Duality and extensions

I Duality

Q is without oriented cycles throughout this talk

The main goal of this talk is to show that for a quiver Q the projectives and injectives are „the same". Here „the same" means that they are equivalent in a categorical sense so let us recall some basic definitions from category theory.

Def 1. Let $\mathcal{C}, \mathcal{D}$ be two categories. Two functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are functorially isomorphic, write $F_1 \cong F_2$ if for every object $M \in \mathcal{C}$ there exists an isomorphism $\eta_M : F_1(M) \to F_2(M)$ s.t. for every morphism $f : M \to N$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F_1(M) \arrow[r, "F_1(f)"] \arrow[d, "\eta_M"'] & F_1(N) \arrow[d, "\eta_N"] \\ F_2(M) \arrow[r, "F_2(f)"] & F_2(N) \end{tikzcd}$$

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called an equivalence of categories if $\exists\, G : \mathcal{D} \to \mathcal{C}$ a functor s.t $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$. Such a functor G is called a quasi-inverse of F.
A contravariant functor F that has a contravariant quasi-inverse is called a duality.

Now we will be talking about a special duality. We start by recalling the opposite quiver Q^{op} of a quiver $Q = (Q_0, Q_1)$. The quiver Q^{op} is obtained by reversing the arrows of Q, i.e $Q^{op} = (Q_0^{op}, Q_1^{op})$, where $Q_0^{op} = Q_0$ and $Q_1^{op} = \{\alpha^{op} \mid \alpha \in Q_1\}$ with $s(\alpha^{op}) = t(\alpha)$ and $t(\alpha^{op}) = s(\alpha)$.

We define now the duality $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk) : \mathrm{rep}\, Q \to \mathrm{rep}\, Q^{op}$ as the contravariant functor such that:

- On objects $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$, we have

$$DM = (DM_i, D\varphi_{\alpha^{op}})_{i \in Q_0, \alpha \in Q_1}, \text{ where } DM_i = \mathrm{Hom}_{\Bbbk}(M_i, \Bbbk) \text{ is}$$
the dual vector space.

For a arrow α in Q we define $D_{\alpha^{op}}$ as the pullback of φ_α

i.e $$D\varphi_\alpha^{op} : DM_{t(\alpha)} \to DM_{s(\alpha)}$$
$$u \longmapsto u \circ \varphi_\alpha .$$

- On morphisms $f: M \to N$ we have $Df: DN \to DM$ defined by $Df(u) = u \circ f$:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "u\circ f"'] & N \arrow[d, "u"] \\ & \mathbb{k} \end{tikzcd} \qquad \begin{pmatrix} f \in \operatorname{rep} Q \\ Df \in \operatorname{rep} Q^{op} \end{pmatrix}$$

Note that the quasi-inverse of D_Q is $D_{Q^{op}}$.

From the definition of $P_Q(i)$ and duality we get the following connection between projectives of Q and injectives of Q^{op}:

Prop 2. For all vert. $i \in Q_0$ we have $D(P_Q(i)) = I_{Q^{op}}(i)$ and in particular the duality restricts to a duality $\operatorname{proj} Q \to \operatorname{inj} Q^{op}$.

Rmk This was the first step in comparing projectives and injectives.

II Nakayama functor

Consider a free representation A given as a direct sum of indecomposable projective representations of Q, i.e. $A = \bigoplus_{j \in Q_0} P(j)$.

Let X be a rep. of Q. We can give $\operatorname{Hom}(X, A)$ the structure of a rep $(M_i, \varphi_\alpha^{op})$ of Q^{op} in the following way:

- For every $i \in Q_0$ define $M_i := \operatorname{Hom}(X, P(i))$
- For every $i \xrightarrow{\alpha} j \in Q_1$, define $\varphi_{\alpha^{op}} : \operatorname{Hom}(X, P(j)) \to \operatorname{Hom}(X, P(i))$
$$f \longmapsto \alpha \circ f$$

i.e.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\varphi_\alpha^{op}(f)"'] & P(j) \arrow[d, "\alpha"] \\ & P(i) \end{tikzcd} \quad \text{commutes}$$

A hand-check of the commutativity of:

$$\begin{tikzcd} \operatorname{Hom}(X', P(j)) \arrow[r, "\varphi_\alpha^{op}"] \arrow[d, "g^* = \operatorname{Hom}(g, P(j))"'] & \operatorname{Hom}(X', P(i)) \arrow[d, "g^* = \operatorname{Hom}(g, P(i))"] \\ \operatorname{Hom}(X, P(j)) \arrow[r, "\varphi_\alpha^{op}"'] & \operatorname{Hom}(X, P(i)) \end{tikzcd} \quad \begin{pmatrix} X, X' \text{ reps of } Q \\ g: X \to X' \text{ morphism} \end{pmatrix}$$

gives us the following:

Prop 3. $\mathrm{Hom}(-,A)$ is a functor from rep Q to rep Q^{op}.

Now we are ready to define a very important functor.

Def 4. The Nakayama functor, which is covariant, is defined as the composition $\nu = D\mathrm{Hom}(-,A) : \mathrm{rep}\, Q \to \mathrm{rep}\, Q$

$$\begin{tikzcd} \mathrm{rep}\,Q \arrow[r, "\mathrm{Hom}(-,A)"] \arrow[rr, bend right, "\nu"'] & \mathrm{rep}\,Q^{op} \arrow[r, "D"] & \mathrm{rep}\,Q \end{tikzcd}$$

From corollary 2.26 from the last talk, $\mathrm{Hom}(-,A)$ is 0 on all reps that don't have projective summands. Therefore, it suffices to consider $M = \mathrm{Hom}(P_Q(i), A)$

In the notation $M = (M_j, \varphi_{\alpha^{op}})_{\substack{j\in Q_0 \\ \alpha \in Q_1}}$ we have $M_j = \mathrm{Hom}(P_Q(i), P_Q(j))$, so M_j has as basis paths $j \to i$ in Q or equivalently, the paths $i \to j$ in Q^{op}. Remains to look at $\varphi_{\alpha^{op}} : M_j \to M_h$. This map by definition sends a path from i to j in Q^{op} to a path from i to h in Q^{op}. Thus, M satisfies the def. of the indec. proj. rep $P_{Q^{op}}(i)$, i.e. $\mathrm{Hom}(P_Q(i), A) \overset{(*)}{=} P_{Q^{op}}(i)$ and $\mathrm{Hom}(-,A)|_{\mathrm{proj}\,Q}$ gives a duality $\mathrm{proj}\, Q \to \mathrm{proj}\, Q^{op}$.

Now, we are ready to prove that projectives and injectives are „the same".

Prop 5. The restriction of ν to proj Q is an equivalence of cats. $\mathrm{proj}\,Q \to \mathrm{inj}\,Q$ whose quasi-inverse is given by

$$\nu^{-1} = \mathrm{Hom}(DA^{op}, -) : \mathrm{inj}\,Q \to \mathrm{proj}\,Q.$$

Moreover, for any $i \in Q_0$,

$$\nu P(i) = I(i)$$

and if c is a path from i to j and $f_c \in \mathrm{Hom}(P(j), P(i))$ is the corresponding mor. then $\nu f_c : I(j) \to I(i)$ is the morphism given by canceling the path c.

Proof: since $\nu = D \circ \mathrm{Hom}(-, A)$ where the functors from the right side are dualities, then ν is an equivalence of cats. To find ν^{-1} we just take the composition of the quasi-inverses of D and $\mathrm{Hom}(-,A)$, i.e.
$\nu^{-1} = \mathrm{Hom}_{Q^{op}}(-, A^{op}) \circ D = \mathrm{Hom}(DA, -)$ since $\mathrm{Hom}(DX, DY) \cong \mathrm{Hom}(X, Y)$, $\forall X, Y \in \mathrm{rep}\, Q$.

So, $\nu P_Q(i) = D\,\mathrm{Hom}(P_Q(i), A) \overset{\text{from}(*)}{=} D(P_{Q^{op}}(i)) = I_Q(i)$.

Finally, let c and f_c be as in the proposition and let f_c^* be the image of f_c under $\mathrm{Hom}(-,A)$.

So, $f_c^*: \mathrm{Hom}(P_Q(i), A) \longrightarrow \mathrm{Hom}(P_Q(j), A))$

$$\begin{array}{ccc} \mathrm{Hom}(P_Q(i), A) & \longrightarrow & \mathrm{Hom}(P_Q(j), A) \\ \| (*) & & \| (*) \\ P_{Q^{op}}(i) & & P_{Q^{op}}(j) \end{array}$$

and $f_c^*: P_{Q^{op}}(i) \to P_{Q^{op}}(j)$ sends y to $c^{op}y$.
Then $\nu f_c = D f_c^*$ sends $D(c^{op} y) = D(y)c$ to $D(y)$, i.e it cancels the path c.

$\boxtimes$

<u>Ex. 6</u> $Q: 1 \longrightarrow 2 \longrightarrow 3$ we build the Auslander-Reiten quiver:

$$\begin{tikzcd}[column sep=small]
 & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
3 \arrow[ur] & & 2 \arrow[ur] & & 1
\end{tikzcd}$$

($P(3) = 0 \to 0 \to \Bbbk$, $P(2) = 0 \to \Bbbk \to \Bbbk$), $P(1) = \Bbbk \to \Bbbk \to \Bbbk$)
$P(3) = 3$ $\quad P(2) = \begin{smallmatrix}2\\3\end{smallmatrix}$ $\quad P(1) = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$

and

$$\begin{tikzcd}[column sep=small]
 & & P(1) = \nu P(3) = I(3) \arrow[dr] & & \\
 & P(2) \arrow[ur] & & \nu P(2) = I(2) \arrow[dr] & \\
P(3) \arrow[ur] & & & & \nu P(1) = I(1)
\end{tikzcd}$$

Ex 7.

$$\begin{tikzcd} & & 4 \\ Q: \; 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 3 \end{tikzcd}$$

proj Q

$$\begin{tikzcd} P(3) = 3 \arrow[dr] & & \\ & \begin{smallmatrix} 2 \\ 3\,4 \end{smallmatrix} = P(2) \arrow[r] & \begin{smallmatrix} 1 \\ 2 \\ 3\,4 \end{smallmatrix} = P(1) \\ P(4) = 4 \arrow[ur] & & \end{tikzcd}$$

inj Q

$$\begin{tikzcd} \nu P(3) = I(3) = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \arrow[dr] & & \\ & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} = \nu P(2) = I(2) \arrow[r] & 1 = I(1) \\ \nu P(4) = I(4) = \begin{smallmatrix} 1 \\ 2 \\ 4 \end{smallmatrix} \arrow[ur] & & \end{tikzcd}$$

Def 8. Let $\mathcal{C}$ and $\mathcal{D}$ be ab. categories. A covariant or contravariant functor $F: \mathcal{C} \to \mathcal{D}$ is called exact if it maps exact sequences in $\mathcal{C}$ to exact sequences in $\mathcal{D}$.

We also have the weaker definitions:

$F: \mathcal{C} \to \mathcal{D}$ covariant is called left exact if for any exact seq

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N$$

the sequence

$$0 \to F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \text{ is exact.}$$

F is called right exact if for any exact. seq.

$$L \xrightarrow{f} M \xrightarrow{g} N \to 0 \quad \text{the seq}$$

$$F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \to 0 \text{ is exact.}$$

Similarly for contravariant functors (see Sc[14]).

One of the important properties of the Nakayama functor is the following:

Prop 9. The Nakayama functor ν is right exact.

Pf: Clear since it is a composition of the left exact functor $\mathrm{Hom}(-, A)$ and the exact (contravariant) functor D. $\square$

Ex 10. From Ex. 8 we get a s.e.s

$$0 \to 3 \xrightarrow{f} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \xrightarrow{g} \begin{smallmatrix}1\\2\end{smallmatrix} \to 0$$

Applying ν yields the exact seq

$$\begin{smallmatrix}1\\2\\3\end{smallmatrix} \xrightarrow{\nu f} 1 \xrightarrow{\nu g} 0 \to 0$$

Note that ν is not exact, since νf is not injective.

Another exact sequence that we get from a minimal projective resolution

$$0 \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

of an indecom. $M \in \operatorname{rep} Q$ is

$$0 \to \tau M \to \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \to 0, \text{ (we applied } \nu)$$

where $\tau M = \ker \nu p_1$ is called the A-R translate of M.
Playing a similar game with a minimal injective resolution, we get the inverse A-R translate.

III Extensions

Let $M \in \operatorname{rep} Q$. Take a projective resolution

$$0 \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0 \quad \text{of } M$$

and let $N \in \operatorname{rep} Q$ be arbitrary. By applying $\operatorname{Hom}(-,N)$ we get the exact seq.:

$$0 \to \operatorname{Hom}(M,N) \xrightarrow{g^*} \operatorname{Hom}(P_0,N) \xrightarrow{f^*} \operatorname{Hom}(P_1,N) \to \operatorname{Ext}^1(M,N) \to 0,$$

where $\operatorname{Ext}^1(M,N) = \operatorname{coker} f^*$ is called the first group of extensions of M and N.

Rmk 11 In arbitrary categories projective resolutions might have the following form:

$$\cdots \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

i.e, they don't stop on the left. Then by applying $Hom(-, N)$ we get a cochain complex:

$$0 \to Hom(M,N) \xrightarrow{f_g^*} Hom(P_0, N) \xrightarrow{f_1^*} \cdots \xrightarrow{f_n^*} Hom(P_n, N) \to \cdots,$$

where $f_i^* f_{i-1}^* = 0$, $\forall i$. Then we can define the i^{th} extension group $Ext^i(M,N) = \ker f_{i+1}^* / \operatorname{im} f_i^*$.

Since in rep Q, minimal projective resolutions end in two steps, we have that Ext^i-groups vanish for $i \geq 2$. However, finding Ext-gps is not easy. We next look at $Ext^1(M,N)$

Def 12. An extension ξ of M by N is a s.e.s. $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$. Two extensions ξ and ξ' are called equivalent if there is a comm. diagram:

$$\begin{tikzcd}
\xi : 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "="] & E \arrow[r, "g"] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, "="] & 0 \\
\xi' : 0 \arrow[r] & N \arrow[r, "f'"] & E' \arrow[r, "g'"] & M \arrow[r] & 0
\end{tikzcd}$$

An extension is split if the s.e.s is split.

Given ξ, ξ' extensions of M by N we define:

$E'' = \{(x,x') \in E \times E' \mid g(x) = g'(x')\}$ and

$F = E'' / \{(f(n), -f'(n)) \in E \oplus E' \mid n \in N\}$. Then we define $\xi + \xi'$ as

$$0 \to N \to F \to M \to 0$$

The set of eq. classes $\mathcal{E}(M,N)$ of extensions of M by N together with the sum of extensions is an abelian gp and the class of split extensions is the zero el. of that gp.

We define an iso. $\mathcal{E}(M,N) \to Ext^1(M,N)$ in the following way: Pick a representative of a class in $\mathcal{E}(M,N)$

$\xi: 0 \to N \xrightarrow{u'} E \xrightarrow{v'} M \to 0$ and a proj. resolution

$0 \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$

Then there exist maps u, f' s.t. the following commutes

$$\begin{tikzcd}
& P_1 \arrow[r, "f"] \arrow[d, "u"] & P_0 \arrow[r, "g"] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "="] & 0 \\
\xi : 0 \arrow[r] & N \arrow[r, "u'"] & E \arrow[r, "v'"] & M \arrow[r] & 0
\end{tikzcd}$$

(see [Sc14] for more details)

The iso $\xi(M,N) \to \mathrm{Ext}^1(M,N)$ is given by

$$\xi \longmapsto u$$

So, $\mathrm{Ext}^1(M,N)$ as a vector space, is isomorphic to the vector space of extensions of M by N.

The reader is welcome to work on examples :).