The $\mathfrak{sl}_3$ web algebra

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Joint work with Marco Mackaay and Weiwei Pan

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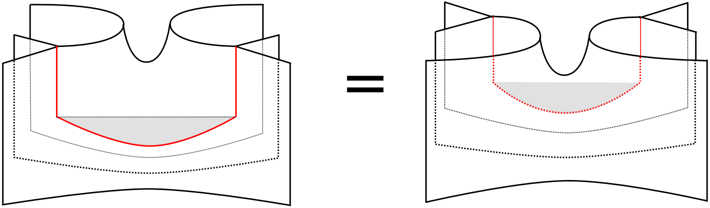


A well-known example

Noether, Hopf, Mayer

Let X be a reasonable finite-dimensional spaces. Then the homology groups $H_k(X)$ are a categorification of the Betti numbers of X and the singular chain complex (C, d_i) is categorification of the Euler characteristic of X.

Note the following common features of the two examples above.

- The Betti numbers and the Euler characteristic can be seen as parts of "bigger, richer" structures.
- In both categorifications it is very easy to "decategorify", i.e. by taking the dimension or the alternating sum of the dimensions.
- Both notions are not obvious, e.g. the first notion of "Betti numbers" was in the year 1857 (B. Riemann!!) and the first notion of "homology groups" was in the year 1925.

The ladder of categories

$\vdots$

forget 3-arrows ↓ / ↑ add 3-arrows

$2-\text{categories} = $ "arrows between arrows"

forget 2-arrows ↓ / ↑ add 2-arrows

$1-\text{categories} = $ usual categories

forget arrows ↓ / ↑ add arrows

$0-\text{categories} = $ sets, vector spaces

size ↓ / ↑ internal structure

$-1-\text{categories} = $ cardinals, numbers

Categorification is ill-defined

Note that the notion of categorification is ill-defined. The rough idea is to replace set theoretical structures by category theoretical structures. So categorification could mean

- An inverse process of some decategorification, e.g.
  - Degroupoidification (Baez, Dolan, Trimble): a functor $D\colon \mathrm{Span}(\mathrm{Gpd}) \to \mathrm{Hilb}$.
  - Grothendieck group $C \mapsto K_0(C)$ constructions (Khovanov, Lauda).
  - Dimension $V \mapsto \dim(V)$ constructions (homology groups).
  - more...
- Common feature: decategorification is easy, categorification is hard.
- Reveals hidden structure.

Today we use decategorification = Grothendieck group.

Webs and foams

If you live in a two-dimensional world, then it is easy to imagine a one-dimensional world, but hard to imagine a three-dimensional world!

The rough idea

The classical picture.

$\mathfrak{sl}_n$-webs ←— Intertwiners —→ $U_q(\mathfrak{sl}_n)$-Tensors

$\mathfrak{sl}_n$-webs —*Kauffman, Kuperberg, MOY*→ $\mathfrak{sl}_n$-knot polynomials

$U_q(\mathfrak{sl}_n)$-Tensors —*Reshetikhin, Turaev*→ $\mathfrak{sl}_n$-knot polynomials

And its categorification.

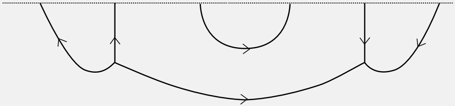
$\mathfrak{sl}_n$-foams ←— **???** —→ $\mathfrak{sl}_n$-string diagrams

$\mathfrak{sl}_n$-foams —*Khovanov, Khovanov−Rozansky*→ $\mathfrak{sl}_n$-knot homologies

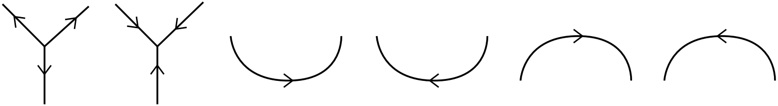
$\mathfrak{sl}_n$-string diagrams —*Webster*→ $\mathfrak{sl}_n$-knot homologies

Basic definitions

Example



A web is an oriented trivalent graph such that any vertex is either a sink or a source. Any web can be obtained by gluing and disjoint union of some basic webs.



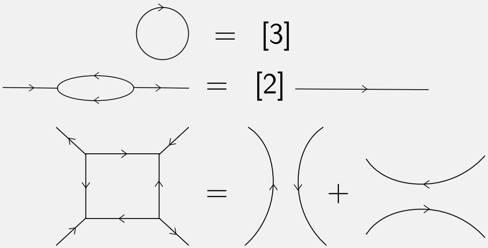
The boundary of a web corresponds to a sign string S, i.e. $+$, if the orientation is pointing in, and $-$ otherwise. The sign string for the example is $S = (++-+--)$.

Definition(Kuperberg)

The web space W_S for a given sign string S is

$$W_S = \mathbb{C}(q)\{w \mid \partial w = S\}/I_S,$$

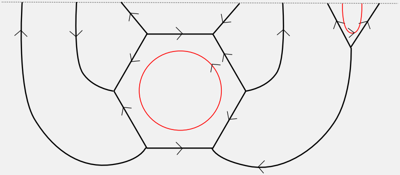
where I_S is generated by the relations



Here $[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-(a-1)}$ is the quantum integer.

Basic definitions

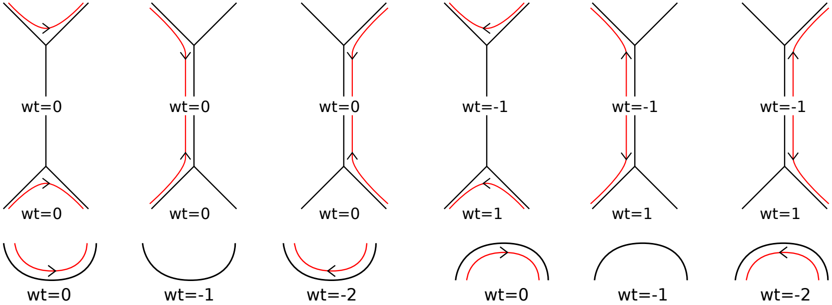
Example



Webs can be coloured with flows: Define a flow f on a web w to be an oriented subgraph that contains exactly two of the three edges incident to each trivalent vertex. The connected components are called the flow lines. At the boundary, the flow lines can be represented by a state string J. By convention, at the i-th boundary edge, we set $j_i = +1$ if the flow line is oriented upward, $j_i = -1$ if the flow line is oriented downward and $j_i = 0$ there is no flow line. So $J = (0, 0, 0, 0, 0, -1, 1)$ in the example.

Basic definitions

Given a web with a flow, denoted w_f attribute a weight to each trivalent vertex and each arc in w_f. The total weight of w_f is by definition the sum of the weights at all trivalent vertices and arcs.



The total weight from the example before is -3.

A sign string $S = (s_1, \dots, s_n)$ corresponds to

$$V_S = V_{s_1} \otimes \cdots \otimes V_{s_n},$$

where V_+ is the fundamental representation and $V_- \cong V_+ \wedge V_+$ its dual. Webs correspond to intertwiners.

Theorem(Kuperberg)

$$W_S \cong \mathrm{Hom}(\mathbb{C}(q), V_S) \cong \mathrm{Inv}(V_S)$$

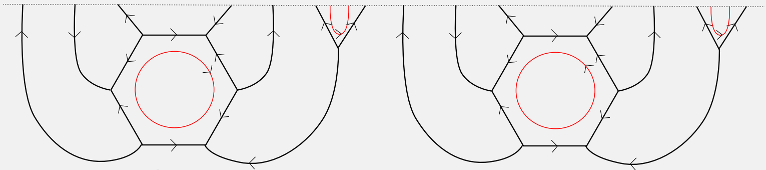
The basis of W_S, denoted B_S, is called web basis of $\mathrm{Inv}(V_S)$. From the relations before, it follows that the webs of B_S are non-elliptic webs, i.e. without circles, digons or squares.

Representation theory of $U_q(\mathfrak{sl}_3)$

Theorem(Khovanov, Kuperberg)

A pair of a sign string $S = (s_1, \ldots, s_n)$ and a state string $J = (j_1, \ldots, j_n)$ correspond to the coefficients of the web basis relative to the standard basis $\{e^{\pm}_{-1}, e^{\pm}_0, e^{\pm}_{+1}\}$ of $V_{\pm}$.

Example



The basis web w_S has a decomposition

$$w_S = \cdots - (q^{-1} + q^{-3})(e^+_0 \otimes e^-_0 \otimes e^+_0 \otimes e^-_0 \otimes e^+_0 \otimes e^+_{-1} \otimes e^+_{+1}) \pm \cdots .$$

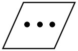
Please, fasten your seat belts!

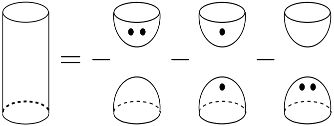
Let's categorify everything!

$\mathfrak{sl}_3$-foams

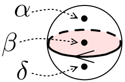
A pre-foam is a cobordism with singular arcs between two webs. Pre-foam composition consists of placing one pre-foam on top of the other. The orientation of the singular arcs is, by convention, as in the diagrams below (called the zip and the unzip respectively):

We allow pre-foams to have dots that can move freely about the facet on which they belong, but we do not allow dot to cross singular arcs.
A foam is a formal $\mathbb{C}$-linear combination of isotopy classes of pre-foams modulo the following relations.

 $= 0$ (3D)

 (NC)

 $=$  $= 0,$  $= -1$ (S)

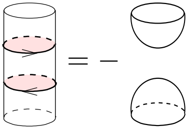


$$= \begin{cases} 1, & (\alpha, \beta, \delta) = (1,2,0) \text{ or a cyclic permutation,} \\ -1, & (\alpha, \beta, \delta) = (2,1,0) \text{ or a cyclic permutation,} \\ 0, & \text{else.} \end{cases}$$ (Θ)

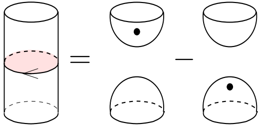
The relations $\ell = (3D, NC, S, \Theta)$ suffice to evaluate any closed foam!

Just to frighten you: more relations

From the relations ℓ follow a lot of identities.

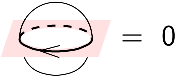


(Bamboo)

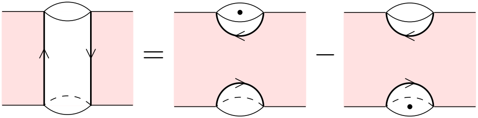


(RD)

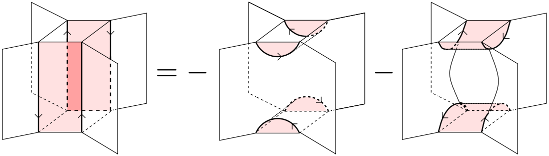
And more relations



(Bubble)

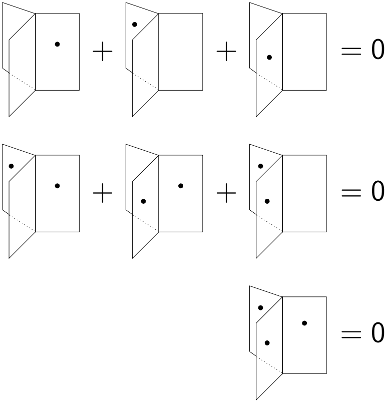


(DR)



(SqR)

And even more relations



(Dot Migration)

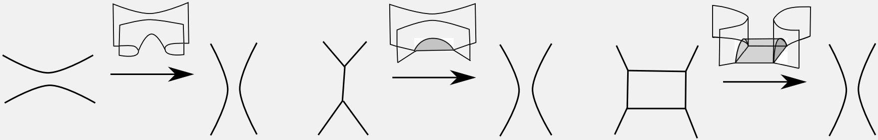
The $\mathfrak{sl}_3$-foam category

Let **Foam**$_3$ be the category of foams, i.e. objects are webs and morphisms are foams between webs.
The category is graded by the $q-$degree of a foam F

$$q(F) = \chi(\partial F) - 2\chi(F) + 2d + b,$$

where d is the number of dots and b is the number of vertical boundary components.

Example

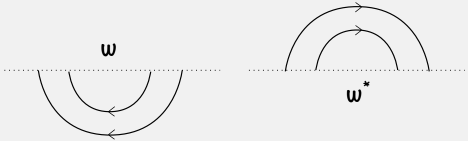


The q-degrees are 2, 1 and 0 respectively.

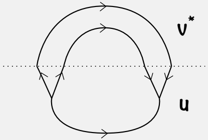
Closed webs and foams

Definition

There is an involution * on the webs.



A closed web is defined by closing of two webs.



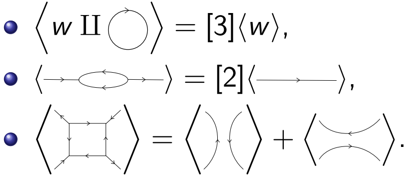
A closed foam is a foam from $\emptyset$ to a closed web.

Definition

The foam homology of a closed web w is defined by

$$\mathcal{F}(w) = \mathbf{Foam}_3(\emptyset, w).$$

$\mathcal{F}(w)$ is a graded complex vector space, whose q-dimension can be computed by the Kuperberg bracket:



The relations above correspond to the decomposition of $\mathcal{F}(w)$ into direct summands.

The $\mathfrak{sl}_3$ web algebra

Definition(MPT)

Let $S = (s_1, \ldots, s_n)$. The $\mathfrak{sl}_3$ web algebra K_S is defined by

$$K_S = \bigoplus_{u,v \in B_S} {}_uK_v,$$

with

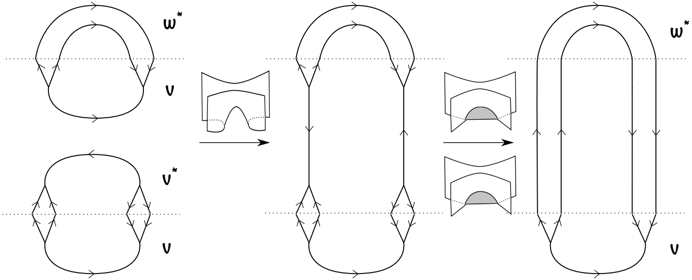
$${}_uK_v := F(u^* v)\{n\}.$$

Multiplication is defined as follows:

$${}_uK_{v_1} \otimes {}_{v_2}K_w \to {}_uK_w$$

is zero, if $v_1 \neq v_2$. If $v_1 = v_2$, use the multiplication foam m_v, e.g.

The $\mathfrak{sl}_3$ web algebra



Proposition(MPT)

The multiplication is associative and unital. The multiplication foam m_v only depends on the isotopy type of v and has q-degree n. Hence, K_S is a finite dimensional, unital and graded algebra.

En(c)hanced sign strings

Definiton

An enhanced sign sequence is a sequence $S = (s_1, \ldots, s_n)$ with $s_i \in \{\circ, -, +, \times\}$, for all $i = 1, \ldots, n$. The corresponding weight $\mu = \mu_S \in \Lambda(n, d)$ is given by the rules

$$\mu_i = \begin{cases} 0, & \text{if } s_i = \circ, \\ 1, & \text{if } s_i = 1, \\ 2, & \text{if } s_i = -1, \\ 3, & \text{if } s_i = \times. \end{cases}$$

Let $\Lambda(n, d)_3 \subset \Lambda(n, d)$ be the subset of weights with entries between 0 and 3. For any enhanced sign string S, we define $\widehat{S}$ by deleting the entries equal to $\circ$ or $\times$.

En(c)hanced sign strings

Moreover for $n = d = 3^k$ we define

$$W_S = W_{\widehat{S}} \text{ and } B_S = B_{\widehat{S}} \text{ and } W_{(3^k)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} W_S$$

one the level of webs and on the level of foams, we define

$$K_S = K_{\widehat{S}} \text{ and } \mathcal{W}_{(3^k)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} K_S - \mathrm{pmod}_{gr}.$$

I will sketch in the following how we obtain one of our main results as a corollary.

Corollary(MPT)

$$K_0(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong W_{(3^k)}.$$

Skew Howe-duality

The natural actions of GL_k and GL_n on

$$\mathrm{Alt}^p(\mathbb{C}^k \otimes \mathbb{C}^n) = \Lambda^p(\mathbb{C}^k \otimes \mathbb{C}^n)$$

are Howe dual (skew Howe duality).
This implies that

$$\mathrm{Inv}_{\mathrm{SL_k}}(\Lambda^{p_1}(\mathbb{C}^k) \otimes \cdots \otimes \Lambda^{p_n}(\mathbb{C}^k)) \cong W(p_1, \ldots, p_n),$$

where $W(p_1, \ldots, p_n)$ denotes the $(p_1, \ldots, p_n)$-weight space of the irreducible GL_n-module $W(k^\ell)$, if $n = k^\ell$.

Quantum groups

Definition

- The algebra $\mathbf{U}_q(\mathfrak{gl}_n)$ is generated by $K_1^{\pm 1}, \dots, K_n^{\pm 1}$ and $E_{\pm 1}, \dots, E_{\pm(n-1)}$ subject to a long list of relations.
- The algebra $\mathbf{U}_q(\mathfrak{sl}_n) \subset \mathbf{U}_q(\mathfrak{gl}_n)$ is generated by $K_i K_{i+1}^{-1}$ and $E_{\pm i}$.
- Their idempotented completions $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ and $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$, are defined by adjoining idempotents 1_λ for any weight $\lambda \in \mathbb{Z}^n$ (and $\lambda \in \mathbb{Z}^{n-1}$ for the special linear group) subject to a long list of relations.

Note that the idempotented complete version are much easier, e.g. it is much easier to write down a nice basis.

A finite-dimensional semi-simple quotient

Lemma(Doty, Giaquinto)

The q-Schur algebra $S_q(n,d)$ is generated by 1_λ, for $\lambda \in \Lambda(n,d)$, and $E_{\pm 1}$, for $i = 1, \dots, n-1$, such that

$$1_\lambda 1_\lambda = \delta_{\lambda,\mu} 1_\lambda,$$

$$\sum_{\lambda \in \Lambda(n,d)} 1_\lambda = 1,$$

$$E_{\pm 1} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm 1},$$

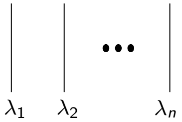
$$E_i E_{-j} - E_{-j} E_i = \delta_{i,j} \sum_{\lambda \in \Lambda(n,d)} [\lambda_i - \lambda_{i+1}] 1_\lambda.$$

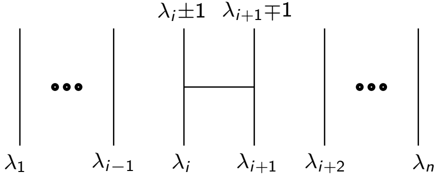
It is finite-dimensional and semi-simple. It is known that

$$S_q(n,n) 1_{3^\ell} / (\mu > (3^\ell)) \cong V_{(3^\ell)}.$$

The action

We defined an action ϕ of $S_q(n,n)$ on $W_{(3^\ell)}$ by

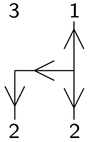
$1_\lambda \mapsto$ 

$E_{\pm i}1_\lambda \mapsto$ 

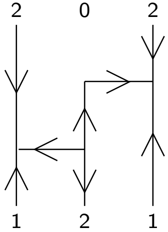
We use the convention that vertical edges labeled 1 are oriented upwards, vertical edges labeled 2 are oriented downwards and edges labeled 0 or 3 are erased. The hard part was to show that this is well-defined.

Exempli gratia

$E_{+1}1_{(22)} \mapsto$



$E_{-2}E_{+1}1_{(121)} \mapsto$



An instance of skew Howe-duality

Lemma

The action ϕ gives rise to an isomorphism

$$\phi\colon V_{(3^\ell)} \to W_{(3^\ell)}$$

of $S_q(n,n)$-modules.

Note that their are categorifications of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ and $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$, denoted as $\mathcal{U}(\mathfrak{sl}_n)$ and $\mathcal{U}(\mathfrak{gl}_n)$, by Khovanov and Lauda.
The idea now is to categorify the whole process!

Some work was already done!

Theorem(Mackaay, Stošić, Vaz)

Define, similar to the uncategorified story, a 2-category $\mathcal{S}(n,d)$. Let $\dot{\mathcal{S}}(n,d)$ be the Karoubi envelope of $\mathcal{S}(n,d)$. Then

$$K_0(\dot{\mathcal{S}}(n,d)) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong S_q(n,d).$$

The following was conjectured by Khovanov and Lauda in 2008. Note that $\mathcal{V} = R_\lambda - \mathrm{pmod}_{\mathrm{gr}}$ for $\lambda \in \Lambda(n,n)^+$ (the algebra R_λ is a quotient of $\mathcal{S}(n,d)$ and is called Khovanov-Lauda-Rouquier algebra).

Theorem(Brundan-Kleshchev, Lauda-Vazirani, Webster, Kang-Kashiwara,...)

As $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ we have

$$K_0(\mathcal{V}_\lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong V_\lambda.$$

The action (categorified)

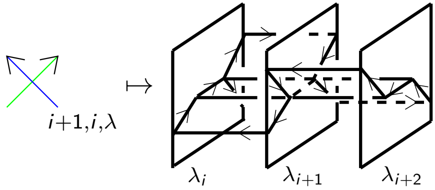
We defined an action ϕ of $\mathcal{S}_q(n,n)$ on $\mathcal{W}_{(3^\ell)}$ by

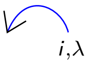
- On objects its the aforementioned action ϕ of $S_q(n,n)$ on $W_{(3^\ell)}$.
- On morphisms we do it, like before, on the generators.

Note that this time everything gets (categorification is "richer", remember?) more complicated, i.e. their are eleven completely different generators instead of two, their are way more relations to check and the pictures are two-dimensional now.

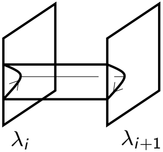
Lets me give two of the definitions for the generators and one example one has to check.

The signs are important!



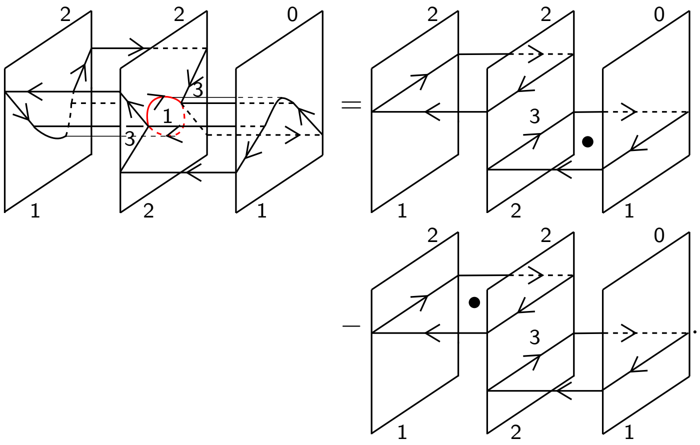


$$\mapsto (-1)^{\lceil \frac{\lambda_i}{2} \rceil + \lfloor \frac{\lambda_{i+1}}{2} \rfloor}$$



But until everything is checked, we get the very nice result that this action is well-defined.

It's horrible!



It's harvest time!

By Rouquier's universality theorem, after pulling back the categorical action, we get

Theorem(MPT)

Let $\mathcal{V}$ be any idempotent complete category, which allows an integrable graded categorical action by $\mathcal{U}(\mathfrak{sl}_n)$ (plus some extra conditions). Then there exists an equivalence of categorical $\mathcal{U}(\mathfrak{sl}_n)$-representations

$$\Phi\colon \mathcal{V}_{(3^k)} \to \mathcal{W}_{(3^k)},$$

and therefore to $\mathcal{V}$.

Note that we are using the $\mathfrak{sl}_3$ web algebra to obtain the result for $\mathcal{U}(\mathfrak{sl}_n)$!

It's harvest time!

Checking all the definitions, we see that we have a commuting square of isomorphisms (bijective isometries even). Hence, we finally get our hands on K_0.

$$\begin{array}{ccc} V_{(3^k)} & \xrightarrow{\gamma_{(3^k)}} & K_0(\mathcal{V}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \\ \downarrow \phi & & \downarrow K_0(\Phi) \\ W_{(3^k)} & \xrightarrow[\psi]{} & K_0(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \end{array}$$

Corollary(MPT)

$$K_0(\mathcal{W}_{(3^k)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong W_{(3^k)}.$$

More to harvest!

The result above leads to the following theorem.

Theorem(MPT)

The two algebras $R_{3\ell}$ and $K_{3\ell}$ are Morita equivalent.

Note that Morita invariant properties can be check in both algebras now.

A trace form on the $\mathfrak{sl}_3$ web algebra

Definition

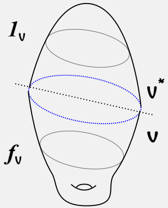
Their is a natural trace form on the algebra K_S. We take, by definition, the trace form

$$\mathrm{tr}\colon K_S \to \mathbb{C}$$

to be zero on ${}_uK_v$, when $u \neq v \in B_S$. For any $v \in B_S$, we define

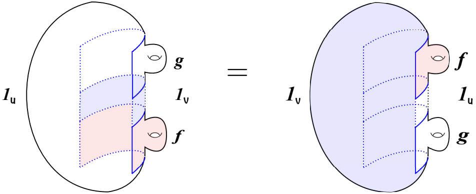
$$\mathrm{tr}\colon {}_vK_v \to \mathbb{C}$$

by closing any foam f_v with 1_v, e.g.



It's Frobenius!

The trace is non-degenerated and symmetric. Both can be seen geometrical, e.g. the fact that $\mathrm{tr}(gf) = \mathrm{tr}(fg)$ holds follows from sliding f around the closure until it appears on the other side of g, e.g.



The non-degenerate trace form on K_S gives rise to a graded (K_S, K_S)-bimodule isomorphism $K_S^\vee \cong K_S\{-2n\}$, i.e. we have

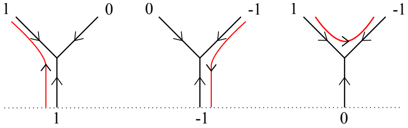
Theorem(MPT)

For any sign string S of length n, the algebra K_S is a graded, symmetric Frobenius algebra of Gorenstein parameter $2n$.

Tableaux and flows

Let p_S be the number of positive entries and n_S the number of negative entries of S. By definition, we have that $d = p_S + 2n_S$. Our key idea is to reduce everything to the case where $n_S = 0$. Fix any state string J of length n, we define a new state string $\widehat{J}$ of length d by the following algorithm:

1. Let ${}_0\widehat{J}$ be the empty string.
2. For $1 \leq i \leq n$, let ${}_i\widehat{J}$ be the result of concatenating j_i to ${}_{i-1}\widehat{J}$ if $\mu_i = 1$. If $\mu_i = 2$ then
    1. concatenate $(1, 0)$ to ${}_{i-1}\widehat{J}$ if $j_i = 1$,
    2. concatenate $(0, -1)$ to ${}_{i-1}\widehat{J}$ if $j_i = -1$,
    3. concatenate $(1, -1)$ to ${}_{i-1}\widehat{J}$ if $j_i = 0$.



Tableaux and flows

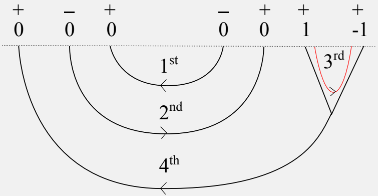
Proposition(MPT)

There is a bijection between $\mathrm{Col}_{\mu}^{\lambda}$ and the set of state strings J such that there exists a $w \in B_S$ and a flow f on w which extends J. The bijection is given by an algorithm.

Example

The tableau on the left gives rise to the web with flow next to it.

1	0	-1
2	1	2
4	3	4
6	5	7



For other choices the same tableau generates the following web with flow.

The center of K_S

Let X^λ_μ be the (λ,μ)-Spaltenstein variety. Note that, if $n_s = 0$, then $X^\lambda_\mu = X^\lambda$, the latter being the Springer fiber associated to λ.
Let $P = \mathbb{C}[x_1,\ldots,x_d]$. If μ is the composition associated to S, then let S_μ be the corresponding parabolic subgroup of the symmetric group S_d and therefore let $P^\mu := P^{S_\mu} \subset P$ be the subring of polynomials which are invariant under S_μ.
For a specific ideal I^λ_μ let $R^\lambda_\mu := P^\mu/I^\lambda_\mu$. Brundan and Ostrik proved that

$$H^*(X^\lambda_\mu) \cong R^\lambda_\mu.$$

The center of K_S

We showed that R^λ_μ acts on K_S and that (as graded complex algebras)

$$R^\lambda_\mu 1 \subset Z(K_S).$$

By a dimension argument (based on Morita equivalence) we get

Theorem(MPT)

$H^*(X^\lambda_\mu)$ is isomorphic (as graded algebras) to $Z(K_S)$. The dimension of the center is $\#\mathrm{Col}^\lambda_\mu$, i.e. the center is parametrised by flows on the boundary line.

Since one can say that X^λ_μ "generalises" Schubert calculus, we say that $Z(K_S)$ "categorifies" a part of the calculations with symmetric polynomials.

Two motivating examples

Example

Let $A = \mathrm{M}_{n\times n}(R)$, i.e. the set of $n \times n$-matrices over R. Set $\mathfrak{P} = \{*\}$ and $\mathcal{T}(*) = \{1, \dots, n\}$. The standard basis of A, i.e. the e_{ij}-matrices, has a very special property, namely that the coefficients for multiplication with a matrix from the right only depend on the row i and vice versa for multiplication from the left. Moreover, for $\mathrm{i}(M) = M^t$, we have $\mathrm{i}(e_{ij}) = e_{ji}$.

Example

Let $A = R[x]/(x^n)$ and $\mathrm{i} = \mathrm{id}$. Then set $\mathfrak{P} = \{0, \dots, n-1\}$ and $\mathcal{T}(k) = \{1\}$. Then the standard basis $c_{11}^k = x^k$ has a very special property, namely that the coefficients for multiplication only depends on higher powers of x (modulo x^n).

The idea of Graham and Lehrer was to "interpolate" between the two extremes.

Cellular algebras

Definition(Graham, Lehrer)

Suppose A is a free algebra over R of finite rank. A cell datum is an order quadruple $(\mathfrak{P}, \mathcal{T}, C, \mathrm{i})$, where $(\mathfrak{P}, \rhd)$ is the weight poset, $\mathcal{T}(\lambda)$ is a finite set for all $\lambda \in \mathfrak{P}$, i is an involution and an injection

$$C\colon \coprod_{\lambda \in \mathfrak{P}} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \to A,\ (s,t) \mapsto c_{st}^{\lambda},$$

such that the c_{st}^{λ} form a R-basis of A with $\mathrm{i}(c_{st}^{\lambda}) = c_{ts}^{\lambda}$ and for all $a \in A$

$$c_{st}^{\lambda} a = \sum_{u \in \mathcal{T}(\lambda)} r_{tu}(a) c_{st}^{\lambda} \pmod{A^{\rhd \lambda}}.$$

The c_{st}^{λ} are called a cellular basis of A (with respect to the involution i).

K_S is cellular!

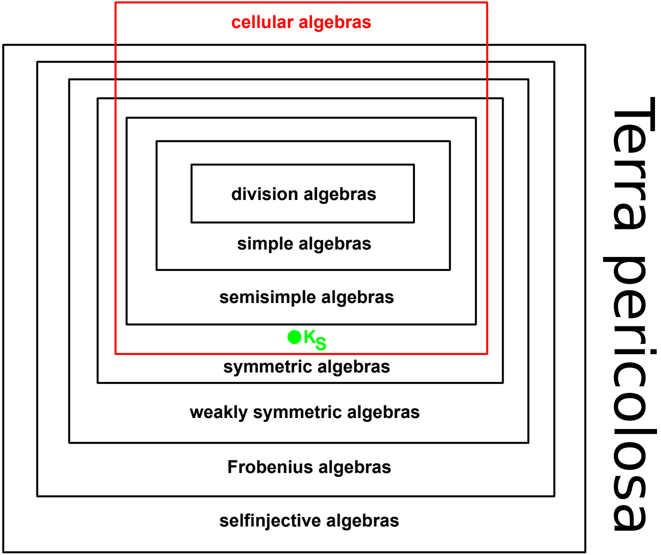
Note that the whole notions of cellularity can be generalised to the concept of graded cellularity. As mentioned before, we know that the algebras $K_{(3^k)}$ and $R_{(3^k)}$ are Morita equivalent. Hu and Mathas showed that latter is a graded cellular algebra. Moreover, König and Xi showed that cellularity is (up to some technicalities with the involutions) an invariant under Morita equivalence. Hence, we have:

Theorem(MPT)

The algebra K_S is a finite dimensional, graded cellular and symmetric Frobenius algebra.

Note that we don't have a cellular basis at the moment (the proof of the invariance of cellularity is not constructive), but we have a good candidate!

The world of algebras



There is still much to do...

Thanks for your attention!