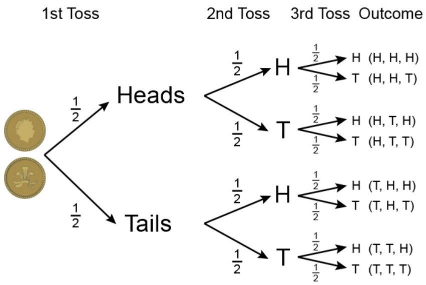
What is...a coin toss run?

Or: Why is this difficult?

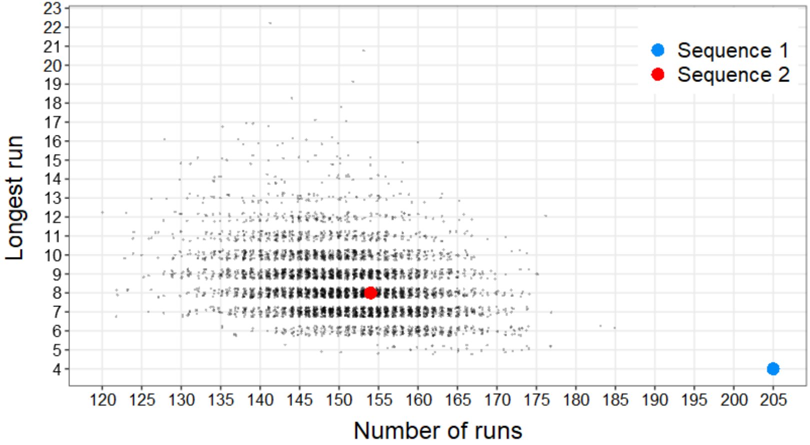
Fair coin tossing



$$p = 1/2$$

- Fair coin toss $=$ heads / tails with probability $1/2$
- Expectation “Everything” about coin tossing should be easy and well-understood
- This video Something obscurely difficult

Runs

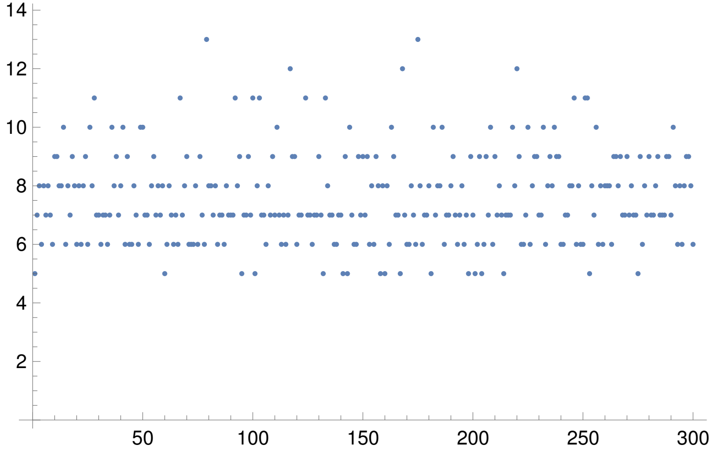


- A classic Humans underestimate the length of runs in coin tossing
- This is often used to distinguish fake from real coin tosses
- Let’s analyze runs mathematically – we will see a surprisingly strange answer

An innocent question

Toss a coin
300 times
Count the
longest 0-run
Repeat 300 times

:



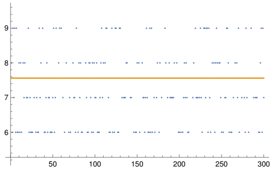
- What is the expected length of the longest run $e_n(heads)$?
- $e_n(heads)$ = here n=number of coin tosses, and we only count head runs
- Sounds easy, right? Well, see above...

Enter, the theorem

$e_n(heads)$ behaves roughly like $\log_2 n - 0.667254$:

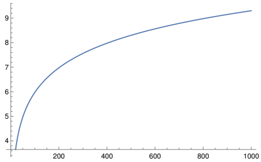
$$e_n(heads) \approx \log_2 n - (\tfrac{2}{3} - \gamma/\ln 2)$$

As before
but now with
the expected value



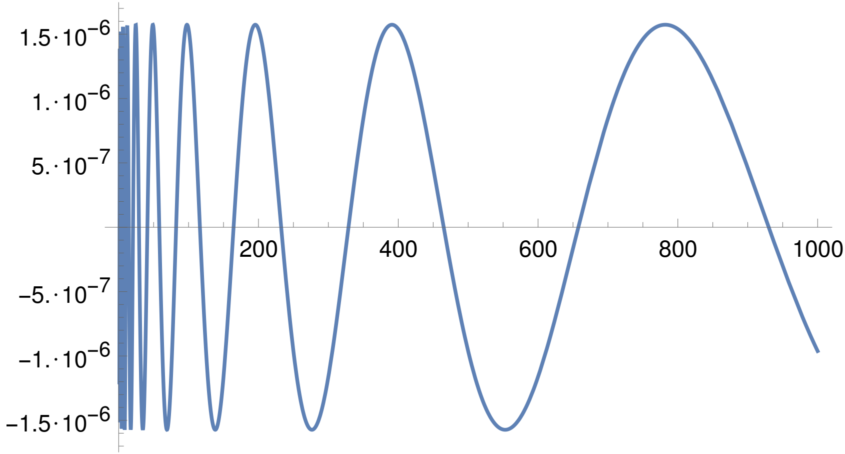
- As potentially expected, we get a log:

$e_n(heads) \approx$



- Unexpected: the offset by 0.667254... (γ = Euler–Mascheroni's gamma)

There is another error term...



- The real formula $e_n(heads) = \log_2 n - (\frac{2}{3} - \gamma/\ln 2) + \delta(n) + o(1)$
- $\delta(n)$ is a oscillating and tiny error function: $|\delta(n)| < 10^{-5}$
- Weird, but this happens often

Thank you for your attention!

I hope that was of some help.