# EXERCISES 3: REPRESENTATION THEORY OF $\otimes$ CATEGORIES

## 1. Exercises

**Exercise 1 (Twisting super vector spaces).** Let $\mathscr{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}$ be the $\mathbb{C}$-linear monoidal category with objects given by the elements of $\mathbb{Z}/2\mathbb{Z}$, $\otimes$ on objects $=$ group multiplication and trivial hom spaces.

Let $\omega\colon \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\to\mathbb{C}^*$ be the 3-cocycle with $\omega(1,1,1)=-1$, and twist $\mathscr{V}\mathrm{ec}^1_{\mathbb{Z}/2\mathbb{Z}}=\mathscr{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}$ on the associator $\alpha$ and unitors $\lambda$, $\rho$ by

$$\alpha_{i,j,k}=\omega(i,j,k)\mathrm{id}_{ijk},\quad \lambda_i=\omega(1,1,i)^{-1}\mathrm{id}_i,\quad \rho_i=\omega(i,1,1)\mathrm{id}_i,$$

and obtain $\mathscr{V}\mathrm{ec}^\omega_{\mathbb{Z}/2\mathbb{Z}}$.

(a) Show that $\mathscr{V}\mathrm{ec}^\omega_{\mathbb{Z}/2\mathbb{Z}}$ is nonstrict and skeletal at the same time.

(b) Show that the additive Grothendieck algebra $[\_]_\oplus$ of the additive closure of $\mathscr{V}\mathrm{ec}^x_{\mathbb{Z}/2\mathbb{Z}}$ is $[\mathscr{V}\mathrm{ec}^x_{\mathbb{Z}/2\mathbb{Z}}]_\oplus\cong\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$, regardless of $x\in\{1,\omega\}$.

(c) Show that $\mathscr{V}\mathrm{ec}^1_{\mathbb{Z}/2\mathbb{Z}}$ is not monoidally equivalent to $\mathscr{V}\mathrm{ec}^\omega_{\mathbb{Z}/2\mathbb{Z}}$.

(d') Classify braiding on $\mathscr{V}\mathrm{ec}^x_{\mathbb{Z}/2\mathbb{Z}}$ for $x\in\{1,\omega\}$ (up to braided equivalence).

**Exercise 2 (Jordan decomposition in prime characteristic).** Let $G=\mathbb{Z}/p\mathbb{Z}$ for some prime $p$, and let $\mathbb{K}=\overline{\mathbb{F}_p}$.

(a) Show that $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ is a fiat monoidal category with one simple and $p$ indecomposable objects.

(b) Compute the cell structure of $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$.

**Exercise 3.** For fixed $n\in\mathbb{Z}_{>1}$, let $\mathscr{S}$ be a fiat monoidal category over $\mathbb{C}$ with indecomposable objects $Z_{l,r}$ for $1\le l\le n$ and $0\le r\le n-1$, with $r$ read modulo $n$. The objects $L_r=Z_{1,r}$ are simple and their projective covers are $P_r=Z_{n,r}$.

The monoidal structure is $M\otimes N\cong N\otimes M$ and

$$L_r\otimes L_{r'}\cong L_{r+r'},\quad L_r\otimes Z_{l,r'}\cong Z_{l,r+r'},\quad L_r\otimes P_{r'}\cong P_{r+r'},$$

$$Z_{l,r}\otimes Z_{l',r'}\cong\begin{cases}\bigoplus_{i=1}^{\min(l,l')}Z_{|l-l'|+2i-1,r+r'-i+\min(l,l')}, & \text{if } l+l'\le n,\\ \bigoplus_{i=1}^{n-\max(l,l')}Z_{|l-l'|+2i-1,r+r'-i+\min(l,l')} \\ \qquad\oplus\bigoplus_{i=1}^{l+l'-n}P_{r+r'-i+1}, & \text{if } l+l'>n,\end{cases}$$

$$Z_{l,r}\otimes P_{r'}\cong\bigoplus_{i=1}^{l}P_{r+r'-i+l},\quad P_r\otimes P_{r'}\cong\bigoplus_{i=1}^{n}P_{r+r'-i+n}.$$

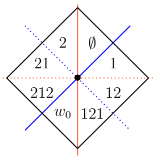
Let us define a two-variable Chebyshev-type polynomial by $U_0(X,Y)=1$, $U_1(X,Y)=X$ and $U_{e+1}(X,Y)=XU_e(X,Y)-YU_{e-1}(X,Y)$ for $e>0$. Let $[\_]_{exact}$ denote the abelian and $[\_]_\oplus$ the additive Grothendieck algebra.

(a) Compare $\mathscr{S}$ with $\mathscr{R}\mathrm{ep}(\mathbb{Z}/n\mathbb{Z},\mathbb{C})$ and $\mathscr{R}\mathrm{ep}(\mathbb{Z}/n\mathbb{Z},\overline{\mathbb{F}_n})$ (the latter assuming that $n$ is prime).

(b) Show that $[\mathscr{S}]_{exact}\cong\mathbb{Z}[Y]/(Y^n-1)$ but $[\mathscr{S}]_\oplus\cong\mathbb{Z}[X,Y]/(Y^n-1,(X-Y-1)U_{n-1}(X,Y))$.

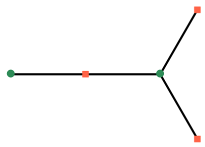
(c) Compute the cell structure of $\mathscr{S}$.

**Exercise 4 (More fun with dihedral groups – $\mathbb{Z}_{\geq 0}$-representations).** As on the previous exercise sheet, let $\emptyset$ denote the unit and let $D_n = \langle 1, 2 | 1^2 = 2^2 = (12)^n = \emptyset \rangle$ be the dihedral group of the $n$ gon.

$D_4 \rightsquigarrow$ 


We also allow $n = \infty$, having the evident meaning, and we consider the KL basis $\{b_w | w \in D_n\}$ as before.

We think of 1 as being colored spinach and 2 as being colored tomato. Take any simple connected bipartite graph $\Gamma = (V, E)$ with spinach $\underline{i}$ and tomato $\overline{i}$ colored vertices such as

$\Gamma =$ 

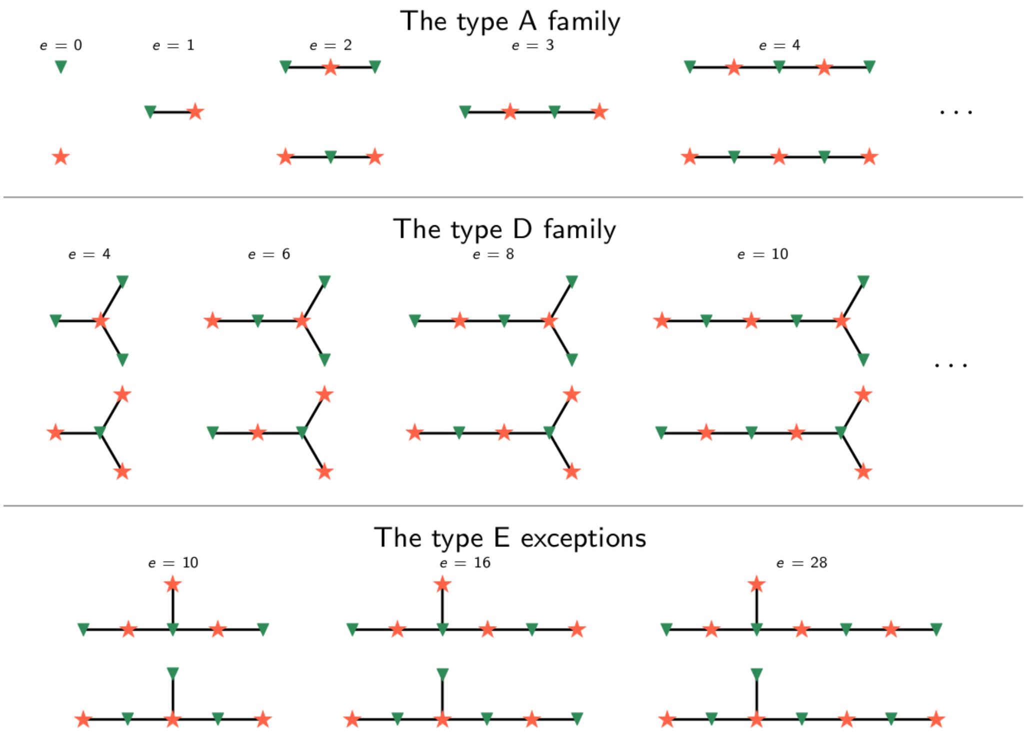
Define an $\mathbb{Z}_{\geq 0}$-representation of $D_n$ on $\mathbb{Z}_{\geq 0}V$ by

$$b_1 \cdot \underline{i} = 2 \cdot \underline{i}, \quad b_1 \cdot \overline{i} = \textstyle\sum_{\underline{j} - \overline{i}} \underline{j}, \quad b_2 \cdot \overline{i} = 2 \cdot \overline{i}, \quad b_2 \cdot \underline{i} = \textstyle\sum_{\overline{j} - \underline{i}} \overline{j},$$

where $a - b$ means $a$ and $b$ are connected.

(a) Verify that the above defines an $\mathbb{Z}_{\geq 0}$-representation of $D_\infty$ for any graph $\Gamma$.

(b*) Let $e = n - 2$. Verify that the above defines an $\mathbb{Z}_{\geq 0}$-representation of $D_n$ if and only if $\Gamma$ is of ADE type




where type $A_m$ shows up for $n = m + 1$, type $D_m$ for $n = 2m - 2$ and types $E_6$, $E_7$ and $E_8$ for $n = 12, 18, 30$, respectively.

(Aside: $e = n - 2$ is often called the level, with 2 being a reference to $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SO}_3(\mathbb{C})$ and the above is essentially a statement about $\mathbb{Z}_{\geq 0}$-representations of the Grothendieck algebras of these Lie Groups.)

(c') If you know Soergel bimodules, then you should be able to guess a categorification.

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- There might be typos on the exercise sheets, my bad. Be prepared.
- Star exercises are a bit trickier; prime exercises use notions I haven't explained.
- SageMath online calculator https://sagecell.sagemath.org/ with the relevant material summarized on https://doc.sagemath.org/html/en/thematic_tutorials/lie/weyl_character_ring.html
- Magma online calculator http://magma.maths.usyd.edu.au/calc/

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**Hints for Exercise 1**

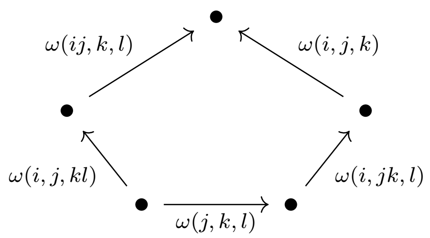
For this exercise observe that all hom spaces are trivial and thus, all calculations are just shifting scalars around.

For the background on (strict and nonstrict monoidal) categories see for example Chapter 2 of https://math.mit.edu/~etingof/egnobookfinal.pdf, and a recollection on braided categories can be found in Chapter 8 of that book.

For a group $G$, one can define a cohomology theory $H^*(G, \mathbb{C}^*)$, called group cohomology. As usual these are constructed from a certain cochain complex and $H^i(G, \mathbb{C}^*) = Z^i(G, \mathbb{C}^*)/B^i(G, \mathbb{C}^*)$, so $i$ cocycles modulo $i$ coboundaries. All we need to know about group cohomology are the 3 cocycles which are functions $\omega\colon G \times G \times G \to \mathbb{C}^*$ satisfying

$$\omega(j,k,l)\omega(i,jk,l)\omega(i,j,k) = \omega(ij,k,l)\omega(i,j,kl),$$

pictorially:

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These 3-cocycles give the obstruction set for twisting a monoidal structure on $\mathcal{V}\mathrm{ec}_G$. Moreover, $\mathcal{V}\mathrm{ec}_G^\omega \cong \mathcal{V}\mathrm{ec}_G^\nu$ if and only if $\omega$ and $\nu$ distinct in $H^3(G, \mathbb{C}^*)$. Monoidal categories of the form $\mathcal{V}\mathrm{ec}_G^\omega$ are nonstrict and skeletal, showing that MacLane's celebrated strictness theorem can not be proven by going to the skeleton.

The category $\mathcal{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}^1$ can be endowed with two braidings, the so-called standard braiding $\beta_{1,1}^{st} = 1$ and the super braiding $\beta_{1,1}^{su} = -1$. These are nonequivalent. For $\mathcal{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}^\omega$ the 3-cocycle $\omega$ only allows one braiding up to equivalence.

**Hints for Exercise 2**

The Jordan decomposition over $\mathbb{C}$ (or rather $\overline{\mathbb{Q}}$ since the Jordan decomposition is unstable over inexact rings) and over $\overline{\mathbb{F}_3}$ can be done using SageMath as above by using:

```
matrix(QQbar,[[0,1,0],[0,0,1],[1,0,0]]).jordan_form(subdivide=False)
matrix(GF(3),[[0,1,0],[0,0,1],[1,0,0]]).jordan_form(subdivide=False)
```

Knowing this, you should be able to give a complete classification of indecomposables modules.

To guess the tensor product rule (and thus, the cell structure) use

```
M=matrix(GF(3),[[1,0],[1,1]]);
M.tensor_product(M).jordan_form()
M=matrix(GF(3),[[1,0,0],[1,1,0],[0,1,1]]);
M.tensor_product(M).jordan_form()
```

**Hints (or rather comments) for Exercise 3**

There is a Hopf algebra $T_n$ realizing the fiat monoidal category $\mathscr{S}$. This algebra is the called Taft algebra and is defined by $T_n = \langle g, x | g^n = 1, x^n = 0, gx = \zeta xg \rangle$ where $\zeta$ is a complex primitive $n$th root of unity.

The Taft algebra is a notorious counterexample in Hopf algebra theory. For example, although $M \otimes N \cong N \otimes M$ holds, the category $\mathscr{S}$ is not braided. For $n = 2$ we also get an example of a fiat monoidal category with four indecomposable objects and infinitely many simples representations.

**Hints for Exercise 4**

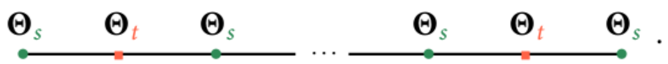
Let $U_k(X)$ be the Chebyshev polynomial defined by $U_0(X) = 1$, $U_1(X) = X$ and $U_{k+1}(X) = XU_k(X) - U_{k-1}(X)$ for $k > 1$. The defining relations of the $b_1$ and $b_2$ generators are the coefficients of these polynomials. That is, define $U_k(b_2, b_1)$ by replacing $X^k$ with an alternating string $\dots b_1 b_2 b_1$ of length $k$ (always having $b_1$ to the right), and define $U_{n-1}(b_2, b_1)$ similarly. Then $U_{n-1}(b_1, b_2) = 0 = U_{n-1}(b_2, b_1)$.

This means the graphs for which one gets a well-defined action must have their spectrum being a subset of the roots of the Chebyshev polynomial. The graphs satisfying this property are the ADE graphs.

If you are up for a challenge, then you can construct the associated simple representations of the Soergel calculus. This is (up to some scaling) straightforward if you have worked with Soergel calculus before:

- ▶ You need an algebra whose category of projectives you would like to act on: take the zigzag algebra associated to the graph $\Gamma$.
- ▶ The projective endofunctors $\Theta$ you need to use for the generating KL basis elements are direct sums of projective endofunctors over the colored vertices:

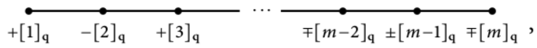
  We sum over the graph of type $A_m$ as (in the case where $m$ is odd):

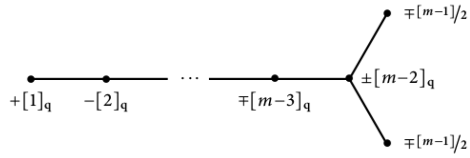
  Spinach = sum over tensoring with spinach projectives of the zigzag algebra and vise versa for tomato.
- ▶ All of the maps in Soergel land are then easy to guess.
- ▶ Warning: For Soergel calculus the scaling is often annoying, and this is the case here as well. The scaling drove me insane. Well, actually not as I was already insane before...

Anyway, the answer is not so bad in the end. You need to rescale everything using the entries of the Perron–Frobenius eigenvector of $\Gamma$, *e.g.* in ADE type:

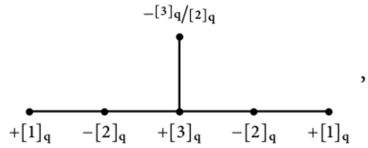
$(A_m)$



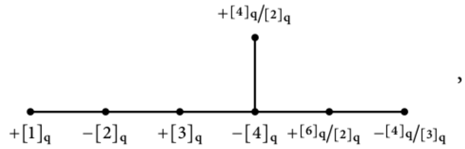

$(D_m)$



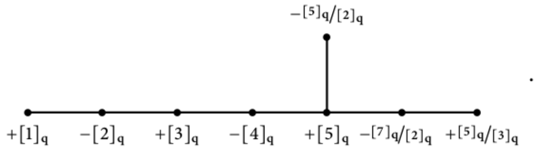

$(E_6)$




$(E_7)$




$(E_8)$




Here $[a]_q$ denote the usual quantum numbers evaluated at an $2n$th primitive complex root of unity. Scale the idempotents of the zigzag algebra using the values associated to the vertices, which are the aforementioned entries of the Perron–Frobenius eigenvector.

► Fingers crossed!