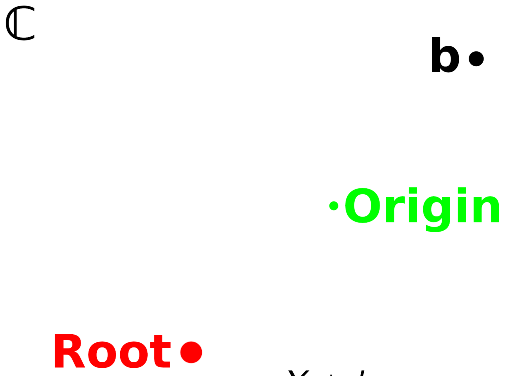
What is...the Abel–Ruffini theorem?

Or: Loops and roots

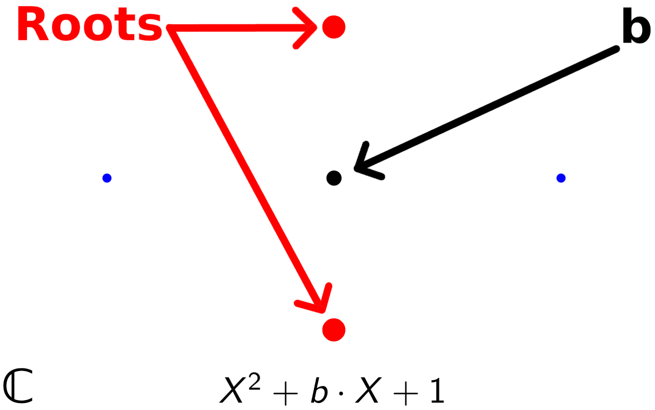


$$X + b$$

- The operations of arithmetic $+, -, \times, \div$ suffice to solve polynomial equations of degree 1:

$$\text{roots} = -b$$

- Suffice = potentially iterated combinations of symbols from $+, -, \times, \div$ and coefficients of $a \cdot X + b$
- Question How far can we push this using only the operations of arithmetic?

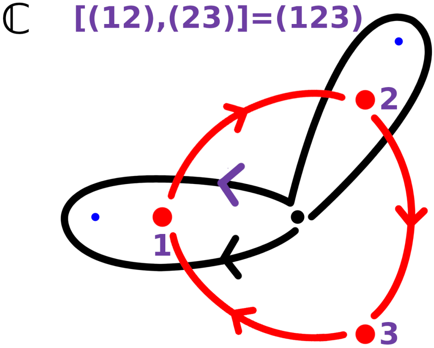


$$X^2 + b \cdot X + 1$$

- Looping b around blue points exchanges the roots
- Looping b around blue points fixes the coefficients
- Square roots can not be defined using the operations of arithmetic:

  There is no quadratic formula built out of a finite combination of $+, -, \times, \div$

Polynomials of degrees $2, 3, 4$ and algebra



$$X^3 + b \cdot X + 1$$

- The commutator $[a, b] = aba^{-1}b^{-1}$ of loops shows that nesting $\sqrt[n]{\_}$ for $n > 2$ is necessary, but that is an operation of algebra
- Allowing the new operation $\sqrt[n]{\_}$ solves the problem:

$$\text{roots} = \frac{1}{2a}\big(-b \pm \sqrt[2]{b^2 - 4ac}\big) \text{ degree 2}$$

degrees $3, 4$ were done around 1500, but are "ugly"

- Question How far can we push this using only the operations of algebra?

Enter, the theorem and proof

For degree 5 and bigger the operations of algebra do not suffice

- Non-trivial k-commutator-loops describe how often $\sqrt[n]{\_}$ needs to be nested, *e.g.*:

$$[[(12),(23)],[(23),(34)]] = (14)(23)$$
$$\Rightarrow$$
$$\text{degree} \geq 4 \text{ needs at least 2-nesting stages}$$

- k-commutator-loops can be arbitrary nested for degree ≥ 5:

$$[(ijk),(klm)] = (jkm)$$

  Needs five symbols!

- No degree ≥ 5 solution formula using finite nested expressions

What the Abel–Ruffini theorem **not** implies

- Algebraic solutions Certain equation can be solved *e.g.*

$$\left(X^5 - 1 = 0\right) \Leftrightarrow \left(X = e^{k \cdot 2\pi i/5}, k \in \{0,1,2,3,4\}\right)$$

- Analytic solutions Using infinitely nested $\sqrt[n]{\_}$ one can write down formulas for roots, *e.g.*

$$\sqrt[2]{1+\sqrt[2]{1+\sqrt[2]{1+\sqrt[2]{1+\cdots}}}} \xrightarrow{\text{converges}} \text{a solution of } X^2 - X - 1$$

...

$$\sqrt[5]{1+\sqrt[5]{1+\sqrt[5]{1+\sqrt[5]{1+\cdots}}}} \xrightarrow{\text{converges}} \text{a solution of } X^5 - X - 1$$

...

$$\sqrt[n]{1+\sqrt[n]{1+\sqrt[n]{1+\sqrt[n]{1+\cdots}}}} \xrightarrow{\text{converges}} \text{a solution of } X^n - X - 1$$

...

Thank you for your attention!

I hope that was of some help. $X^5 - X - 1$