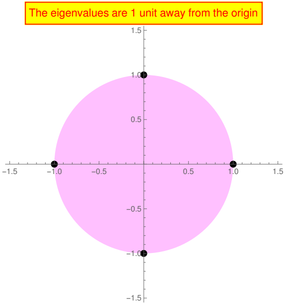
(a) Draw circles $R(i)$ of radius $\sum_{j,j\neq i} |a_{i,j}|$ around a_{ii} The row circles

(b) Draw circles $C(j)$ of radius $\sum_{i,i\neq j} |a_{i,j}|$ around a_{ii} The column circles

(c) Any eigenvalue will be in one of the circles $R(i)$; dually, any eigenvalue will be in one of the circles $C(j)$

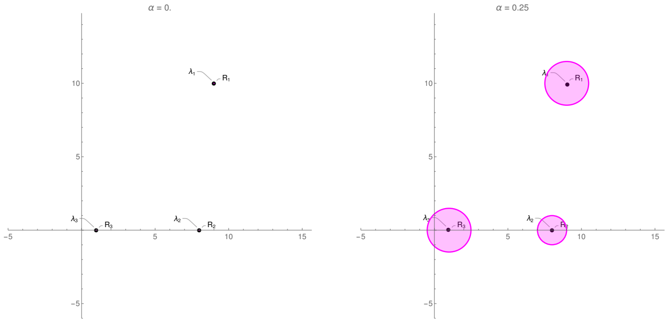
The radius bound is optimal, *e.g.*:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

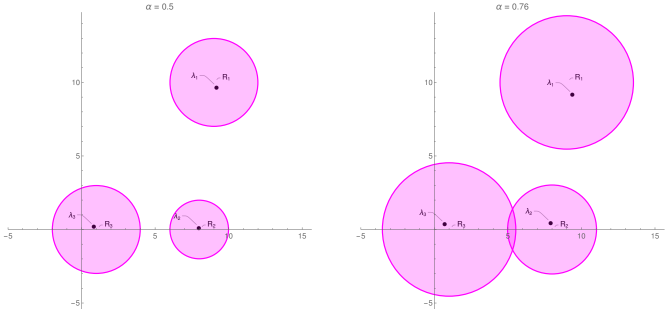


Smooth behavior

$$M(\alpha) = \begin{pmatrix} 1 & -3\alpha & 3\alpha \\ 0 & 8 & -4\alpha \\ 6\alpha & 0 & 9+10i \end{pmatrix}$$



$$M(1/2) = \begin{pmatrix} 1 & -3/2 & 3/2 \\ 0 & 8 & -4/2 \\ 6/2 & 0 & 9+10i \end{pmatrix}$$



Thank you for your attention!

I hope that was of some help.