EXERCISES 13: LECTURE FOUNDATIONS OF MATHEMATICS

Exercise 1. Write down all axioms of ZF set theory which do not give you the creeps. (Meaning, repeat the what kind of axioms one has.)

Exercise 2. Show that separability implies that subsets are actually sets.

Exercise 3. Show that the class of all sets can not be a set.
Hint: use Exercise 2, and remember Russell.

Exercise 4. Note that this exercise is very difficult.

(a) Show that the well-order theorem implies the axiom of choice.
Hint: Let $\mathfrak{A}$ be a set of non-empty sets. Then the set $\bigcup \mathfrak{A}$ can be well-ordered. Find a choice function.

(b) Show that Zorn's lemma implies the well-order theorem.
Hint: Consider the set $\mathfrak{W}(X)$ of all subsets of X together with a well-order, i.e. pairs $(A, <_A)$ where $A \subset X$ and $<_A$ is a well-order. $\mathfrak{W}(X)$ is a partially ordered set ($(A, <_A) < (B, <_B)$ holds, by definition, if $<_A$ is obtained by restriction from $<_B$ and there exists $b \in B$ such that $A = \{a \in B \mid a <_B b\}$), on which one can apply Zorn's lemma. Show that this implies $(X, <_X) \in \mathfrak{W}(X)$.

(c) Show that the axiom of choice implies Zorn's lemma.
Hint: only for the valiant.

No submission of the exercise sheet.