Cyclotomic quiver Hecke algebras I

Quiver Hecke algebras and categorification

Andrew Mathas

University of Sydney

Categorification, representation theory and symplectic geometry
Hausdorff Research Institute for Mathematics
November 2017

Outline of lectures

1. Quiver Hecke algebras and categorification
   - Basis theorems for quiver Hecke algebras
   - Categorification of $U_q(\mathfrak{g})$
   - Categorification of highest weight modules
2. The Brundan-Kleshchev graded isomorphism theorem
   - Seminormal forms and semisimple KLR algebras
   - Lifting idempotents
   - Cellular algebras
3. The Ariki-Brundan-Kleshchev categorification theorem
   - Dual cell modules
   - Graded induction and restriction
   - The categorification theorem
4. Recent developments
   - Consequences of the categorification theorem
   - Webster diagrams and tableaux
   - Content systems and seminormal forms

Generalised Cartan matrices

Let I be a (finite) indexing set

Let $C = (C_{ij})_{i,j\in I}$ be a symmetrizable generalised Cartan matrix:
$\iff$ $c_{ii} = 2$, $c_{ij} \le 0$ if $i \ne j$, $c_{ij} = 0 \iff c_{ji} = 0$
and DC is symmetric for a diagonal matrix $D = \mathrm{diag}(d_i|i \in I)$, for $d_i > 0$

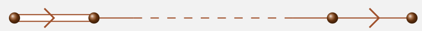
We assume that C is indecomposable in the sense that if $\emptyset \ne J \subsetneq I$ then there exist $i \in I \setminus J$ and $j \in J$ such that $c_{ij} \ne 0$

A Cartan datum $(P, P^\vee, \Pi, \Pi^\vee)$ for C consists of:

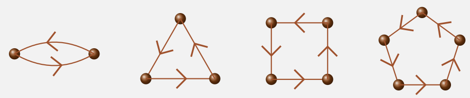
- A weight lattice P with basis fundamental weights $\{\,\Lambda_i \mid i \in I\,\}$
- Dual weight lattice $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$
- Simple roots $\Pi = \{\,\alpha_i \mid i \in I\,\}$
- Simple coroots $\Pi^\vee = \{\,h_i \mid i \in I\,\} \subset P^\vee$
- A pairing such that $\langle h_i, \alpha_j\rangle = c_{ij}$ and $\langle h_i, \Lambda_j\rangle = \delta_{ij}$

Let $\mathfrak{h}^* = \mathbb{Q} \otimes_{\mathbb{Z}} P \implies$ symmetric bilinear form on $\mathfrak{h}^*$: $(\alpha_i, \alpha_j) = d_i c_{ij}$
Let $P^+ = \bigoplus_{i\in I} \mathbb{N}\Lambda_i$ be the dominant weight lattice
and $Q^+ = \bigoplus_{i\in I} \mathbb{N}\alpha_i$ the positive root lattice
The positive root $\alpha = \sum_i a_i\alpha_i \in Q^+$ has height $\mathrm{ht}(\alpha) = \sum_i a_i$

Symmetrizable quivers

Type	Cartan	Quiver
A_e	$\begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}$	
C_e	$\begin{pmatrix} 2 & -2 & \cdots & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}$	

$+\ B_e, D_e, E_6, E_7, E_8, F_4, G_2$ (finite types)

A_∞		
$A_e^{(1)}$		 $\dots$
$C_e^{(1)}$		

$+\ B_e^{(1)}, D_e^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}\ A_{2e}^{(2)}, A_{2e-1}^{(2)}, D_{e+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$

If there are e_{ij} edges from i to j then for $i \ne j$
$$c_{ij} = \begin{cases} -e_{ij}, & \text{if } e_{ij} > e_{ji} \\ -1, & \text{if } e_{ij} < e_{ji} \\ -e_{ij} - e_{ji}, & \text{otherwise} \end{cases}$$

Graded modules and graded algebras

In these talks, graded will always means $\mathbb{Z}$-graded

A graded module M is a module with a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$

If $m \in M_d$ then m is homogeneous of degree d and we write $\deg m = d$

The graded dimension of M is $\dim_q M = \sum_d (\dim M_d) q^d \in \mathbb{N}[q, q^{-1}]$
$\implies$ $\dim M = (\dim_q M)_{|q=1}$ (assuming only finitely many $M_d \neq 0$)

If $s \in \mathbb{Z}$ let $q^s M$ be the graded module that is equal to M but with the degree shifted upwards by s so that $(q^s M)_d = M_{d-s}$.

More generally, if $f(q) = \sum_s f_s q^s \in \mathbb{N}[q, q^{-1}]$ let

$$f(q)M = \bigoplus_s (q^s M)^{\oplus f_s} \quad \implies \quad \dim_q f(q)M = f(q) \dim_q M$$

An algebra A is graded if $A = \bigoplus_{k\in\mathbb{Z}} A_k$ and $A_k A_l \subseteq A_{k+l}$

A graded A-module is a graded module M such that $A_k M_d \subseteq M_{k+d}$

A map $f : M \longrightarrow N$ is homogeneous of degree d if $\deg f(m) = d + \deg m$
$\implies$ $\operatorname{Hom}(M, N) = \bigoplus_d \operatorname{Hom}(M, N)_d$,
where $\operatorname{Hom}(M, N)_d$ is the space of homogeneous maps of degree d

In the graded category, all isomorphisms are homogeneous of degree zero

Quiver Hecke algebras – the Q-polynomials

Let $\Bbbk = \bigoplus_d \Bbbk_d$ be a positively graded commutative ring

Fix polynomials $(Q_{ij}(u, v))_{i,j\in I}$ in $\Bbbk[u, v]$ with $Q_{ij}(u, v) = Q_{ji}(v, u)$ and

$$Q_{ij}(u,v) = \begin{cases} \sum_{a,b} q_{ab} u^a v^b, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

where $q_{ab} = q_{a,b,i,j} \in \Bbbk_{-2(\alpha_i,\alpha_j)-a(\alpha_i,\alpha_i)-b(\alpha_j,\alpha_j)}$ and $q_{-c_{ij}0} \in \Bbbk_0^\times$

For $1 \le m < n-1$ define $\partial Q_{ijk}(u, v, w) = \delta_{ik} \frac{Q_{ij}(u,v) - Q_{ij}(w,v)}{w-u}$

Examples

$$Q_{ij}(u,v) = \begin{cases} -(u-v)^2 & \text{if } i \leftrightarrows j, \\ u - v^2, & \text{if } i \Longrightarrow j \\ u^2 - v, & \text{if } i \Longleftarrow j \\ u - v, & \text{if } i \longrightarrow j \\ v - u, & \text{if } i \longleftarrow j \\ 1, & \text{if } i \not\!\!- j \\ 0, & \text{if } i = j \end{cases} \implies \partial Q_{iji} = \begin{cases} u + w - 2v & \text{if } i \leftrightarrows j, \\ -(u+w), & \text{if } i \Longrightarrow j \\ u + w, & \text{if } i \Longleftarrow j \\ -1, & \text{if } i \longrightarrow j \\ 1, & \text{if } i \longleftarrow j \\ 0, & \text{otherwise} \end{cases}$$

Quiver Hecke algebras

The symmetric group $\mathfrak{S}_n$ acts on I^n by place permutations:

$$w\mathbf{i} = (i_{w(1)}, \dots, i_{w(n)}), \text{ for } w \in \mathfrak{S}_n \text{ and } \mathbf{i} \in I^n$$

For $\alpha \in Q^+$ let $I^\alpha = \{\, \mathbf{i} \in I^n \mid \alpha = \alpha_{i_1} + \cdots + \alpha_{i_n} \,\}$, where $n = \operatorname{ht}(\alpha)$

Definition (Khovanov-Lauda, Rouquier 2008)

The quiver Hecke algebra, or KLR algebra, $\mathscr{R}_\alpha$ is the unital associative $\Bbbk$-algebra generated by $\{\, 1_{\mathbf{i}} \mid \mathbf{i} \in I^\alpha \,\} \cup \{\, \psi_r \mid 1 \le r < n \,\} \cup \{\, y_r \mid 1 \le r \le n \,\}$ subject to the relations:

- $1_{\mathbf{i}} 1_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}} 1_{\mathbf{i}}$, $\quad \sum_{\mathbf{i}\in I^\alpha} 1_{\mathbf{i}} = 1$, $\quad \psi_r 1_{\mathbf{i}} = 1_{s_r \mathbf{i}} \psi_r$,
- $y_r 1_{\mathbf{i}} = 1_{\mathbf{i}} y_r$, $\quad y_r y_t = y_t y_r$, $\quad \psi_r^2 1_{\mathbf{i}} = Q_{i_r, i_{r+1}}(y_r, y_{r+1}) 1_{\mathbf{i}}$
- $\psi_r y_t = y_t \psi_r$ if $s \neq r, r+1$, $\quad \psi_r \psi_t = \psi_t \psi_r$ if $|r - t| > 1$
- $(\psi_r y_{r+1} - y_r \psi_r) 1_{\mathbf{i}} = \delta_{i_r, i_{r+1}} 1_{\mathbf{i}} = (y_{r+1} \psi_r - \psi_r y_r) 1_{\mathbf{i}}$
- $(\psi_{r+1} \psi_r \psi_{r+1} - \psi_r \psi_{r+1} \psi_r) 1_{\mathbf{i}} = \partial Q_{i_r, i_{r+1}, i_{r+1}}(y_r, y_{r+1}, y_{r+1}) 1_{\mathbf{i}}$

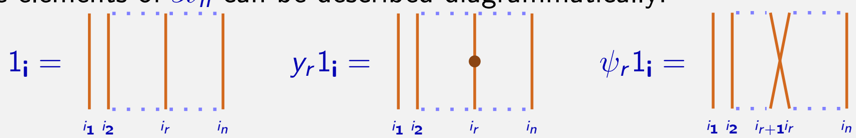
Let $\mathscr{R}_n = \bigoplus_{\alpha\in Q_n^+} \mathscr{R}_\alpha$, where $Q_n^+ = \{\, \alpha \in \mathbb{Q}^+ \mid \operatorname{ht}(\alpha) = n \,\}$

Importantly, $\mathscr{R}_n$ is graded with the grading determined by

$$\deg 1_{\mathbf{i}} = 0, \quad \deg y_r 1_{\mathbf{i}} = (\alpha_{i_r}, \alpha_{i_r}), \quad \text{and} \quad \deg \psi_r 1_{\mathbf{i}} = -(\alpha_{i_r}, \alpha_{i_{r+1}})$$

Diagrammatic presentation for $\mathscr{R}_n$

The elements of $\mathscr{R}_n$ can be described diagrammatically:



If D and E are diagrams then the diagram $D \circ E$ is zero if the residues of the strings do no match up and, when the residues coincide, $D \circ E$ is obtained by putting D on top of E and then rescaling using isotopy

The relations become "local" operations on the diagrams that describe how to move dots and strings past crossings.

For example, the relation $y_{r+1}\psi_r 1_{\mathbf{i}} = (\psi_r y_r + \delta_{i_r i_{r+1}}) 1_{\mathbf{i}}$ and the braid relation (in the simply laced case when $e \neq 2$), can be written as:

A spanning set

A reduced expression for $w \in \mathfrak{S}_n$ is a word $w = s_{a_1} \dots s_{a_k}$ with k minimal, where $s_a = (a, a+1)$

If $w = s_{a_1} \dots s_{a_k}$ is reduced set $\psi_w = \psi_{a_1} \dots \psi_{a_k}$

Warning In general, ψ_w depends on the choice of reduced expression!

Proposition

The algebra $\mathscr{R}_n$ is spanned by the following set of elements:
$$\{\, y_1^{k_1} \dots y_n^{k_n} \psi_w 1_{\mathbf{i}} \mid k_1, \dots, k_n \in \mathbb{N}, w \in \mathfrak{S}_n \text{ and } \mathbf{i} \in I^n \,\}$$

Proof By definition, $\mathscr{R}_n$ is spanned by all diagrams. Using the relations, we can move all of the dots to the top of the diagram, giving the result.

The nil Hecke algebra

Let $\alpha = n\alpha_i$, for some $i \in I \implies I^\alpha = \{(i, \dots, i)\}$ (omit $1_{(i^n)}$ below)

$\implies \mathscr{R}_n$ is generated by $y_1, \dots, y_n, \psi_1, \dots, \psi_{n-1}$ with relations

$$y_r y_t = y_t y_r, \quad \psi_r^2 = 0, \quad \psi_r\psi_{r+1}\psi_r = \psi_{r+1}\psi_r\psi_{r+1},$$
$$\psi_r\psi_t = \psi_r\psi_t \text{ if } |r-s| > 1, \ \psi_r y_t - y_{s_r(t)}\psi_r = \delta_{r+1,t} - \delta_{r,t}$$

$\implies$ the ψ_r satisfy the $\mathfrak{S}_n$-braid relations $\implies$ ψ_w depends only on w

Further, $\deg y_r = (\alpha_i, \alpha_i) = 2d_i = -\deg \psi_t$

Moreover, the ψ_r's satisfy the relations of the nil Hecke algebra NH_n

$\implies$ there is an action of NH_n on the polynomial ring $\Bbbk[\underline{x}] = \Bbbk[x_1, \dots, x_n]$, where y_t acts as multiplication by x_t and ψ_r acts as a *Demazure operator*

$$\partial_r f = \frac{{}^{s_r}f - f}{x_r - x_{r+1}}, \quad \text{where } {}^{s_r}f(x_1, \dots, x_n) = f(x_1, \dots, x_{r+1}, x_r, \dots, x_n)$$

Exercise Check that $\partial_r^2 = 0$, $\partial_r\partial_{r+1}\partial_r = \partial_{r+1}\partial_r\partial_{r+1}$ and that
$$\partial_r\partial_t = \partial_t\partial_r \text{ if } |r-t| > 1$$

$\implies$ if $w = s_{a_1} \dots s_{a_k}$ then $\partial_w = \partial_{a_1} \dots \partial_{a_k}$, reduced, depends only on w

For $w \in \mathfrak{S}_n$ define the (Schubert polynomial) $\mathbb{p}_w = \partial_w(x_2x_3^2 \dots x_n^{n-1})$

Symmetric polynomials and the coinvariant algebra

Let $\mathrm{Sym}_n = \Bbbk[\underline{x}]^{\mathfrak{S}_n}$ be the ring of symmetric polynomials in $\Bbbk[\underline{x}]$

By the product rule, $\partial_r(fg) = \partial_r(f)g + f\partial_r(g)$

$\implies$ ∂_w is a Sym_n-module endomorphism of $\Bbbk[\underline{x}]$

Theorem

The polynomial ring $\Bbbk[\underline{x}]$ is a free Sym_n-module with basis $\{\, \mathbb{p}_w \mid w \in \mathfrak{S}_n \,\}$.

Sketch Let $w_{[1,n]}$ be the longest element of $\mathfrak{S}_n$. We claim $\mathbb{p}_{w_{[1,n]}} = 1$. Now, $w_{[1,n]} = s_{n-1} \dots s_1 w_{[2,n]}$, where $w_{[2,n]}$ is the longest element of $\mathfrak{S}_{\{2,\dots,n\}}$, so

$$\begin{aligned}\mathbb{p}_{w_{[1,n]}} &= \partial_{n-1} \dots \partial_1 \partial_{w_{[2,n]}}(x_2x_3^2 \dots x_n^{n-1}) \\ &= \partial_{n-1} \dots \partial_1 (x_2 \dots x_n)\partial_{w_{[2,n]}}(x_3 \dots x_n^{n-2}) = 1\end{aligned}$$

by induction. Now suppose that $f = \sum_w \lambda_w \mathbb{p}_w = 0$, for $\lambda_w \in \mathrm{Sym}_n$ and let $v \in \mathfrak{S}_n$ be of minimal length such that $\lambda_v \neq 0$. Applying $\partial_{w_{[1,n]}v^{-1}}$ to f shows that $\lambda_v = 0$.

Counting graded dimensions, with x_r in degree $2d_i$, completes the proof

Quiver Hecke algebra basis theorem

Theorem (Khovanov-Lauda, Rouquier)

Let $\alpha = Q_n^+$. Then $\mathscr{R}_\alpha$ has basis
$$\{\, y_1^{k_1} \dots y_n^{k_n} \psi_w 1_{\mathbf{i}} \mid k_r \geq 0, w \in \mathfrak{S}_n, \mathbf{i} \in I^\alpha \,\}$$

Sketch When $\alpha = n\alpha_i$, for $i \in I$, the proof reduces to the nil-Hecke case

The general case is similar in spirit. Using the relations you check that $\mathscr{R}_\alpha$ has a faithful polynomial representation

$$\Bbbk[\underline{x}]_\alpha = \bigoplus\nolimits_{\mathbf{i} \in I^\alpha} \Bbbk[\underline{x}_{\mathbf{i}}]$$

where $y_1, \dots, y_n$ act by multiplication, $1_{\mathbf{i}}\Bbbk[\underline{x}_{\mathbf{j}}] = \delta_{\mathbf{ij}}\Bbbk[\underline{x}_{\mathbf{i}}]$ and $\psi_r 1_{\mathbf{i}}$ acts on $\Bbbk[\underline{x}_{\mathbf{i}}]$ via

$$\psi_r f(\underline{x}_{\mathbf{i}}) = \begin{cases} \partial_r f(\underline{x}_{\mathbf{i}}), & \text{if } i_r = i_{r+1} \\ {}^{s_r}f(\underline{x}_{\mathbf{i}}), & \text{if there is an edge from } i_r \text{ to } i_{r+1}, \\ Q_{i_r i_{r+1}}(y_{r+1}, y_r)\,{}^{s_r}f(\underline{x}_{\mathbf{i}}), & \text{otherwise} \end{cases}$$

The faithfulness of the $\mathscr{R}_\alpha$ action and the freeness of $\Bbbk[\underline{x}]_\alpha$ implies that the elements in the spanning set are linearly independent

Cyclotomic quiver Hecke algebras

Fix a dominant weight $\Lambda \in P^+$ and for each $i \in I$ fix a monic polynomial $\kappa_i(u) \in \Bbbk[u]$ of degree (h_i, Λ) of the form:

$$\kappa_i(u) = \sum_{d=0}^{(h_i,\Lambda)} k_d u^{(h_i,\Lambda)-d}, \qquad \text{where } k_d \in \Bbbk_{d(\alpha_i,\alpha_i)}$$

Definition (Khovanov-Lauda, Rouquier, Brundan-Stroppel, Brundan-Kleshchev)

Let $\Lambda \in P^+$ and $\alpha \in Q_n^+$. The Cyclotomic quiver Hecke algebra, or Cyclotomic KLR algebra, $\mathscr{R}_\alpha^\Lambda$ is the quotient of $\mathscr{R}_\alpha$ by the two-sided ideal generated by $\{\, \kappa_{i_1}(y_1)1_{\mathbf{i}} \mid \mathbf{i} \in I^\alpha \,\}$. Set $\mathscr{R}_n^\Lambda = \bigoplus_{\alpha \in Q_n^+} \mathscr{R}_\alpha^\Lambda$

We abuse notation and identify the elements $\psi_1, \dots, \psi_{n-1}$, $y_1, \dots, y_n$, $1_{\mathbf{i}}$, and ψ_w of $\mathscr{R}_n$ with their images in $\mathscr{R}_n^\Lambda$

Corollary

The algebra $\mathscr{R}_n^\Lambda$ is spanned by the elements
$$\{\, y_1^{k_1} \dots y_n^{k_n} \psi_w 1_{\mathbf{i}} \mid k_r \geq 0, w \in \mathfrak{S}_n, \mathbf{i} \in I^\alpha \,\}$$

It is *not* obvious how to find a smaller spanning set

Finiteness of cyclotomic quiver Hecke algebras

Proposition

The algebra $\mathscr{R}_n^\Lambda$ is finitely generated as a $\Bbbk$-module

Proof It is enough to show that for any $\mathbf{i} \in I^n$ there exists a monic polynomial $h_r(u) \in \Bbbk[u]$ such that $h_r(y_r)1_{\mathbf{i}} = 0$. By definition, such a polynomial exists when $r = 1$. Hence, by induction, it is enough to show how to construct $h_{r+1}(u)$ from $h_r(u)$

Case $i_r \neq i_{r+1}$: Let $h_r'(u)$ be such that $h_r'(y_r)1_{s_r\mathbf{i}} = 0$. Then

$$h_r'(y_{r+1})Q_{i_r i_{r+1}}(y_r, y_{r+1})1_{\mathbf{i}} = h_r'(y_{r+1})\psi_r^2 1_{\mathbf{i}} = \psi_r h_r(y_r)1_{s_r\mathbf{i}}\psi_r = 0$$

Case $i_r = i_{r+1}$: Let $\varphi_r = \psi_r(y_r - y_{r+1})1_{\mathbf{i}} = \Big((y_{r+1} - y_r)\psi_r - 2\Big)1_{\mathbf{i}}$.

$$\implies \varphi_r\psi_r 1_{\mathbf{i}} = -2\psi_r 1_{\mathbf{i}} \quad \implies \quad (1 + \varphi_r)^2 = 1_{\mathbf{i}}$$
$$\implies y_{r+1}1_{\mathbf{i}} = (1+\varphi_r)y_r(1+\varphi_r)1_{\mathbf{i}} \quad \implies \quad h_r(y_{r+1})1_{\mathbf{i}} = 0 \qquad \square$$

Currently, bases for $\mathscr{R}_n^\Lambda$ can be found in the literature only in type A

The nil Hecke algebra case

Fix $i \in I$ and take $\alpha = n\alpha_i$, $\Lambda = n\Lambda_i$ and $\kappa_i(u) = u^n$. Then $\mathscr{R}_\alpha^\Lambda$ is a cyclotomic quotient of the quiver Hecke algebra $\mathscr{R}_\alpha$ that we considered in the nil Hecke case.

Recall that $\mathbb{p}_1 = x_2 x_3^2 \dots x_{n-1}^{n-1}$. For $v, w \in \mathfrak{S}_n$ define

$$\psi_{vw} = \psi_{v^{-1}} y_2 y_3^2 \dots y_{n-1}^{n-1} \psi_w$$
$$\implies \deg \psi_{vw} = d_i\Big(n(n-1) - 2\ell(v) - 2\ell(w)\Big)$$

Proposition

The algebra $\mathscr{R}_\alpha^\Lambda$ has graded cellular basis $\{\, \psi_{vw} \mid v, w \in \,\}$. In particular, $\mathscr{R}_\alpha^\Lambda$ has a unique irreducible module, up to grading shift

To prove this you need to explicitly describe how the y_r's act on the irreducible module after which you can show that multiplication by $y_2 y_3^2 \dots y_{n-1}^{n-1}$ sends the "bottom" basis element to the "top" basis element and so, in particular, is non-zero

Induction and restriction functors

For $\beta \in Q^+$ and $i \in I$ define $1_{\beta,1} = \sum_{\mathbf{j} \in I^\alpha} 1_{\mathbf{j}i}$. Define functors:

$$e_i : \mathscr{R}_{\beta+\alpha_i}\text{-Mod} \longrightarrow \mathscr{R}_\beta\text{-Mod};\ M \mapsto 1_{\beta,i}M$$
$$f_i : \mathscr{R}_\beta\text{-Mod} \longrightarrow \mathscr{R}_{\beta+\alpha_i}\text{-Mod};\ N \mapsto \mathscr{R}_{\beta_a+\alpha_i} \otimes_{\mathscr{R}_\beta} N$$

Proposition (Khovanov-Lauda, Rouquier)

The functors (e_i, f_i) are an adjoint pair.

In particular, these functors are exact and send projectives to projectives

There are natural cyclotomic analogues of these functors:

$$e_i^\Lambda : \mathscr{R}_{\beta+\alpha_i}^\Lambda\text{-Mod} \longrightarrow \mathscr{R}_\beta^\Lambda\text{-Mod};\ M \mapsto 1_{\beta,i}M$$
$$f_i^\Lambda : \mathscr{R}_\beta^\Lambda\text{-Mod} \longrightarrow \mathscr{R}_{\beta+\alpha_i}^\Lambda\text{-Mod};\ N \mapsto \mathscr{R}_{\beta_a+\alpha_i}^\Lambda \otimes_{\mathscr{R}_\beta^\Lambda} N$$

Theorem (Kashiwara, Rouquier)

The functors $(e_i^\Lambda, f_i^\Lambda)$ are an adjoint pair.

Theorem (Kang-Kashiwara, Li)

Let $\Bbbk$ be a commutative graded ring. Then $\mathscr{R}_\alpha^\Lambda$ is free as a $\Bbbk$-module.

Grothendieck groups

Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$, for q an indeterminate

Let $\mathrm{Rep}(\mathscr{R}_n^\Lambda)$ be the Grothendieck group of the finitely generated graded $\mathscr{R}_n^\Lambda$-modules, modulo short exact sequences

So, $\mathrm{Rep}(\mathscr{R}_n^\Lambda)$ is the $\mathbb{A}$-module generated by symbols $[M]$, as M runs over the isomorphism classes of finitely generated $\mathscr{R}_n^\Lambda$-modules, with relations

- $[qM] = q[M]$ (q acts as grading shift)
- $[M] = [L] + [N]$, whenever $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact

Similarly, let $\mathrm{Proj}(\mathscr{R}_n^\Lambda)$ be the split, or projective, Grothendieck group of finitely generated projective $\mathscr{R}_n^\Lambda$modules modulo direct sums

Observe that e_i^Λ and f_i^Λ induce linear endomorphisms of

$$\mathrm{Rep}(\mathscr{R}^\Lambda) = \bigoplus_{n\ge 0} \mathrm{Rep}(\mathscr{R}_n^\Lambda) \quad\text{and}\quad \mathrm{Proj}(\mathscr{R}^\Lambda) = \bigoplus_{n\ge 0} \mathrm{Proj}(\mathscr{R}_n^\Lambda)$$

given by $e_i^\Lambda[M] = [e_i^\Lambda M]$ and $f_i^\Lambda[M] = [f_i^\Lambda M]$

Similarly, we have Grothendieck groups $\mathrm{Rep}(\mathscr{R}_n)$ and $\mathrm{Proj}(\mathscr{R}_n)$ and the functors e_i and f_i induce endomorphisms of

$$\mathrm{Rep}(\mathscr{R}) = \bigoplus_{n\ge 0} \mathrm{Rep}(\mathscr{R}_n) \quad\text{and}\quad \mathrm{Proj}(\mathscr{R}) = \bigoplus_{n\ge 0} \mathrm{Proj}(\mathscr{R}_n)$$

Quantum groups

The quantum group $U_q(\mathfrak{g})$ associated with $(C, P, P^\vee, \Pi, \Pi^\vee)$ is the unital associative $\mathbb{Q}(q)$-algebra with generators $\{ E_i, F_i, K_i^\pm \mid i \in I \}$, subject to the relations:

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = 1, \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}}$$

$$K_iE_jK_i^{-1} = q^{d_ic_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-d_ic_{ij}}F_j$$

$$\textstyle\sum_{0\le c\le 1-c_{ij}} (-1)^c \left[\!\left[{1-c_{ij} \atop c}\right]\!\right]_i E_i^{1-c_{ij}-c}E_jE_i^c = 0$$

$$\textstyle\sum_{0\le c\le 1-c_{ij}} (-1)^c \left[\!\left[{1-c_{ij} \atop c}\right]\!\right]_i F_i^{1-c_{ij}-c}F_jF_i^c = 0$$

where $q_i = q^{d_i}$, $[\![m]\!]_i! = \prod_{k=1}^m (q^k - q^{-k})/(q - q^{-1})$, and $\left[\!\left[{a \atop b}\right]\!\right]_i = [\![b]\!]_i!/[\![a]\!]_i![\![b-a]\!]_i!$ for integers $a < b, m \in \mathbb{N}$.

Let $U_q^+(\mathfrak{g}) = \langle E_i \mid i \in I\rangle$ and $U_q^-(\mathfrak{g}) = \langle F_i \mid i \in I\rangle$

$\Longrightarrow$ There is a PBW decomposition $U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g})$

Finally, the Lusztig integral form of $U_q(\mathfrak{g})$ is the $\mathbb{A}$-subalgebra $U_{\mathbb{A}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ generated by the quantised divided powers $E_i^{(k)} = E_i^k/[\![k]\!]_i!$ and $F_i^{(k)} = F_i^k/[\![k]\!]_i!$ for $k \ge 0$ and $i \in I$

Categorification of $U_q^-(\mathfrak{g})$

Theorem (Khovanov-Luda, Rouquier)

Suppose that $\Bbbk$ is a field. Then there are $\mathbb{A}$-algebra isomorphisms

$$U_{\mathbb{A}}^-(\mathfrak{g}) \cong \mathrm{Proj}(\mathscr{R}) \quad\text{and}\quad (U_{\mathbb{A}}^-(\mathfrak{g}))^\vee \cong \mathrm{Rep}(\mathscr{R})$$

In fact, these are isomorphisms of twisted bialgebras where the multiplication on $\mathrm{Rep}(\mathscr{R})$ and $\mathrm{Proj}(\mathscr{R})$ is induced by the convolution product: if $M \in \mathscr{R}_\alpha$-Mod and $N \in \mathscr{R}_\beta$-Mod then

$$M \circ N = \mathscr{R}_{\alpha+\beta}1_{\alpha,\beta} \otimes_{\mathscr{R}_\alpha\otimes\mathscr{R}_\beta} M \otimes N$$

Canonical bases

The Grothendieck groups $\mathrm{Rep}(\mathscr{R})$ and $\mathrm{Proj}(\mathscr{R})$ come equipped with distinguished bases:

$\mathrm{Rep}(\mathscr{R}_n) = \langle\, [D] \mid D$ self-dual irreducible $\mathscr{R}_n$-modules $\rangle$

$\mathrm{Proj}(\mathscr{R}_n) = \langle\, [P] \mid P$ self-dual indecomposable projective $\mathscr{R}_n$-modules $\rangle$

Warning: different dualities are used in $\mathrm{Rep}(\mathscr{R}_n)$ and in $\mathrm{Proj}(\mathscr{R}_n)$. We will give more precise details later

On the quantum group side, $U_q^-(\mathfrak{g})$ and $U_q^-(\mathfrak{g})^\vee$ also come equipped with distinguished bases: Lusztig's canonical basis and dual canonical basis or, equivalently, Kashiwara's upper and lower global crystal bases

Theorem (Varagnolo-Vasserot, Brundan-Stroppel, Brundan-Kleshchev, Webster)

Assume that $\Bbbk$ is a field of characteristic zero and that C is a symmetric Cartan matrix ($d_i = 1$ for all $i \in I$). Then canonical basis of $U_q^-(\mathfrak{g})$ coincides with the basis of self-dual projective indecomposable modules and the dual canonical basis coincides with the basis of self-dual irreducible modules.

Categorification of highest weight modules

For each dominant weight $\Lambda \in P^+$ there is a unique irreducible integral highest weight module $L(\Lambda)$ for $U_q(\mathfrak{g})$.

Let $v_\Lambda \in L(\Lambda)$ be a highest weight vector and define $L_{\mathbb{A}}(\Lambda) = U_{\mathbb{A}}(\mathfrak{g})v_\Lambda$ and $L_{\mathbb{A}}(\Lambda)^\vee = \bigoplus L_{\mathbb{A}}(\Lambda)^\vee_\mu$, where $L_{\mathbb{A}}(\Lambda)^\vee_\mu = \mathrm{Hom}_{\mathbb{A}}(L_{\mathbb{A}}(\Lambda)_\mu, \mathbb{A})$

Theorem (Kang-Kashiwara, Webster)

Let C be a generalised symmetrizable Cartan matrix. Then

$$L_{\mathbb{A}}(\Lambda) \cong \bigoplus_{n\geq 0} \mathit{Proj}(\mathscr{R}_n^\Lambda) \quad \text{and} \quad L_{\mathbb{A}}(\Lambda)^\vee \cong \bigoplus_{n\geq 0} \mathit{Rep}(\mathscr{R}_n^\Lambda)$$

Prior to this result, Lauda and Vazirani proved the weaker statement that the irreducible $\mathscr{R}_n^\Lambda$-modules categorify the crystal of $L(\Lambda)$

Canonical bases for integrable highest weight modules

Combining the last two results proves the following:

Corollary (Varagnolo-Vasserot, Brundan-Stroppel, Brundan-Kleshchev, Webster)

Assume that $\Bbbk$ is a field of characteristic zero and that C is a symmetric Cartan matrix ($d_i = 1$ for all $i \in I$). Then:

- *The canonical basis of $L_{\mathbb{A}}(\Lambda)$ coincides with* $\{\, [P] \mid \text{self dual projective indecomposable } \mathscr{R}_n^\Lambda\text{-module, } n \geq 0 \,\}$
- *The dual canonical basis of $L_{\mathbb{A}}(\Lambda)^\vee$ coincides with* $\{\, [D] \mid \text{self dual irreducible } \mathscr{R}_n^\Lambda\text{-modules, } n \geq 0 \,\}$

In general, this result cannot hold for non-symmetric Cartan matrices because there are known examples where the structure constants for the canonical bases are polynomials with negative coefficients

Further reading I

- J. Brundan, *Quiver Hecke algebras and categorification*, in Advances in representation theory of algebras, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013, 103–133.
- J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484.
- J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$*, Represent. Theory, **15** (2011), 170–243. arXiv:0812.1090.
- S.-J. Kang and M. Kashiwara, *Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras*, Invent. Math., **190** (2012), 699–742.
- M. Kashiwara, *Biadjointness in cyclotomic Khovanov-Lauda-Rouquier algebras*, Publ. Res. Inst. Math. Sci., **48** (2012), 501–524.
- M. Khovanov and A. D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory, **13** (2009), 309–347.

Further reading II

- ——, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc., **363** (2011), 2685–2700.
- A. D. Lauda and M. Vazirani, *Crystals from categorified quantum groups*, Adv. Math., **228** (2011), 803–861.
- G. Li, *Integral Basis Theorem of cyclotomic Khovanov-Lauda-Rouquier algebras of Type A*, J. Algebra, 2017, accepted Oct. 2016. arXiv:1412.3747.
- R. Rouquier, *2-Kac-Moody algebras*, 2008, preprint. arXiv:0812.5023.
- ——, *Quiver Hecke algebras and 2-Lie algebras*, Algebra Colloq., **19** (2012), 359–410.
- M. Varagnolo and E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math., **659** (2011), 67–100.
- B. Webster, *Canonical bases and higher representation theory*, Compos. Math., **151** (2015), 121–166. arXiv:1209.0051.