What is...the pea and the sun paradox?

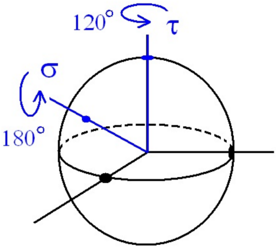
Or: One orange equals two oranges

This “paradox” is true

An orange may be separated into a finite number of pieces and reassembled into two oranges identical to the original

- You can do with 5 pieces
- No reshaping needed Rotations are the key
- There are some set theoretical issues Axiom of choice (AoC)

Rotation of $\mathbb{R}^3$ are crazy!



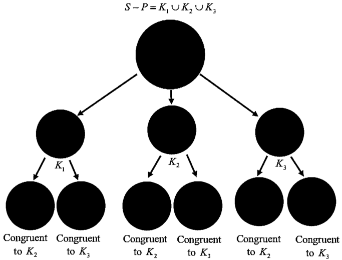
$G = \langle \tau, \sigma \rangle$

- The rotations τ and σ form an infinite group G
- Partition G into three subsets G_1, G_2 and G_3

$$G \to G_1 = \{1, \sigma\tau, \sigma\tau^2, \dots\} \quad G_2 = \{\sigma, \tau, \tau\sigma\tau, \dots\} \quad G_3 = \{\tau^2, \tau\sigma, \tau^2\sigma\tau, \dots\}$$

- We get associated poles P on S^2

Enter, (AoC)



- G partitions $S^2 \setminus P$ into orbits, take one piece per orbit and obtain C AoC
- $K_i = G_i \circlearrowleft C$, and we get the Hausdorff partition of the sphere S^2

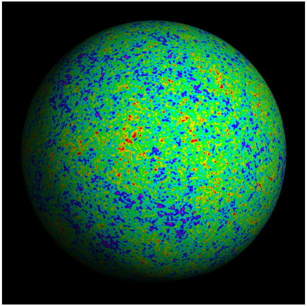
$$S^2 \setminus P = K_1 \cup K_2 \cup K_3, \quad K_1 \approx K_2 \approx K_3 \approx (K_2 \cup K_3)$$

- This is weird: K_i are "$\frac{1}{3}$ of $S^2 \setminus P$" but $K_2 \cup K_3$ is also "$\frac{1}{3}$ of $S^2 \setminus P$"
- Use "$\frac{1}{3}$ of $S^2 \setminus P$" $K_2 \cup K_3$ to create a second copy of $S^2 \setminus P$
- Thicken the whole story to the orange (a.k.a. 3-ball)

Enter, the theorem

Banach–Tarski A solid ball may be separated into a finite number of pieces and reassembled into two solid balls identical in shape and volume to the original

- The partitions from before are a bit like cosmic microwave background:



$K_1 \leftrightsquigarrow$ red/yellow

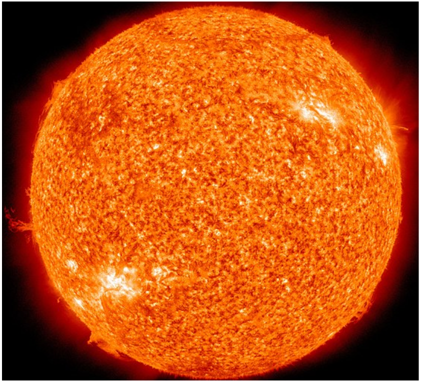
$K_2 \leftrightsquigarrow$ blue

$K_3 \leftrightsquigarrow$ green

- This theorem can not be proven without (AoC)
- One can beef the theorem up and obtain infinitely many copies

The pea and the sun

$\approx$



If A and B are any two bounded 3d sets with non-empty interiors than $A \approx B$

Thank you for your attention!

I hope that was of some help.