# EXERCISES 10: LECTURE FOUNDATIONS OF MATHEMATICS

**Exercise 1.** As a reminder: a total order on a set $X$ is called well-ordered from below (above) if there exists a smallest (biggest) element for any non-empty subset of $X$ with respect to the fixed total order.

Let $X$ a totally ordered set which is well-ordered from below and above. Show that $X$ is finite.

**Exercise 2.** Find an order on $\mathbb{Q}$ such that $\mathbb{Q}$ is well-ordered from below.

**Exercise 3.** Let $X$ a set. Define

$$\Delta\colon \mathfrak{P}(X) \times \mathfrak{P}(X) \to \mathfrak{P}(X),\ \Delta(A, B) = (A \cup B) \setminus (A \cap B),$$
$$\cap\colon \mathfrak{P}(X) \times \mathfrak{P}(X) \to \mathfrak{P}(X),\ \cap(A, B) = A \cap B.$$

Show that $(\mathfrak{P}(X), \Delta, \cap)$ is a unital ring.

**Exercise 4.** Let $\mathbb{Z}^{\mathbb{N}}$ be the set of all maps $\mathbb{N} \to \mathbb{Z}$. For $f, g \in \mathbb{Z}^{\mathbb{N}}$ define an addition $+$ and a multiplication $*$ via

$$+\colon \mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}},\ +(f, g)(x) = f(x) + g(x),$$
$$*\colon \mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}},\ *(f, g)(x) = \sum_{ab=x} f(a)g(b),$$

where the sum runs over all $a, b \in \mathbb{N}$ with $ab = x$. Show that $(\mathbb{Z}^{\mathbb{N}}, +, *)$ is a commutative, unital ring.

**Submission of the exercise sheet**: 25.Nov.2019 before the lecture. **Return of the exercise sheet**: 28.Nov.2019 during the exercise sessions.