THE CLASSICAL THEORY IV
SOERGEL BIMODULES, THE BEGINNINGS

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SOME ALGEBRAIC NOTATIONS

- DEFINITION: A $\mathbb{Z}$**-graded vector space** is a vector space M with decomposition $M := \bigoplus_{i\in\mathbb{Z}} M^i$ into subspaces M^i.The M^i are the **graded pieces of $\boldsymbol{M}$**.
- DEFINITION: A **homogeneous element with degree $\boldsymbol{i}$** is an element $m \in M$ that is contained in some M^i.
- DEFINITION: Given a graded object M and $i \in \mathbb{Z}$, define $M(i)$ with graded pieces $\boldsymbol{M(i)^j := M^{i+j}}$.
- DEFINITION: A **graded submodule of $\boldsymbol{M}$** is a submodule of M which is generated by homogeneous elements.
- DEFINITION: A graded R-module M is **free**, if it has an R-basis that consists of homogeneous elements of M.

- DEFINITION: A **Coxeter system** $(\boldsymbol{W}, \boldsymbol{S})$ is a group W and a finite set $S \subset$W. Its **geometric representation** $\boldsymbol{V}$ over $\mathbb{R}$ is a real vector space with basis $\{\alpha_s | s \in S\}$
- DEFINITIONS: The basis elements α_s are called **simple roots**.
- LEMMA: V has dimension $|S|$ and is equipped with symmetric bilinear form $(\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}}$,
  - $m: S \times S \longrightarrow \mathbb{N} \cup \{\infty\}$ a symmetric function
    - $m_{ss} = 1$ for all $s \in S$.
    - For $s \neq t \in S, m_{st} = m_{ts} \in \{2,3, \ldots\} \cup \{\infty\}$
  - $W = \langle s \in S | (st)^{m_{st}} = id$ for any $s, t \in S$ with $m_{st} < \infty \rangle$
- DEFINITION: Define an action $W \longrightarrow V$ where the elements are $s \in S$ by

$$\boldsymbol{s(\alpha_t) = \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s}$$

- DEFINTION: Let $I \subset S$.Then the **standard parabolic subgroup** $\boldsymbol{W_I} := \langle I \rangle \subset W$.
- DEFINTION: If that parabolic subgroup W_I is a finite group, then I is **finitary**.
- DEFINTION: Let R be the **symmetric algebra** of $\boldsymbol{V}$.This means that

$$R = Sym(V) = \oplus_{i \in \mathbb{Z}_{\geq 0}} Sym^i(V)$$

- DEFINTION: The **ring of $\boldsymbol{W_I}$-invariants of R**, (denoted by $\boldsymbol{R^I}$) is:

$$R^I = \{f \in R | w \cdot f = f \text{ for all } w \in W_I\}$$

  - Then R^S are the invariants under the entire Coxeter group.
  - We write R^s instead of $R^{\{s\}}$.

CHEVALLEY-SHEPHARD-TODD THEOREM (CST)

- THEOREM **(Chevalley-Shephard-Todd, CST)**:
  For $I \subset S$ finitary, R^I is a polynomial ring. R then is a graded free module of finite rank over R^I.
- The ring of invariants of a finite group is a polynomial ring
  $\Leftrightarrow$ group generated by pseudoreflections.
- DEFINITION: A **pseudoreflection** is an invertible linear transformation g of V with finite order and such that $V^g = \{v \in V | gv = v\}$ is a subspace of dimension $n - 1$.
- CST is an algebraic foundation upon which the theory of Soergel bimodules is built.
- CST is a generalization of the theory of symmetric polynomials.

- EXAMPLE:
  Let $W = S_5$ which acts on $\mathbb{R}[x_1, \ldots, x_5]$. Let $I = \{s_1, s_3, s_4\}$.
  Then we have

$$R^I = \mathbb{R}[z_1, \ldots, z_5] = \mathbb{R}[x_1 + x_2, x_1x_2, x_3 + x_4 + x_5, x_3x_4 + x_3x_5 + x_4x_5, x_3x_4x_5]$$

$\Longrightarrow R^I$ has 5 algebraically independent generators (in different degrees)

$\Longrightarrow R^I$ is a polynomial ring.

DEMAZURE OPERATOR

- RECALL: $s(\alpha_t) = \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s$, with $(\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}}$.
- LEMMA: Let (W, S). Then $\forall s \in S, R^s$ is generated by α_s^2 and $\alpha_t + \left(\cos\frac{\pi}{m_{st}}\right)\alpha_s$ for all $t \in S \setminus \{s\}$.
  Hence $R = R^s \oplus R^s\alpha_s$ is a splitting of R into s-invariants and s-antiinvariants.
- DEFINITION: An element is said to be ***s*-invariant** if $sf = f$ and ***s*-antiinvariant** if $sf = -f$.
- DEFINITION: Let $s \in S$. The **Demazure operator $\boldsymbol{\partial_s}$** is a graded map

$$\partial_s : R \longrightarrow R^s(-2), f \longmapsto \frac{f - s(f)}{\alpha_s}$$

  - The s-antiinvariants are generated by α_s (because $R = R^s \oplus R^s\alpha_s$)
  - $f - s(f)$ is divisible by α_s
  - ∂_s is well defined.

- LEMMA: The fraction $\frac{f-s(f)}{\alpha_s}$ is s-invariant
- LEMMA: For any $f \in R$, $\partial_s(f\alpha_s) = \frac{f\alpha_s - s(f\alpha_s)}{\alpha_s} = f + s(f)$ (I)

  and $\alpha_s \partial_s(f) = f - s(f)$ (II)
  - (I) $\Longrightarrow s(f)$: $\partial_s(f\alpha_s) - f = s(f)$ (III)
  - (III) $\Longrightarrow$ (II): $\alpha_s\partial_s(f) = f - \partial_s(f\alpha_s) + f$ (IV)
  - (IV) $\Longrightarrow f$: $f = \partial_s\left(f\frac{\alpha_s}{2}\right) + \frac{\alpha_s}{2}\partial_s(f)$

$\Longrightarrow$ Isomorphism $R \to R^s \oplus R^s(-2), f \longmapsto \left(\partial_s\left(f\frac{\alpha_s}{2}\right), \partial_s(f)\right)$ with inverse given by $(g,h) \longmapsto g + h\frac{\alpha_s}{2}$.

$\Longrightarrow$ Demazure operator can be used to make the R^s-module splitting R into the direct sum of R^s and $R^s \cdot \alpha_s$.

- DEFINITION: An **expression** of $w \in W$ is a word $\underline{w} = (s_1, \dots, s_n)$.
- DEFINITION: An expression $\underline{w}$ is **reduced** if the length of w is n ($\ell(w) = n$)
- DEFINITION: Demazure operator for $w \in W$ with reduced expression $\underline{w} = (s_1, \dots, s_n)$:

$$\boldsymbol{\partial_w := \partial_{s_1} \cdots \partial_{s_n}}$$

- LEMMA: Let $s \in S$.

1. ∂_s is an R^s-bimodule map
2. $s \circ \partial_s = \partial_s$ and $\partial_s \circ s = -\partial_s$
3. $\partial_s^2 = 0$
4. Twisted Leibniz rule: For $f, g \in R$, we have $\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$
5. $\left\{1, \frac{\alpha_s}{2}\right\}$ is a basis for R over R^s, with dual basis $\left\{\frac{\alpha_s}{2}, 1\right\}$, because of $(f, g)_s \longmapsto \partial_s(fg)$
6. Braid relations: $s, t \in S$ distinct with $m_{st} < \infty$. Then $\overbrace{\partial_s\partial_t\partial_s \dots}^{m_{st}} = \partial_t\partial_s\partial_t \dots$

- PROOF OF 2.): [$s \circ \partial_s = \partial_s$ and $\partial_s \circ s = -\,\partial_s$]
  - $s \circ \partial_s = s\big(\partial_s(f)\big) = s\left(\frac{f-s(f)}{\alpha_s}\right) = -\frac{1}{\alpha_s} s\big(f - s(f)\big) = -\frac{s(f)}{\alpha_s} + \frac{s\big(s(f)\big)}{\alpha_s} = \frac{f-s(f)}{\alpha_s} = \partial_s$
    - $s(\alpha_s) = -\alpha_s$
    - $s(f + g) = s(f) + s(g)$
    - $s\big(s(f)\big) = s(-f) = -s(f) = f$
  - $\partial_s \circ s = \partial_s\big(s(f)\big) = \frac{s(f)-s\big(s(f)\big)}{\alpha_s} = \frac{s(f)-f}{\alpha_s} = -\frac{f-s(f)}{\alpha_s} = -\partial_s$
- PROOF OF 3.): [$\partial_s^2 = 0$]
  - $\partial_s^2 = \partial_s\big(\partial_s(f)\big) = \partial_s\left(\frac{f-s(f)}{\alpha_s}\right) = \frac{\frac{f-s(f)}{\alpha_s} - s\left(\frac{f-s(f)}{\alpha_s}\right)}{\alpha_s} = \frac{f-s(f)}{\alpha_s^2} + \frac{s\big(f-s(f)\big)}{\alpha_s^2} = 0$
    - $s\big(f - s(f)\big) = -f + s(f)$

- PROPOSITION: $\partial_w(f) = 0 \Leftrightarrow f$ is w-invariant $(w \cdot f = f)$
- EXAMPLE: Let $W = S_5$ act on $\mathbb{R}[x_1, \ldots, x_5]$. Let $I = \{s_1, s_3, s_4\}$
  - $R^I = \mathbb{R}[z_1, z_2, z_3, z_4, z_5] = \mathbb{R}[x_1 + x_2, x_1x_2, x_3 + x_4 + x_5, x_3x_4 + x_3x_5 + x_4x_5, x_3x_4x_5]$
  - Here: $\alpha_{s_1} = x_1 - x_2, \alpha_{s_2} = x_2 - x_3$
  - For $z_1 = x_1 + x_2$:

$$\partial_{s_1}(x_1 + x_2) = \partial_{s_1}(x_1) + \partial_{s_1}(x_2) = \frac{x_1 - s_1(x_1)}{\alpha_{s_1}} + \frac{x_2 - s_1(x_2)}{\alpha_{s_1}} = \frac{x_1 - s_1(x_1)}{x_1 - x_2} + \frac{x_2 - s_1(x_2)}{x_1 - x_2} = \frac{x_1 - x_2}{x_1 - x_2} + \frac{x_2 - x_1}{x_1 - x_2} = 1 + (-1) = 0$$

  - For $z_2 = x_1x_2$:

$$\partial_{s_2}(x_1x_2) = \partial_{s_2}(x_1)x_2 + s_2(x_1)\partial_{s_2}(x_2) = \frac{x_1 - s_2(x_1)}{\alpha_{s_2}}x_2 + s_2(x_1)\frac{x_2 - s_2(x_2)}{\alpha_{s_2}} = \frac{x_1 - x_2}{x_2 - x_3}x_2 + x_2\frac{x_2 - x_1}{x_2 - x_3} = \frac{0}{x_2 - x_3} = 0$$

  - For $z_3 = x_3 + x_4 + x_5$:

$$\partial_{s_3}(x_3 + x_4 + x_5) = \partial_{s_3}(x_3) + \partial_{s_3}(x_4) + \partial_{s_3}(x_5) = \frac{x_3 - s_3(x_3)}{x_3 - x_4} + \frac{x_4 - s_3(x_4)}{x_3 - x_4} + \frac{x_5 - s_3(x_5)}{x_3 - x_4} = \frac{x_3 + x_4 + x_5 - s_3(x_3) - s_3(x_4) - s_3(x_5)}{x_3 - x_4} = 0$$

$\Longrightarrow \partial_{s_i}(z_i) = 0$ for all $i \in \{1,2,3,4,5\} \Longrightarrow \prod_{i=1}^{5} \partial_{s_i} = 0 \overset{Prop.}{\Longrightarrow} R^I$ is s-invariant

BOTT-SAMELSON BIMODULES

- DEFINITION: For $s \in S$, $\boldsymbol{B_s}$ is the graded R-bimodule $B_s \coloneqq R \otimes_{R^s} R(1)$
  - B_s belongs to R-gbim.
- DEFINITION: ***R*-gbim** is the category of graded R-bimodules.
  - shift factor (n) for each integer n which sends $M \longmapsto M(n)$.
  - tensor product $- \otimes_R -$, hence the category of graded R-bimodules is per definition a monoidal category.
- LEMMA: Tensor product and grading shift commute.
  - For graded R-bimodules M and N and $n \in \mathbb{Z}$ we have the following canonical identifications:
$$\big(M(n)\big) \otimes_R N = M \otimes_R \big(N(n)\big) = (M \otimes_R N)(n)$$
  - $MN \coloneqq M \otimes_R N$

- DEFINITION: An **element in $\boldsymbol{B_s}$** can be represented as

$$\sum_i f_i \otimes g_i = \sum_i f_i|_s g_i \text{ for some appropriate } f_i, g_i \in R$$

- LEMMA: $f|_s 1 = 1|_s f \Leftrightarrow f$ is s-invariant.
  - $1|_s 1$ has degree -1 and $1|_{s_1} 1|_{s_2} 1|_{s_3} \cdots |_{s_n} 1$ is of degree $-\ell(\underline{w})$.
  - B_s is graded free as a left respectively right R-module and its graded rank is $(v + v^{-1})^l$
- DEFINITION: The **Bott-Samelson bimodule** corresponding to $\underline{w} = (s_1, \dots, s_n)$ is the graded R-bimodule

$$BS(\underline{w}) := B_{s_1} B_{s_2} \dots B_{s_n} = B_{s_1} \otimes B_{s_2} \otimes \cdots \otimes B_{s_n}$$

  - Canonical isomorphism: $BS(\underline{w}) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R\left(\ell(\underline{w})\right)$
- DEFINITION: An **element of $\boldsymbol{BS(\underline{w})}$** is of the form

$$\sum_i f_i \otimes g_i \otimes \cdots \otimes h_i = \sum_i f_i|_{s_1} g_i\,|_{s_2} \cdots |_{s_n} h_i \text{ for some } f_i, g_i, \dots, h_i \in R$$

- EXAMPLE: Let $W = A_2$.
  - Then the $BS\left(\underline{s_1 s_2 s_1}\right)$ decomposes into the direct sum $B_{s_1 s_2 s_1} \oplus B_{s_1} = B_{s_1} B_{s_2} B_{s_1}$,
  - $B_{s_1 s_2 s_1} = R \otimes_{R^W} R(3)$ is the submodule generated by $1 \otimes 1 \otimes 1$.
  - More generally: If W is a dihedral group generated by (s, t) and $l(w') < l(w)$ where $w' = sw$, then $B_s B_{w'} = B_w \oplus B_{tw'}$
- SOME PROPERTIES (Bott-Samelson-bimodules)
  - Let $\underline{u}, \underline{v}$ be two expressions. Then $BS(\underline{u}) BS(\underline{v}) = BS(\underline{uv})$ (closed under tensor product)
  - $f g_1 \otimes_{R^{s_{i_1}}} g_2 \otimes_{R^{s_{i_2}}} \cdots \otimes_{R^{s_{i_n}}} g_n = g_1 \otimes_{R^{s_{i_1}}} g_2 \otimes_{R^{s_{i_2}}} \cdots \otimes_{R^{s_{i_n}}} g_n f$ for $f \in R^W$

  $\Longrightarrow$ every Soergel bimodule is an $R \otimes_{R^W} R$-module.

- EXAMPLE: Product of two Bott-Samelson-bimodules. Using that
  - $R = R^s \oplus R^s \alpha_s = R^s \oplus R^s(-2)$
  - $B_s B_s = R \otimes_{R^s} R \otimes_{R^s} R = R \otimes_{R^s} \big(R^s \oplus R^s(-2)\big) \otimes_{R^s} R = B_s(1) \oplus B_s(-1)$.
  - This is analogous to the relation: $b_s^2 = (v + v^{-1}) b_s$, where b_s is an element of the Kazhdan-Lusztig-basis
- LEMMA: Bott-Samelson bimodules are not closed under taking grading shifts or direct sums.
  - $B_s \simeq R \otimes_{R^s} \big(R^s \oplus R^s(-2)\big)(1) \simeq R(1) \oplus R(-1) \Longrightarrow B_s$ is graded free as a left R-module (resp as a right R-module).
- LEMMA: Any Bott-Samelson bimodule is graded free of finite rank as a left respectively right R-module.

- EXAMPLE: Consider $c_{id} := 1 \otimes 1$ of degree -1 and $c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$ of degree 1.
  $\Longrightarrow$ These elements form a basis of B_s as a left (or right) R-module.
- LEMMA: For any $f \in R, f \cdot c_s = c_s \cdot f$
- LEMMA: For any $f \in R, f \cdot c_{id} = c_{id} \cdot s(f) + c_s \cdot \partial_s(f)$ (Polynomial forcing relation)
  - PROOF:

$$f \cdot c_{id} = f \cdot (1 \otimes 1)$$

and

$$c_{id} \cdot s(f) + c_s \cdot \partial_s(f) = (1 \otimes 1) \cdot s(f) + \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \cdot \frac{f - s(f)}{\alpha_s} = (1 \otimes 1) \cdot s(f) + 2\alpha_s (1 \otimes 1) \cdot \frac{f - s(f)}{2\alpha_s}$$

$$= (1 \otimes 1) \cdot s(f) + (1 \otimes 1) \cdot f - (1 \otimes 1) s(f) = f \cdot (1 \otimes 1)$$

SOERGEL BIMODULES

- DEFINITION: A **Soergel bimodule** is a finite direct sum of shifts of summands of Bott-Samelson bimodules in $\mathbb{BS}Bim$ (category of Bott-Samelson bimodules)
- LEMMA: Soergel bimodules are closed under grading shifts.
- DEFINITION: The **category of Bott-Samelson bimodules $\mathbb{BS}\boldsymbol{Bim}$** is the monoidal category (category equipped with the tensor product of R-bimodules)
- DEFINITION: The **category of Soergel bimodules $\mathbb{S}$Bim** is the strictly full subcategroy of R-gbim consisting of Soergel bimodules
  - Is the smallest strictly full subcategory of R-gbim containing R and B_s for all $s \in S$ that is closed under tensor product, direct sums, direct summands and shifts.
- DEFINITION: $\mathbb{S}\mathrm{Bim}$ is **strictly full** if it is closed under isomorphisms.

- LEMMA: Soergel Bimodules are closed under tensor products: $\mathbb{S}\mathrm{Bim}_{\underline{u}} \otimes_s \mathbb{S}\mathrm{Bim}_{\underline{v}} = \mathbb{S}\mathrm{Bim}_{\underline{uv}}$
- LEMMA: For a graded left R-module M (free of finite rank), any graded summand N of M is also graded free.
- LEMMA:
  - Any Bott-Samelson bimodule is graded free of finite rank as a left respectively right R-module.
  - Any Soergel bimodule is graded free as a left respectively right R-module.

- DEFINITION: An object M of an additive category is called **indecomposable** if it cannot be expressed as a direct sum $M' \oplus M''$ for nonzero subobjects M', M''.
- LEMMA: Suppose that M is a graded R-bimodule which is generated as an R-bimodule by a homogeneous element $m \in M$. This then implies that M is indecomposable.
- EXAMPLE: R and $B_s = R \otimes_{R^s} R(1)$ are indecomposable
- LEMMA: The category of Soergel bimodules is an additive category such that every object is isomorphic to a direct sum of indecomposable objects and such decomposition is unique up to isomorphism and permutation of summands.

EXAMPLES

- Symmetric group S_3, $R = \mathbb{R}[x, y, z]$
  - s interchanges x and y: $s \cdot f(x, y, z) = f(y, x, z)$ → $R^s = \mathbb{R}[x + y, xy, z]$
  - r interchanges y and z: $s \cdot f(x, y, z) = f(x, z, y)$ → $R^r = \mathbb{R}[x, y + z, yz]$

  $\Longrightarrow R^{s,t} = \mathbb{R}[x + y + z, xy + xz + yz, xyz]$
  - Grading: x,y and z in degree 2, x^2 and xy in degree 4, $3xy^2z^7$ in degree 20.
  - Define the ring R shifted down by one $R(1) \Longrightarrow x$ is in degree 1, x^2 in degree 3 and $3xy^2z^7$ in degree 19.
- SOME EASY EXAMPLES:
  - R and $B_s := R \otimes_{R^s} R(1)$
  - $B_{sr} = B_s \otimes_R B_r$ and $B_{srs} = R \otimes_{R^{s,r}} R(3)$.
- In S_3, the category of Soergel bimodules the indecomposable set is $\{R, B_s, B_r, B_{sr}, B_{rs}, B_{srs}\}$

- COMPARISON: Hecke algebra $\leftrightarrow$ Soergel bimodules
  - B_s, B_sB_r, B_{srs} are analogous objects to the elements b_s, b_{sr} and $b\ _{srs}$ respectively in the Hecke algebra.
  - The product (resp. direct sum) between Soergel bimodules as an analogue of product (resp. sum) in the Hecke algebra.
  - Shifting the degree of a Soergel bimodule by one can be seen as multiplying the corresponding element in the Hecke algebra by v.
- RECALL: Hecke algebra $\mathcal{H}(S_3)$ is free over $\mathbb{Z}[v, v^{-1}]$ with basis $\{1, b_s, b_r, b_{sr}, b_{rs}, b_{srs}\}$.

- R, B_s and B_r: distinct and indecomposable.
- R is generated by the subrings R^s and $R^r \Longrightarrow B_sB_r \cong R\otimes_{R^s} R\otimes_{R^r}R(2)$ and $B_rB_s \cong R\otimes_{R^r} R\otimes_{R^s}R(2)$
  - both are generated by $1\otimes1\otimes1 \Longrightarrow$ B_s and B_r are not isomorphic $\Longrightarrow$ $B_{sr} := B_sB_r \ncong B_rB_s =: B_{rs}$
  - B_sB_r and B_rB_s are indecomposables
- Isomorphism $B_sB_s \cong B_s(1)\oplus B_s(-1) \Longrightarrow B_sB_s$ is clearly decomposable. (B_tB_t is decomposable)
- Look at the possibility $B_{srs} = R \otimes_{R^{s,r}} R(3) \Longrightarrow$ add B_{srs} to our indecomposables
  - generated by $1\otimes1$ in degree -3.
  - Isomorphism $B_sB_rB_s \backsimeq B_{srs}\oplus B_s \Longrightarrow B_{srs}$ actually is in $\mathbb{S}Bim$.
- $B_sB_{srs} \backsimeq B_{srs}(1)\oplus B_{srs}(-1) \backsimeq B_rB_{srs} \Longrightarrow B_{srs}$ is not isomorphic to any grading shift of indecomposables
- List of distinct indecomposables up to grading shift is complete and is given by the set $\{R, B_s, B_r, B_{sr}, B_{rs}, B_{srs}\}$.

- EXAMPLE: Category of Soergel bimodules in S_3 is stable under product
- If $p \in R$, then $p - s \cdot p \in (y-x)R^s$,
  - For example if $p = 3xy^2z^7 + yz, p - s \cdot p = 3xy^2z^7 + yz - 3yx^2z^7 + xz = (y-x)(3xyz^7 + z)$.
  - true because the polynomial $p - s \cdot p$ vanishes in the hyperplane defined by the equation $y = x$.
  - same result for r: $p - r \cdot p \in \ (z-y)R^r$.
- Define $\alpha_s := y - x$ and $\alpha_r := z - y$. Define $P_s(p) = \frac{p+s\cdot p}{2} \in R^s$ and $\partial_s(p) = \frac{p-s\cdot p}{2\alpha_s} \in R^s$.

  $\Longrightarrow$ Then $p = P_s(p) + \alpha_s \partial_s(p) \Longrightarrow$ isomorphism of graded R^s-bimodules $R \cong R^s \oplus R^s(-2)$.
- $B_s B_s \cong R \otimes_{R^s} R \otimes_{R^s} R(2) \cong R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R = B_s(1) \oplus B_s(-1)$ ($\leftrightarrow b_s b_s = v b_s + v^{-1} b_s$)
- $B_s B_{srs} \cong R \otimes_{R^s} R \otimes_{R^s} R(4) \cong B_{srs}(1) \oplus B_{srs}(-1)$ ($\leftrightarrow b_s b_{srs} = v b_s + v^{-1} b_{srs}$)
- Same comparison for all products of elements of the set $\{R, B_s, B_r, B_{sr}, B_{rs}, B_{srs}\}$.

THANK YOU FOR YOUR ATTENTION