1.5 Unitarizability of finite-dimensional modules (Arno)

Recall this very important statement:
Every finite dimensional g-module is semisimple.

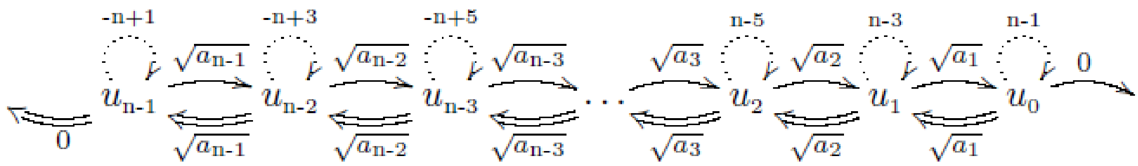
The correspondence
$\mathbf{e}^\star = \mathbf{f}$, $\mathbf{f}^\star = \mathbf{e}$, $\mathbf{h}^\star = \mathbf{h}$
uniquely extends to a *skew-linear* involution $\star$ on g in the sense that
$(\lambda x)^\star = \underline{\lambda} x^\star$ and $(xy)^\star = y^\star x^\star$
for all $x,y \in$ g and $\lambda \in$ C where _ denotes the complex conjugation. This involution satisfies
$[x^\star, y^\star] = [y, x]^\star$
for all $x, y \in$ g and hence is a (skew) *antiinvolution* of the Lie algebra g.
The involution $\star$ induces an involution on the set $\{E, F, H\}$, which we will denote by the same symbol.

A g-module V is called *unitarizable* with respect to the involution $\star$
provided that there exists a (positive definite) Hermitian inner product $(\cdot, \cdot)$ on V such that
$(X(v),w) = (v,X^\star(w))$ for all $v,w \in V$ and $X \in \{E, F, H\}$

Theorem 1.5.1. *Every finite-dimensional* g*-module is unitarizable.*

Exercise 1.5.2. A direct sum $V \oplus W$ of two g-modules V and
W is unitarizable if and only if each summand is unitarizable.
'<=' H_V,H_W the Hermitian innerproducts, $H=\{\{H_V,0\},\{0,H_w\}\}$
'=>' dimV=n, dimW=m, consider $H_V=(h_{ij})_{1\leq i,j\leq n}$, $H_W=(h_{ij})_{n+1\leq i,j\leq n+m}$

*Proof. E*very finite-dimensional g-module decomposes into a direct sum of modules $\mathbf{V}(n)$, $n \in$ N. Hence Exercise 1.5.2 implies that it is enough to prove the statement of the theorem for the modules $\mathbf{V}(n)$, $n \in$ N.
Assume that $n \in$ N and the module $\mathbf{V}(n)$ is given by (1.9). Note that all $a_i > 0$ and define
$c_0 = 1$, $c_i = c_{i-1}/\sqrt{(a_i)}$, $i = 1, . . . , n - 1$.
Then $u_i = c_i v_i$ defines a diagonal change of basis in $\mathbf{V}(n)$. In the basis $\{u_i\}$ the action of E, F and H is given by:



(1.24)

Let $(\cdot, \cdot)$ be the inner product on $\mathbf{V}(n)$ with respect to which the basis $\{u_0, . . . , u_{n-1}\}$ is orthonormal. From (1.24) it follows by a direct calculation that in this basis the linear operators E, F and H satisfy (1.23).
This proves that $\mathbf{V}(n)$ is unitarizable. □

The antiinvolution $\star$ is not the only antiinvolution on g. The correspondence
$\mathbf{e}^\diamond = \mathbf{e}$, $\mathbf{f}^\diamond = \mathbf{f}$, $\mathbf{h}^\diamond = -\mathbf{h}$
uniquely extends to a linear involution $\diamond$ on g. This involution satisfies
$[x^\diamond, y^\diamond] = [y, x]^\diamond$
for all $x, y \in$ g and hence is an antiinvolution of the Lie algebra g. The involution $\diamond$ induces an involution on the set $\{E, F, \pm H\}$, which we will denote by the same symbol. A g-module V is called a $\diamond$*-module* provided that there exists a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on V such that
$(X(v),w) = (v,X^\diamond(w))$ for all $v,w \in V$ and $X \in \{E, F, H\}$.

Exercise 1.5.3. Let V be a non-trivial simple finite-dimensional $\diamond$-module with the corresponding symmetric bilinear form $(\cdot, \cdot)$. $(\cdot, \cdot)$ is neither positive nor negative definite.

Exercise 1.5.4. A direct sum $V \oplus W$ of two $\diamond$-modules is a $\diamond$-module.

Theorem 1.5.5. *Every finite-dimensional* g*-module is a* $\diamond$*-module.*

Proof. Every finite-dimensional g-module decomposes into a direct sum of modules $\mathbf{V}^{(n)}$, $n \in \mathrm{N}$. Hence Exercise 1.5.4 implies that it is enough to prove the statement of the theorem for the modules $\mathbf{V}^{(n)}$, $n \in \mathrm{N}$. Assume that $n \in \mathrm{N}$ and the module $\mathbf{V}^{(n)}$ is given by (1.9). Let $(\cdot, \cdot)$ be the symmetric bilinear form on $\mathbf{V}^{(n)}$ which is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{tikzcd}
{} & v_{n-1} \arrow[l, bend left, "0"] \arrow[r, bend left, "a_{n-1}"] \arrow[loop above, "-n+1"] & v_{n-2} \arrow[l, bend left, "1"] \arrow[r, bend left, "a_{n-2}"] \arrow[loop above, "-n+3"] & v_{n-3} \arrow[l, bend left, "1"] \arrow[r, bend left, "a_{n-3}"] \arrow[loop above, "-n+5"] & \dots \arrow[l, bend left, "1"] \arrow[r, bend left, "a_3"] & v_2 \arrow[l, bend left, "1"] \arrow[r, bend left, "a_2"] \arrow[loop above, "n-5"] & v_1 \arrow[l, bend left, "1"] \arrow[r, bend left, "a_1"] \arrow[loop above, "n-3"] & v_0 \arrow[l, bend left, "1"] \arrow[r, bend left, "0"] \arrow[loop above, "n-1"] & {}
\end{tikzcd} \tag{1.9}$$

in the basis $\{v_0, v_1, \dots, v_{n-1}\}$. From (1.9) it follows by a direct calculation that in this basis the linear operators E, F and H satisfy (1.25). This proves that $\mathbf{V}^{(n)}$ is a $\diamond$-module and completes the proof. □

Proposition 1.5.6. *Let V be a simple finite-dimensional* g*-module. Then the Hermitian inner product with respect to which the module V is unitarizable is unique up to a positive real scalar.*

Proof. Let $V = \mathbf{V}^{(n)}$, $n \in \mathbf{N}$, and $(\cdot, \cdot)$ be a Hermitian inner product on V with respect to which V is unitarizable. Consider the basis $\{u_0, \dots, u_{n-1}\}$ from the proof of Theorem 1.5.1. The vectors in this basis are eigenvectors of the self-adjoint linear operator H corresponding to pairwise different eigenvalues. Hence $(u_i, u_j) = 0$ for all $i \neq j$. Let $(u_0, u_0) = c$. Then c is a positive real number. Let us prove that $(u_i, u_i) = c$ for all $i \in \{0, \dots, n-1\}$ by induction on i. The basis of the induction is trivial. For all $i \in \{1, \dots, n-1\}$ we have

$$\begin{array}{rcl} (u_i, u_i) & \overset{(1.24)}{=} & \left(\frac{1}{\sqrt{a_i}} F(u_{i-1}), \frac{1}{\sqrt{a_i}} F(u_{i-1})\right) \\ \text{(by (1.23))} & = & \frac{1}{a_i}(u_{i-1}, E(F(u_{i-1}))) \\ \text{(by (1.24))} & = & \frac{1}{a_i}(u_{i-1}, a_i u_{i-1}) \\ & = & (u_{i-1}, u_{i-1}) \\ \text{(by induction)} & = & c. \end{array}$$

Proposition 1.5.7. *Let V be a simple finite-dimensional* g*-module. Then the non-degenerate symmetric bilinear form with respect to which the module V is a $\diamond$-module is unique up to a non-zero complex scalar.*

Proof. Let $V=\mathbf{V}(n)$, $n \in \mathbf{N}$, and $(\cdot,\cdot)$ be a non-degenerate symmetric bilinear form on V with respect to which V is a $\diamond$-module. Consider the basis $\{v_0, v_1, \dots, v_{n-1}\}$ from (1.9). For $i, j \in \{0, \dots, n-1\}$ by (1.25) we have $(H(v_i), v_j) = -(v_i, H(v_j))$. As all elements of our basis are eigenvectors to H with pairwise different eigenvalues, it follows that $(v_i, v_j) \neq 0$ implies that the eigenvalues λ_i and λ_j of v_i and v_j, respectively, satisfy $\lambda_i = -\lambda_j$. Hence $(v_i, v_j) \neq 0$ implies $i = n - 1 - j$. Let $(v_0, v_{n-1}) = c$. As $(\cdot,\cdot)$ is non-degenerate, we have $c \neq 0$. Let us show by induction on i that $(v_i, v_{n-1-i}) = c$ for all $i \in \{0, 1, \dots, n-1\}$. For all $i \in \{1, \dots, n-1\}$

$$\begin{aligned}
(v_i, v_{n-1-i}) &\overset{(1.9)}{=} \left(F(v_{i-1}), \frac{1}{a_{n-i}} E(u_{n-i})\right) \\
\text{(by (1.25))} &= \frac{1}{a_{n-i}}(v_{i-1}, F(E(v_{n-i}))) \\
\text{(by (1.9))} &= \frac{1}{a_{n-i}}(v_{i-1}, a_{n-i} v_{n-i}) \\
&= (v_{i-1}, v_{n-i}) \\
\text{(by induction)} &= c.
\end{aligned}$$

For a direct sum of simple modules the description of bilinear forms analogous to Propositions 1.5.6 and 1.5.7 will be more complicated. In particular, as an obvious observation one could point out that it is possible to independently rescale the restrictions of the bilinear form to pairwise orthogonal direct summands.

1.8.16. (a) The correspondence

$$\mathbf{e}^{\natural} = -\mathbf{e},\ \mathbf{f}^{\natural} = -\mathbf{f},\ \mathbf{h}^{\natural} = -\mathbf{h}$$

uniquely extends to an antiinvolution $\natural$ on g.

(b) For every $n \in \mathrm{N}$ there exists a unique (up to a non-zero scalar) non-degenerate bilinear form $(\cdot,\cdot)_n$ on $\mathbf{V}(n)$ such that

$$(X(v), w)_n = (v, X^{\natural}(w))_n$$

for all $X \in \{\pm E, \pm F, \pm H\}$ and all $v, w \in \mathbf{V}(n)$.

1.6 Bilinear forms on tensor products

Let V and W be two vector spaces and $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be bilinear forms on V and W respectively. Then the assignment

$$(v \otimes w, v' \otimes w') = (v, v')_1 \cdot (w, w')_2 \quad (1.26)$$

extends to a bilinear form on the tensor product $V \otimes W$.

Exercise 1.6.1. Check that the form $(\cdot,\cdot)$ is symmetric provided that both $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are symmetric; that the form $(\cdot,\cdot)$ is non-degenerate provided that both $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are non-degenerate; and that the form $(\cdot,\cdot)$ is Hermitian provided that both $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are Hermitian.

Proposition 1.6.2. *Assume that V and W are unitarizable modules (resp. $\diamond$-modules) with respect to the forms $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ respectively. Then $V \otimes W$ is unitarizable (resp. a $\diamond$-module) with respect to $(\cdot,\cdot)$.*

Proof. We prove the statement for unitarizable modules. For $\diamond$-modules the proof is similar. Because of Exercise 1.6.1 it is sufficient to check (1.23) for $X \in \{E, F, H\}$. For $v, v' \in V$ and $w, w' \in W$ we have

$$\begin{aligned}
(X(v \otimes w), v' \otimes w') &\overset{(1.17)}{=} (X(v) \otimes w + v \otimes X(w), v' \otimes w') \\
\text{(by linearity)} &= (X(v) \otimes w, v' \otimes w') + (v \otimes X(w), v' \otimes w') \\
\text{(by (1.26))} &= (X(v), v')_1 \cdot (w, w')_2 + (v, v')_1 (X(w), w')_2 \\
\text{(by (1.23))} &= (v, X^{\star}(v'))_1 \cdot (w, w')_2 + (v, v')_1 (w, X^{\star}(w'))_2 \\
\text{(by (1.26))} &= (v \otimes w, X^{\star}(v') \otimes w') + (v \otimes w, v' \otimes X^{\star}(w'))
\end{aligned}$$

(by linearity) $= (v \otimes w, X_\star(v') \otimes w' + v' \otimes X_\star(w'))$
(by (1.17)) $= (v \otimes w, X_\star(v' \otimes w'))$.
The claim follows.

We know that the tensor product of two simple finite-dimensional gmodules is not simple in general (see Theorem 1.4.5). Hence the bilinear form making this tensor product module unitarizable or a $\diamond$-module is usually not unique. However, we would like to finish this section with a description of one invariant, which turns out in the real case.

Exercise 1.6.3. Consider the real Lie algebra $\mathrm{sl}_2(\mathbb{R})$. Show that (1.9) still defines on the real span $\mathbf{V}^{(n)}$ of $\{v_0, \dots, v_{n-1}\}$ the structure of a simple $\mathrm{sl}_2(\mathbb{R})$-module.

Check that the analogues of Theorem 1.4.5 and all the above results from Sections 1.5 and 1.6 are true for $\mathrm{sl}_2(\mathbb{R})$ with the same proofs. After Exercise 1.6.3 one could point out one striking difference between the real versions of Proposition 1.5.6 and Proposition 1.5.7. It is the possibility of the sign change in the assertion of Proposition 1.5.7 (note that two forms which differ by a sign change cannot be obtained from each other by a base change in the original module). Let us call the form on $\mathbf{V}^{(n)}$, described in the proof of Proposition 1.5.7, *standard*, and the form, obtained from the standard form by multiplying with -1, *non-standard*. Our main result in this section is the following:

Recall:

Theorem 1.4.5. *Let $m, n \in \mathbb{N}$ be such that $m \leq n$. Then*

$$\mathbf{V}^{(n)} \otimes \mathbf{V}^{(m)} \cong \mathbf{V}^{(n-m+1)} \oplus \mathbf{V}^{(n-m+3)} \oplus \dots \oplus \mathbf{V}^{(n+m-3)} \oplus \mathbf{V}^{(n+m-1)}. \quad (1.18)$$

Theorem 1.6.4. *Let $m, n \in \mathbb{N}$, $m \leq n$; $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be standard forms on $\mathbf{V}^{(m)}$ and $\mathbf{V}^{(n)}$ respectively; and $(\cdot,\cdot)$ be the form on $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(m)}$ given by* (1.26). *Then, up to multiplication with a positive real number, for $i = 0, 1, \dots, m$ the restriction of $(\cdot,\cdot)$ to the direct summand $\mathbf{V}^{(n+m-1-2i)}$ of $\mathbf{V}^{(n)} \otimes \mathbf{V}^{(m)}$ is standard for all even i and non-standard for all odd i.*
(wihtout proof)

1.8.22. For every finite-dimensional g-module V we define the function $\mathrm{ch}_V : \mathbb{Z} \to \mathbb{N}_0$ as follows:
$\mathrm{ch}_V(\lambda) = \dim V_\lambda$, $\lambda \in \mathbb{Z}$.
(a) Show that $\mathrm{ch}_V(\lambda) = 0$ for all $\lambda \in \mathbb{Z}$ such that $|\lambda|$ is big enough.
(b) Show that $\mathrm{ch}_V(\lambda) = \mathrm{ch}_V(-\lambda)$ for all $\lambda \in \mathbb{Z}$.
(c) Show that $\mathrm{ch}_V(\lambda) \geq \mathrm{ch}_V(\mu)$ for all elements $\lambda, \mu \in \mathbb{Z}$ of the same parity such that $0 \leq |\lambda| \leq |\mu|$.
(d) Show that $\mathrm{ch}_V = \mathrm{ch}_W$ if and only if $V \cong W$.
(e) Show that for any function $\mathrm{ch} : \mathbb{Z} \to \mathbb{N}_0$, which has the properties, described in (a)–(c) above, there exists a unique (up to isomorphism) g-module V such that $\mathrm{ch} = \mathrm{ch}_V$.

Proof: a) since V has only finitely many eigenvalues, the statement follows.
b) for all irr rep clearly $\mathrm{ch}_V(\lambda) = \mathrm{ch}_V(-\lambda)$ hence also for V
c) for all irr rep, they have eigenvalues of some parity with abs(λ)<n with dim(eigenspace)=1
d), e) define $N_i = \mathrm{ch}_V(i) - \mathrm{ch}_V(i+2)$ it follows, that $V \cong N_1 V^{(1)} \oplus \dots \oplus N_k V^{(k)} \oplus \dots$