2.2. Projective Resolutions & Radicals of Projectives

First, we'll do a short recap about things we know so far:

- there exist simple, projective & injective representations of a quiver Q
- the representations $S(i), P(i)$ & $I(i)$ are indecomposable
  $\Rightarrow$ we'll show in this chapter that there are no other indecomposable projective or injective representations
- the sums of proj. (resp. inj.) representations are proj. (resp. inj.) & the summands of proj. (resp. inj.) repr. are proj. (resp. inj.)
  $\Rightarrow$ we'll show any proj. repr. $P \cong P(i_1) \oplus P(i_2) \oplus \dots \oplus P(i_t)$
- proj. repr. can be used to describe the vector spaces of an arbitrary repr. M using the Hom functor: $\mathrm{Hom}(P(i), M) \cong M_i$
  $\Rightarrow$ we'll see another way of describing arbitrary repr. M using projective resolutions

So let's start with two definitions:

Def. 2.3. Let $M \in \mathrm{rep}Q$.

(1) A projective resolution of M is an exact sequence $\dots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$, where each P_i is a proj. repr.

(2) An injective resolution of M is an exact sequence
$0 \to M \to I_0 \to I_1 \to I_2 \to I_3 \to \dots$, where each I_i is an inj. repr.

Next, we're going to see that such resolutions always exist.

Thm. 2.15 Let $M \in \operatorname{rep} Q$.

(1) There exists a projective resolution of M of the form:
$0 \to P_1 \to P_0 \to M \to 0$.

(2) There exists an injective resolution of M of the form:
$0 \to M \to I_0 \to I_1 \to 0$.

The proof is quite technical & Long, therefore I'll just present the idea behind the proof.

Proof IDEA part 1:

Construction of the so called standard projective resolution of $M = (M_i, \varphi_i)$.

(1) Define $P_0 = \bigoplus_{i \in Q_0} d_i P(i)$ & $P_1 = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha))$
where $d_i = \dim M_i$.

(2) Introduce specific bases for P_1, P_0 & M:
- $\forall i \in Q_0$, let $\{m_{i1}, \dots m_{id_i}\}$ be a basis for M_i
$\Rightarrow B'' := \{m_{ij} \mid i \in Q_0, j = 1, 2, \dots, d_i\}$ is a basis for M

- $B := \{c_{ij} \mid i \in Q_0,\ c_i$ path s.t. $s(c_i) = i$ & $j = 1, \dots, d_i\}$ basis for P_0
- $B' := \{b_{\alpha j} \mid \alpha \in Q_1,\ b_\alpha$ path s.t. $s(b_\alpha) = t(\alpha)$ & $j = 1, \dots, d_{s(\alpha)}\}$ basis for P_1

(3) Define g & f on these bases:
Let $g : P_0 \to M$ s.t. $\forall c_{ij} \in B$ $g(c_{ij}) := \varphi_{c_i}(m_{ij}) \in M_{t(c_i)}$
Let $f : P_1 \to P_0$ s.t. $\forall b_{\alpha j} \in B'$ $f(b_{\alpha j}) := (\alpha b_\alpha)_j - b_\alpha^M$, where αb_α is the path from $s(\alpha) \to t(b_\alpha)$ & $b_\alpha^M = \sum_{l=1}^{d_{t(\alpha)}} \Theta_l b_{\alpha l}$, where Θ_l are the scalars that occur when writing $\varphi_\alpha(m_{s(\alpha)j})$ in the basis $\{m_{t(\alpha)l} \mid l = 1, \dots, d_{t(\alpha)}\}$ of $M_{t(\alpha)}$ $\Rightarrow \varphi_\alpha(m_{s(\alpha)j}) = \sum_{l=1}^{d_{t(\alpha)}} \Theta_l m_{t(\alpha)l}$

(4) Prove that $0 \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ is a short exact sequence
i.e.
- g surjective
- $\operatorname{im} f = \ker g$
- f injective

(5) Conclude:
- since sums of proj. repr. are projective $\Rightarrow P_1, P_0$ projective
- $0 \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ is a short exact sequence by (4)

$\Rightarrow 0 \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ is a proj. resolution of M ■

Rk: (1) There exist other proj. resolutions than the standard resolution as well.
(2) In the next chapter 2.3. we'll hear about the so called Nakayama functor, which maps projective resolutions to injective ones, therefore part (2) of Thm. 2.15 can be deduced from that.

Exp. 2.5 quiver Q: $1 \longrightarrow 2 \longleftarrow 3$

representations
- $S(3)$: $0 \longrightarrow 0 \longleftarrow k$ $\qquad S(3) = 3$
- M' : $k \longrightarrow k \longleftarrow k$ $\qquad M' = \begin{smallmatrix}1 & & 3\\ & 2 & \end{smallmatrix}$

(I) Find a proj. resol. for $M = S(3) = 3$.
Therefore, we'll use the standard resolution defined in the proofidea of thm. 2.15.

$P_0 = \bigoplus_{i \in Q_0} d_i P(i)$ & $P_1 = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha))$

Since $M = S(3) = 3$: $0 \rightarrow 0 \leftarrow k$ we can see that
$d_1 = \dim(M_1) = \dim(0) = 0 = d_2$ & $d_3 = \dim(M_3) = \dim(k) = 1$

$\Rightarrow \underline{P_0} = d_1 \cdot P(1) \oplus d_2 \cdot P(2) \oplus d_3 \cdot P(3) =$
$= 0P(1) \oplus 0P(2) \oplus 1P(3) = P(3) = \underline{\begin{smallmatrix}3\\2\end{smallmatrix}}$: $0 \rightarrow k \leftarrow k$

$\underline{P_1} = d_1 \cdot P(2) \oplus d_3 \cdot P(2) =$
$= 0P(2) \oplus 1P(2) = P(2) = \underline{2}$: $0 \rightarrow k \leftarrow 0$

Hence we get the standard proj. resolution:
$$0 \rightarrow \underbrace{2}_{P_1} \rightarrow \underbrace{\begin{smallmatrix}3\\2\end{smallmatrix}}_{P_0} \rightarrow \underbrace{3}_{S(3)} \rightarrow 0.$$

(II) $M' = \begin{smallmatrix}1 & & 3\\ & 2 & \end{smallmatrix}$ $\Rightarrow$ $d_1 = d_2 = d_3 = 1$

$\underline{P_0} = \bigoplus_{i \in Q_0} d_i P(i) = 1 \cdot P(1) \oplus 1 \cdot P(2) \oplus 1 \cdot P(3) = \underline{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}3\\2\end{smallmatrix}}$

since
$P(1)$: $k \rightarrow k \leftarrow 0$
$P(2)$: $0 \rightarrow k \leftarrow 0$
$P(3)$: $0 \rightarrow k \leftarrow k$

$\underline{P_1} = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) = 1 \cdot P(2) \oplus 1 \cdot P(2) = \underline{2 \oplus 2}$

Hence we get the standard proj. resolution:
$$0 \rightarrow 2 \oplus 2 \rightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \rightarrow \begin{smallmatrix}1 & & 3\\ & 2 & \end{smallmatrix} \rightarrow 0$$

Exp 2.6

quiver Q:
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & 4 \end{tikzcd}$$

representation M:
$$\begin{tikzcd} & & k \\ k \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] & k^2 \arrow[ur, "{[0\ 1]}"] \arrow[dr, "{[1\ 1]}"'] & \\ & & k \end{tikzcd}$$
$M = \begin{smallmatrix} & 1 & \\ 2 & & 2 \\ 3 & & 4 \end{smallmatrix}$

$M = \begin{smallmatrix} & 1 & \\ 2 & & 2 \\ 3 & & 4 \end{smallmatrix} \Rightarrow d_1 = d_3 = d_4 = 1 \ \& \ d_2 = 2$

$$\begin{aligned} P_0 = \bigoplus_{i \in Q_0} d_i P(i) &= d_1 \cdot P(1) \oplus d_2 P(2) \oplus d_3 P(3) \oplus d_4 P(4) \\ &= 1 \cdot P(1) \oplus 2 \cdot P(2) \oplus 1 \cdot P(3) \oplus 1 \cdot P(4) \\ &= \begin{smallmatrix} & 1 & \\ & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus \left(\begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix}\right) \oplus 3 \oplus 4 \end{aligned}$$

since $P(1)$: $k \to k \begin{smallmatrix} \nearrow k \\ \searrow k \end{smallmatrix}$ $\qquad$ $P(2)$: $0 \to k \begin{smallmatrix} \nearrow k \\ \searrow k \end{smallmatrix}$

$P(3)$: $0 \to 0 \begin{smallmatrix} \nearrow k \\ \searrow 0 \end{smallmatrix}$ $\qquad$ $P(4)$: $0 \to 0 \begin{smallmatrix} \nearrow 0 \\ \searrow k \end{smallmatrix}$

$$\begin{aligned} P_1 = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)) &= d_1 P(2) \oplus d_2 \cdot P(3) \oplus d_2 P(4) \\ &= 1 \cdot P(2) \oplus 2 \cdot P(3) \oplus 2 \cdot P(4) \\ &= \begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus (3 \oplus 3) \oplus (4 \oplus 4) \end{aligned}$$

Hence we get the standard projective resolution:
$$0 \to \begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus (3 \oplus 3) \oplus (4 \oplus 4) \to \begin{smallmatrix} & 1 & \\ & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus \left(\begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix}\right) \oplus 3 \oplus 4 \to \begin{smallmatrix} & 1 & \\ 2 & & 2 \\ 3 & & 4 \end{smallmatrix} \to 0$$

Rk: (1) the proj. resol. found for M' in Exp. 2.5 is not minimal as we can eliminate a direct summand „2" in both P_0 & P_1 & the resulting sequence $0 \to 2 \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \to 3 \to 0$ is still a proj. resolution

(2) in Exp. 2.6 we can eliminate the three direct summands „3" „4" „$\begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix}$" appearing both in P_0 & P_1 s.t. $0 \to 3 \oplus 4 \to \begin{smallmatrix} & 1 & \\ & 2 & \\ 3 & & 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 2 & & 2 \\ 3 & & 4 \end{smallmatrix} \to 0$ is still a proj. resolution, thus the found result in Exp. 2.6 is not minimal either

To talk about the minimality of resolutions, we'll need further definitions.

Def. 2.4. Let $M \in \operatorname{rep} Q$.

(1) A projective cover of M is a proj. representation P together with a surj. morphism $g: P \to M$ with the property that: $\forall g': P' \to M$ surj. morph. with P' proj. $\exists h: P \twoheadrightarrow P'$ surj. morph. s.t. the diagram

$$\begin{tikzcd} & P' \arrow[dl, dotted, "\exists h"'] \arrow[d, "g'"] & \\ P \arrow[r, "g"'] & M \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd}$$

commutes i.e. $g \circ h = g'$.

(2) An injective envelope of M is an inj. representation I together with an inj. morph. $f: M \to I$ with the property that: $\forall f': M \to I'$ inj. morph. with I' inj. $\exists h: I \hookrightarrow I'$ inj. morph. s.t. the diagram

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f'"'] & I \arrow[dl, dotted, "\exists h"] \\ & I' & \end{tikzcd}$$

commutes i.e. $h \circ f = f'$.

Def. 2.5.

(1) A proj. resolution $\cdots \to P_3 \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ is called minimal if $f_0 : P_0 \to M$ is a proj. cover & $f_i : P_i \to \ker f_i$ is a proj. cover $\forall i > 0$.

(2) An injective resolution $0 \to M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \xrightarrow{f_3} I_3 \to \cdots$ is called minimal if $f_0 : M \to I_0$ is an inj. envelope & $\forall i > 0$ $f_i : \operatorname{coker} f_{i-1} \to I_i$ is an inj. envelope as well.

We will show that proj. covers are unique up to isomorphism.

Prop. 2.17. Let $g : P \to M$ be a proj. cover of M & let $g' : P' \to M$ be a surj. morphism with P' projective. Then P is isomorphic to a direct summand of P'.

We will take this proposition for granted & use it to prove the following important proposition.
(you can find the proof of Prop. 2.17. in the bonus material)

Prop. 2.18. Let $g : P \to M$ & $g' : P' \to M$ both be proj. covers of $M \in \operatorname{rep} Q$. Then $P \cong P'$.

Proof: We know that $g: P \to M$ is a proj. cover & $g': P' \to M$ is as well. By the definition of proj. covers we have that P' is proj. & g' is a surj. morphism. Thus we can use Prop. 2.17. to deduce that P is isomorphic to a direct summand of P'. Vice versa we get from Prop. 2.17. that P' is a direct summand of P.

Therefore, we can conclude that $P \cong P'$. ∎

Rk: Dual statements to Prop. 2.17. & 2.18. about inj. envelopes hold as well. i.e. if $f: M \to I$ & $f': M \to I'$ are inj. envelopes, then $I \simeq I'$. (proof: Bonus material)

Next, we will talk about free representations, whose prototypes are the sum of the indecomposable projective representations.

Def. 2.6.: Let $A = \bigoplus_{i \in Q_0} P(i)$. A representation $F \in \mathrm{rep}\, Q$ is called *free* if $F \cong A \oplus \dots \oplus A$.

Prop. 2.20.: $M \in \mathrm{rep}\, Q$ is projective iff $\exists\, F \in \mathrm{rep}\, Q$ free s.t. M is isomorphic to a direct summand of F.

To prove this statement, we'll have to recall a few results of earlier talks.

Recall:

(1) Thm. 1.2. (Krull-Schmidt): Let Q quiver, $M \in \operatorname{rep} Q$, then $M \cong M_1 \oplus M_2 \oplus .. \oplus M_t$, where $M_i \in \operatorname{rep} Q$ are indecomposable & unique up to order.

(2) Prop. 2.8.: $S(i)$, $P(i)$ & $I(i)$ are indecomposable.

(3) Prop. 2.7.: $P, P' \in \operatorname{rep} Q$. Then $P \oplus P'$ proj. iff P, P' proj.

(4) Cor. 2.4.: If P proj., then any exact seq. $0 \to L \to M \xrightarrow{g} P \to 0$ splits.

(5) Prop. 1.8. (b): Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact seq. in $\operatorname{rep} Q$. If f section, then $\operatorname{im} f = \ker g$ is a direct summand of M.

Proof (Prop. 2.20.):

($\Leftarrow$) Let's assume there exists a free repr. $F \in \operatorname{rep} Q$. i.e. $F \cong A \oplus .. \oplus A$ where $A = \bigoplus_{i \in Q_0} P(i)$.
By assumption M is isomorphic to a direct summand of F. Since every direct summand of F is a direct sum of $P(i)$'s, which are proj. we get by Prop. 2.7. that every direct summand of F is proj.
Using that $P(i)$'s are indecomposable (Prop. 2.8.) & the uniqueness of the decomposition of M by the Krull-Schmidt thm. we can deduce that M is projective as well.

($\Rightarrow$) Let's assume that $M = (M_i, \varphi_\alpha)$ is projective. Denote $d_i = \dim(M_i)$ $\forall i \in Q_0$. Using the standard proj. resolution $P_0 = \bigoplus_{i \in Q_0} d_i P(i)$ we get a surj. morphism $g: P_0 \to M$. Hence we can build a short exact sequence of the form $0 \to \ker g \to P_0 \xrightarrow{g} M \to 0$.

Since M is proj. by assumption we get by Cor. 2.4. that above seq. splits and thus by Prop. 1.8. M is isomorphic to a direct summand of $P_0 = \bigoplus_{i \in Q_0} d_i P(i)$. Now we can choose F large enough s.t. P_0 is a direct summand of F and thus we found a free repr. $F \in \operatorname{rep} Q$ s.t. M is isomorphic to a direct summand of F. ∎

From this Proposition the next CRUCIAL Corollary follows directly.

Cor. 2.21.: Any proj. representation $P \in \operatorname{rep} Q$ is a direct sum of $P(i)$'s, that is $P \cong P(i_1) \oplus \ldots \oplus P(i_t)$ with $i_1, \ldots, i_t$ not necessarily distinct.

Now we will introduce a particular subrepresentation of $P(i)$, which we will use to show that subrepresentations of proj. repr. are projective, too.

Def. 2.7.: Let $P(i) = (P(i)_j, \varphi_\alpha)$ be the proj. repr. at vertex i. The radical of $P(i)$ is the repr. $\operatorname{rad} P(i) = (R_j, \varphi'_\alpha)$ defined by

$$R_i = 0, \quad R_j = P(i)_j \text{ if } i \neq j \quad \& \quad \varphi'_\alpha = \begin{cases} 0 & \text{if } s(\alpha) = i \\ \varphi_\alpha & \text{otherwise} \end{cases}$$

At last we'll have a look at a few results concerning rad $P(i)$ without giving the proofs. (proof (idea)s : bonus material)

Lemma 2.22.: Any proper subrepresentation of $P(i)$ is contained in rad $P(i)$.

Lemma 2.23.: If $P(i)$ is simple, then rad $P(i)=0$. If $P(i)$ is not simple, then rad $P(i)$ is projective.

Thm. 2.24.: Subrepresentations of proj. repr. in rep$Q$ are projective.

Rk: Categories with the property that subrepresentations inherit the projectivity are called *hereditary*.

Cor. 2.26.: Let $f: M \to P$ be a nonzero morphism with $M \in \text{rep}Q$ indecomposable & $P \in \text{rep}Q$ projective. Then M is proj. & f inj.

The last Corollary tells us that we must start with the proj. repr. if we want to construct the Auslander-Reiten quiver of Q & moreover, that these repr. are partially ordered by inclusion.

Bonus material

Problem 2.3.1:

Compute a proj. resolution for each simple repr. $S(i)$ for each of the following quivers.

(1)
$$\begin{tikzcd}[row sep=small] & & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[dr] & 4 & 5 \arrow[l] \\ & & & 6 \arrow[ur] & & 7 & \end{tikzcd}$$

As in Exp. 2.5. & 2.6. we'll compute the standard projective resolution where $P_0 = \bigoplus_{i \in Q_0} d_i P(i)$ & $P_1 = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha))$.

$\boxed{i=1}$ $M = S(1) = 1 \Rightarrow d_1 = 1$ & $\forall i = 2, .., 7 : d_i = 0$

$P_0 = d_1 P(1) \oplus \dots \oplus d_7 P(7) = 1 \cdot P(1) = \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 4\,7 \end{smallmatrix}$

$P_1 = d_1 P(2) \oplus d_2 P(3) \oplus d_3 P(4) \oplus d_6 P(3) \oplus d_3 P(7) \oplus d_5 P(4) = 1 \cdot P(2) = \begin{smallmatrix} 2 \\ 3 \\ 4\,7 \end{smallmatrix}$

$\Rightarrow$ proj. resol.: $0 \to \begin{smallmatrix} 2 \\ 3 \\ 4\,7 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 4\,7 \end{smallmatrix} \to 1 \to 0$

$\boxed{i=2}$ $M = S(2) = 2 \Rightarrow d_2 = 1$ & $\forall i \in \{1,3,4,5,6,7\} : d_i = 0$

$P_0 = 1 \cdot P(2) = \begin{smallmatrix} 2 \\ 3 \\ 4\,7 \end{smallmatrix}$ $\qquad P_1 = 1 \cdot P(3) = \begin{smallmatrix} 3 \\ 4\,7 \end{smallmatrix}$

$\Rightarrow$ proj. resol.: $0 \to \begin{smallmatrix} 3 \\ 4\,7 \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 3 \\ 4\,7 \end{smallmatrix} \to 2 \to 0$

$\boxed{i=3}$ $M = S(3) = 3 \Rightarrow d_3 = 1$ & $\forall i \in \{1,2,4,5,6,7\} : d_i = 0$

$P_0 = 1 \cdot P(3) = \begin{smallmatrix}3\\4\ 7\end{smallmatrix}$ $\quad P_1 = 1 \cdot P(4) \oplus 1 \cdot P(7) = 4 \oplus 7$

$\Rightarrow$ proj. resol.: $0 \to 4 \oplus 7 \to \begin{smallmatrix}3\\4\ 7\end{smallmatrix} \to 3 \to 0$

$\boxed{i=4}$ $M = S(4) = 4 \Rightarrow d_4 = 1$ & $\forall i \in \{1,2,3,5,6,7\} : d_i = 0$

$P_0 = 1 \cdot P(4) = 4$ $\quad P_1 = 0$

$\Rightarrow$ proj. resol.: $0 \to 0 \to 4 \to 4 \to 0$

$\boxed{i=5}$ $M = S(5) = 5 \Rightarrow d_5 = 1$ others: 0

$P_0 = 1 \cdot P(5) = \begin{smallmatrix}5\\4\end{smallmatrix}$ $\quad P_1 = 1 \cdot P(4) = 4$

$\Rightarrow$ proj. resol.: $0 \to 4 \to \begin{smallmatrix}5\\4\end{smallmatrix} \to 5 \to 0$

$\boxed{i=6}$ $M = S(6) = 6 \Rightarrow d_6 = 1$ others: 0

$P_0 = 1 \cdot P(6) = \begin{smallmatrix}6\\3\\4\ 7\end{smallmatrix}$ $\quad P_1 = 1 \cdot P(3) = \begin{smallmatrix}3\\4\ 7\end{smallmatrix}$

$\Rightarrow$ proj. resol.: $0 \to \begin{smallmatrix}3\\4\ 7\end{smallmatrix} \to \begin{smallmatrix}6\\3\\4\ 7\end{smallmatrix} \to 6 \to 0$

$\boxed{i=7}$ $M = S(7) = 7 \Rightarrow d_7 = 1$ others: 0

$P_0 = 1 \cdot P(7) = 7$ $\quad P_1 = 0$

$\Rightarrow$ proj. resol.: $0 \to 0 \to 7 \to 7 \to 0$

(2) $1 \leftleftarrows 2 \leftleftarrows 3$

$\boxed{i=1}$ $M = S(1) = 1 \quad \Rightarrow d_1 = 1 \;;\; d_2 = d_3 = 0$

$P_0 = d_1 \cdot P(1) \oplus d_2 P(2) \oplus d_3 P(3) = P(1) = 1$

$P_1 = d_2 P(1) \oplus d_2 P(1) \oplus d_3 P(2) \oplus d_3 P(2) = 0$

$\Rightarrow$ proj. resol.: $0 \to 0 \to 1 \to 1 \to 0$

$\boxed{i=2}$ $M = S(2) = 2 \quad \Rightarrow d_2 = 1 \;;\; d_1 = d_3 = 0$

$P_0 = 1 \cdot P(2) = \begin{smallmatrix} 2 \\ 1\,1 \end{smallmatrix} \qquad P_1 = P(1) \oplus P(1) = 1 \oplus 1$

$\Rightarrow$ proj. resol: $0 \to 1 \oplus 1 \to \begin{smallmatrix} 2 \\ 1\,1 \end{smallmatrix} \to 2 \to 0$

$\boxed{i=3}$ $M = S(3) = 3 \quad \Rightarrow d_3 = 1 \;;\; d_1 = d_2 = 0$

$P_0 = 1 \cdot P(3) = \begin{smallmatrix} & 3 & \\ 2 & & 2 \\ 1\,1 & & 1\,1 \end{smallmatrix} \qquad P_1 = P(2) \oplus P(2) = \begin{smallmatrix} 2 \\ 1\,1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1\,1 \end{smallmatrix}$

$\Rightarrow$ proj. resol. : $0 \to \begin{smallmatrix} 2 \\ 1\,1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1\,1 \end{smallmatrix} \to \begin{smallmatrix} & 3 & \\ 2 & & 2 \\ 1\,1 & & 1\,1 \end{smallmatrix} \to 3 \to 0$

(3)
$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] \arrow[d] & 3 \arrow[dl] \\ & 4 & \end{tikzcd}$$

$\boxed{i=1}$ $M = S(1) = 1 \quad \Rightarrow d_1 = 1 \;;\; d_2 = d_3 = d_4 = 0$

$P_0 = d_1 P(1) \oplus \ldots \oplus d_4 P(4) = 1 \cdot P(1) = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ 3\,4 & & 4 \\ 4 & & \end{smallmatrix}$

$$P_1 = d_1 P(2) \oplus d_1 P(3) \oplus d_2 P(3) \oplus d_2 P(4) \oplus d_3 P(4)$$
$$= P(2) \oplus P(3) = \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ 4 & & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}$$

$\Rightarrow$ proj. resol.: $0 \to \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ 4 & & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \to \begin{smallmatrix} & & 1 & \\ & 2 & & 3 \\ 3 & 4 & & 4 \\ 4 & & & \end{smallmatrix} \to 1 \to 0$

$\boxed{i=2}$ $M = S(2) = 2 \quad \Rightarrow d_2 = 1 \ ; \ d_1 = d_3 = d_4 = 0$

$P_0 = 1 \cdot P(2) = \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ 4 & & \end{smallmatrix} \qquad P_1 = P(3) \oplus P(4) = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4$

$\Rightarrow$ proj. resol.: $0 \to \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus 4 \to \begin{smallmatrix} & 2 & \\ 3 & & 4 \\ 4 & & \end{smallmatrix} \to 2 \to 0$

$\boxed{i=3}$ $M = S(3) = 3 \quad \Rightarrow d_3 = 1 \ ; \ d_1 = d_2 = d_4 = 0$

$P_0 = 1 \cdot P(3) = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \qquad P_1 = 1 \cdot P(4) = 4$

$\Rightarrow$ proj. resol.: $0 \to 4 \to \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \to 3 \to 0$

$\boxed{i=4}$ $M = S(4) = 4 \quad \Rightarrow d_4 = 1 \ ; \ d_1 = d_2 = d_3 = 0$

$P_0 = 1 \cdot P(4) = 4 \qquad P_1 = 0$

$\Rightarrow$ proj. resol.: $0 \to 0 \to 4 \to 4 \to 0$

Prop. 2.17. Let $g: P \to M$ be a proj. cover of M & Let $g': P' \to M$ be a surj. morphism with P' projective. Then P is isomorphic to a direct summand of P'.

To prove this proposition, we again recall Prop. 1.8. (b) & cor 2.4.

Recall:

(1) Cor. 2.4.: If P proj., then any exact seq. $0 \to L \to M \xrightarrow{g} P \to 0$ splits.

(2) Prop. 1.8. (b): Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact seq. in repQ. If f section, then $\operatorname{im} f = \ker g$ is a direct summand of M.

Proof: By the definition of proj. covers we can deduce that there exists a surj. morphism $h: P' \to P$. Using this morphism we can build an exact seq. in the following way: $0 \to \ker h \xrightarrow{f} P' \xrightarrow{h} P \to 0$. As P is proj. Cor. 2.4. yields that above seq. splits i.e. $f: \ker h \to P'$ is a section. Hence by Prop. 1.8. $P' \cong \ker h \oplus P$, thus the result follows. ∎

Dual statements to Prop. 2.17. & 2.18. for inj. envelopes:

Prop. 2.17. (2): Let $f: M \to I$ be an inj. envelope of M & let $f': M \to I'$ be an inj. morphism with I' injective. Then I is isom. to a direct summand of I'.

Proof: By the definition of inj. envelopes there must exist $h: I \hookrightarrow I'$ injective. Thus we have an exact seq. $0 \to I \xhookrightarrow{h} I' \to \operatorname{coker} h \to 0$ which splits since I injective. Hence $I' \cong \operatorname{coker} h \oplus I$, therefore the result follows. ∎

Prop. 2.18. (2): Let $f: M \to I$ & $f': M \to I'$ be inj. envelopes of M. Then $I \cong I'$.

Proof: Applying Prop. 2.17. (2) twice we get that I is isom. to a direct summand of I' & I' is isom. to a direct summand of I, hence it must hold that $I \cong I'$. ∎

Lemma 2.22.: Any proper subrepresentation of $P(i)$ is contained in $\operatorname{rad} P(i)$.

Proof: Let $M = (M_i, \varphi_\alpha)$ be a subrepresentation of $P(i) = (P(i)_j, \varphi_\alpha)$. Hence there exists an inj. morph. $f: M \hookrightarrow P(i)$.
Clearly, if $M_i = 0$, we have that $f(M_i) \subset \operatorname{rad} P(i)$ as $\operatorname{rad} P(i) = (R_j, \varphi_\alpha')$ with $R_i = 0$ & $R_j = P(i)_j$ if $i \neq j$.
Thus, let us suppose that $M_i \neq 0$. We're going to show that in this case f is an isomorphism & therefore M is not a PROPER subrepresentation.
Since $P(i)_i \cong k$ & $f: M \hookrightarrow P(i)$ inj. it follows $M_i \cong k$ as well. Hence there is an element $m_i \in M_i$ s.t. $f_i(m_i) = e_i$.
Now let c be a path from i to j, where j is any vertex.

Then, $c \overset{\text{Rk. 2.1.}}{=} \varphi_c(e_i) = \varphi_c(f_i(m_i)) \overset{f \text{ morph.}}{=} f_j(\varphi_c(m_i)) \in \operatorname{im} f_j$.
Since c is an arbitrary element of the basis of $P(i)_j$ & c lies in the image of f_j, we get that f is surjective. Hence f is an isomorphism & M is not a proper subrepresentation of $P(i)$. ∎

Lemma 2.23: If $P(i)$ is simple, then $\operatorname{rad} P(i) = 0$. If $P(i)$ is not simple, then $\operatorname{rad} P(i)$ is projective.

Proof:

Case 1: $P(i)$ simple

As $P(i)$ is simple by assumption we know that $P(i)_i \cong k$ & $P(i)_j = 0$ $\forall j \neq i$. By the def. of $\operatorname{rad} P(i)$ we have $R_i = (\operatorname{rad} P(i))_i = 0$ & $R_j = P(i)_j = 0$.
Therefore, $\operatorname{rad} P(i) = 0$.

Case 2: $P(i)$ not simple.

For that case, we'll show that $\operatorname{rad} P(i)$ is isomorphic to $P := \bigoplus_{\alpha : s(\alpha) = i} P(t(\alpha))$, which is projective, because it is a direct sum of proj. representations.
For any $j \neq i$, $R_j = (\operatorname{rad} P(i))_j = P(i)_j$ has as a basis the set of paths from i to j.
Let us define a morph. $f = (f_j)_{j \in Q_0} : \operatorname{rad} P(i) \to P$ on this basis by setting $f_j(i|\alpha, \beta_1, \ldots, \beta_s|j) = (t(\alpha)|\beta_1, \ldots, \beta_s|j)$.
Thus, f_j sends the basis of R_j to a basis of P_j for any $j \in Q_0$ & therefore f is an isomorphism.
i.e. $\operatorname{rad} P(i)$ is projective. ∎

Thm. 2.24.: Subrepresentations of proj. repr. in rep$Q$ are projective.

Proof IDEA: Proof by induction on $d = \sum_{i \in Q_0} d_i$, where $(d_i)_{i \in Q_0}$ is the dimension vector of the proj. repr. P.

- B.C. $d=1 \Rightarrow P$ simple ✓
- Fix some $d > 1$ & assume that subrepr. of proj. repr. in rep$Q$ with dimension strictly smaller than d are projective.
- Let $u: M \to P$ inclusion morphism, use $P \cong P(i_1) \oplus \dots \oplus P(i_t) \Rightarrow u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}$, $\operatorname{im} u_j \subset P(i_j)$
- Deduce $M \cong \operatorname{im} u_1 \oplus \dots \oplus \operatorname{im} u_t$
- Show $\operatorname{im} u_j$ proj. for each j: use that subrepr. of $P(i_j)$ are contained in $\operatorname{rad}(P_{i_j})$, whose dim. is strictly smaller than d.
- Conclude, by induction, that $\operatorname{im} u_j$ is projective, which completes the proof. ∎

Cor. 2.26.: Let $f: M \to P$ be a nonzero morphism with $M \in$ rep$Q$ indecomposable & $P \in$ rep$Q$ projective. Then M is proj. & f inj.

Proof: As $\operatorname{im} f$ is a subrepr. of P, thm. 2.24. yields that $\operatorname{im} f$ is projective. Therefore, by Cor. 2.4., the short exact seq. $0 \to \ker f \to M \to \operatorname{im} f \to 0$ splits. Prop. 1.8. implies that $\operatorname{im} f$ is isom. to a direct summand of M. Since M is indecomposable we get $M \cong \operatorname{im} f$ & thus M is projective. Last, we can deduce that $\ker f = 0$ & therefore, f is injective. ∎