

Duality and extensions

I Duality

Q is without oriented cycles throughout this talk

The main goal of this talk is to show that for a quiver Q the projectives and injectives are "the same". Here "the same" means that they are equivalent in a categorical sense so let us recall some basic definitions from category theory.

Def 1. Let \mathcal{C}, \mathcal{D} be two categories. Two functors $F_1, F_2: \mathcal{C} \rightarrow \mathcal{D}$ are functorially isomorphic, write $F_1 \cong F_2$ if for every object $M \in \mathcal{C}$ there exists an isomorphism $\eta_M: F_1(M) \rightarrow F_2(M)$ s.t. for every morphism $S: M \rightarrow N$ in \mathcal{C} the following diagram commutes:

$$\begin{array}{ccc} F_1(M) & \xrightarrow{F_1(S)} & F_1(N) \\ \eta_M \downarrow & & \downarrow \eta_N \\ F_2(M) & \xrightarrow{F_2(S)} & F_2(N) \end{array}$$

A covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called an equivalence of categories if $\exists G: \mathcal{D} \rightarrow \mathcal{C}$ a functor s.t. $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$. Such a functor G is called a quasi-inverse of F .

A contravariant functor F that has a contravariant quasi-inverse is called a duality.

Now we will be talking about a special duality. We start by recalling the opposite quiver Q^{op} of a quiver $Q = (Q_0, Q_1)$. The quiver Q^{op} is obtained by reversing the arrows of Q , i.e. $Q^{op} = (Q_0^{op}, Q_1^{op})$, where $Q_0^{op} = Q_0$ and $Q_1^{op} = \{d^{op} \mid d \in Q_1\}$ with $s(d^{op}) = t(d)$ and $t(d^{op}) = s(d)$.

We define now the duality $D = \text{Hom}_k(-, k): \text{rep } Q \rightarrow \text{rep } Q^{op}$ as the contravariant functor such that:

- On objects $M = (M_i, \rho_d)_{i \in Q_0, d \in Q_1}$, we have

$DM = (DM_i, D\rho_{d^{op}})_{i \in Q_0, d \in Q_1}$, where $DM_i = \text{Hom}_k(M_i, k)$ is the dual vector space.

For an arrow f in Q we define $D_{d^{op}}$ as the pullback of f

i.e. $D_{d^{op}}: DM_{t(d)} \rightarrow DM_{s(d)}$
 $u_1 \rightarrow u_0 \circ f$

- On morphisms $f: M \rightarrow N$ we have $Df: DN \rightarrow DM$ defined by $Df(u) = u \circ f$:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow u \circ f & \downarrow u \\ & & K \end{array} \quad \left(\begin{array}{l} f \in \text{rep } Q \\ Df \in \text{rep } Q^{op} \end{array} \right)$$

Note that the quasi-inverse of D_Q is $D_{Q^{op}}$.

From the definition of $P_Q(i)$ and duality we get the following connection between projectives of Q and injectives of Q^{op} :

Prop 2. For all $\text{vert. } i \in Q_0$ we have $D(P_Q(i)) = I_{Q^{op}}(i)$ and in particular the duality restricts to a duality $\text{proj } Q \rightarrow \text{inj } Q^{op}$.

Remark This was the first step in comparing projectives and injectives.

II Nakayama Functor

Consider a free representation A given as a direct sum of indecomposable projective representations of Q , i.e. $A = \bigoplus_{i \in Q_0} P(i)$.

Let X be a rep. of Q . We can give $\text{Hom}(X, A)$ the structure of a rep. $(M_i, \varphi_{i, Q^{op}})$ of Q^{op} in the following way:

- For every $i \in Q_0$ define $M_i := \text{Hom}(X, P(i))$
- For every $i \xrightarrow{f} j \in Q_1$, define $\varphi_{i, Q^{op}}: \text{Hom}(X, P(j)) \rightarrow \text{Hom}(X, P(i))$
 $f \longmapsto f \circ f$

$$\text{i.e. } \begin{array}{ccc} X & \xrightarrow{f} & P(j) \\ & \searrow \varphi_{i, Q^{op}}(f) & \downarrow u \\ & & P(i) \end{array} \quad \downarrow u \text{ commutes}$$

A hand-check of the commutativity of:

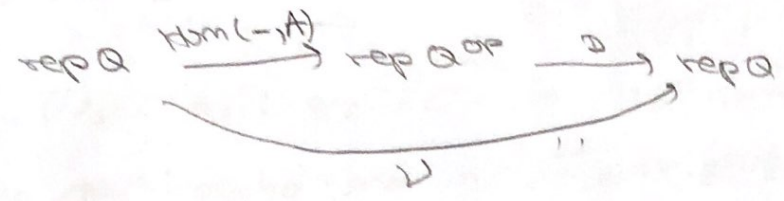
$$\begin{array}{ccc} \text{Hom}(X', P(j)) & \xrightarrow{\varphi_{i, Q^{op}}} & \text{Hom}(X', P(i)) \\ \downarrow g^* = \text{Hom}(g, P(j)) & & \downarrow g^* = \text{Hom}(g, P(i)) \\ \text{Hom}(X, P(j)) & \xrightarrow{\varphi_{i, Q^{op}}} & \text{Hom}(X, P(i)) \end{array} \quad \left(\begin{array}{l} X, X' \text{ reps of } Q \\ g: X \rightarrow X' \text{ morphism} \end{array} \right)$$

gives us the following:

Prop 3. $\text{Hom}(-, A)$ is a functor from $\text{rep } Q$ to $\text{rep } Q^{\text{op}}$.

Now we are ready to define a very important functor.

Def 4. The Nakayama functor, which is covariant, is defined as the composition $\nu = D\text{Hom}(-, A) : \text{rep } Q \rightarrow \text{rep } Q$



From corollary 2.26 from the last talk, $\text{Hom}(-, A)$ is 0 on all reps that don't have projective summands. Therefore, it suffices to consider $M = \text{Hom}(P_{\alpha}(i), A)$

In the notation $M = (M_j, \varphi_{j, h}^{\text{op}})$ so M_j has as basis paths $i \rightarrow j$ in Q^{op} . Remains to look at $\varphi_{j, h}^{\text{op}} : M_j \rightarrow M_h$. This map by definition sends a path from i to j in Q^{op} to a path from i to h in Q^{op} . Thus, M satisfies the def. of the indec. proj. rep $P_{Q^{\text{op}}}(i)$ i.e. $\text{Hom}(P_{\alpha}(i), A) \stackrel{(*)}{=} P_{Q^{\text{op}}}(i)$ and $\text{Hom}(-, A)|_{\text{proj } Q}$ gives a duality $\text{proj } Q \rightarrow \text{proj } Q^{\text{op}}$.

Now, we are ready to prove that projectives and injectives are "the same".

Prop 5. The restriction of ν to $\text{proj } Q$ is an equivalence of cats. $\text{proj } Q \rightarrow \text{inj } Q$ whose quasi-inverse is given by

$$\nu^{-1} = \text{Hom}(D A^{\text{op}}, -) : \text{inj } Q \rightarrow \text{proj } Q.$$

Moreover, for any $i \in Q_0$,

$$\nu P(i) = I(i)$$

and if c is a path from i to j and $f_c \in \text{Hom}(P(j), P(i))$ is the corresponding morphism then $\nu f_c : I(j) \rightarrow I(i)$ is the morphism given by canceling the path c .

Proof: since $\mathcal{D} = \mathcal{D} \circ \text{Hom}(-, A)$ where the functors from the right side are dualities, then \mathcal{D} is an equivalence of cats. To find \mathcal{D}^{-1} we just take the composition of the quasi-inverses of \mathcal{D} and $\text{Hom}(-, A)$, i.e.

$$\mathcal{D}^{-1} = \text{Hom}_{\mathcal{Q}^{\text{op}}}(-, A^{\text{op}}) \circ \mathcal{D} = \text{Hom}(\mathcal{D}A, -) \text{ since } \text{Hom}(\mathcal{D}X, \mathcal{D}Y) \cong \text{Hom}(X, Y), \forall X, Y \in \text{rep } \mathcal{Q}.$$

$$\text{So, } \mathcal{D} P_{\mathcal{Q}}(i) = \mathcal{D} \text{Hom}(P_{\mathcal{Q}}(i), A) \stackrel{\text{from (*)}}{=} \mathcal{D}(P_{\mathcal{Q}^{\text{op}}}(i)) = \mathcal{I}_{\mathcal{Q}}(i).$$

Finally, let c and f_c be as in the proposition and let f_c^* be the image of f_c under $\text{Hom}(-, A)$.

$$\text{So, } f_c^* : \text{Hom}(P_{\mathcal{Q}}(i), A) \rightarrow \text{Hom}(P_{\mathcal{Q}}(j), A)$$

$$\parallel (*) \qquad \qquad \qquad \parallel (*)$$

$$P_{\mathcal{Q}^{\text{op}}}(i) \qquad \qquad \qquad P_{\mathcal{Q}^{\text{op}}}(j)$$

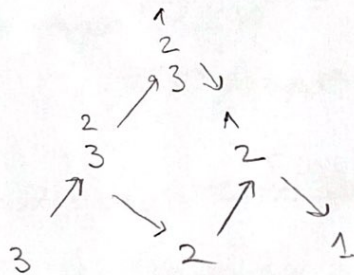
and $f_c^* : P_{\mathcal{Q}^{\text{op}}}(i) \rightarrow P_{\mathcal{Q}^{\text{op}}}(j)$ sends y to copy_y .

Then $\mathcal{D} f_c = \mathcal{D} f_c^*$ sends $\mathcal{D}(\text{copy}_y) = \mathcal{D}(y)c$ to $\mathcal{D}(y)$, i.e. it cancels the path c .

□

Ex. 6

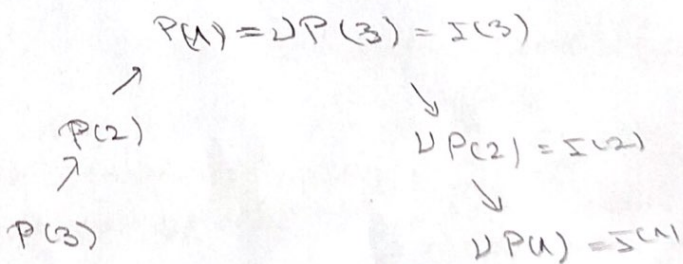
$\mathcal{Q} : 1 \rightarrow 2 \rightarrow 3$ we build the Auslander-Reiten quiver :



$$(P(3) = 0 \rightarrow 0 \rightarrow K, \quad P(2) = 0 \rightarrow K \rightarrow K, \quad P(1) = K \rightarrow K \rightarrow K)$$

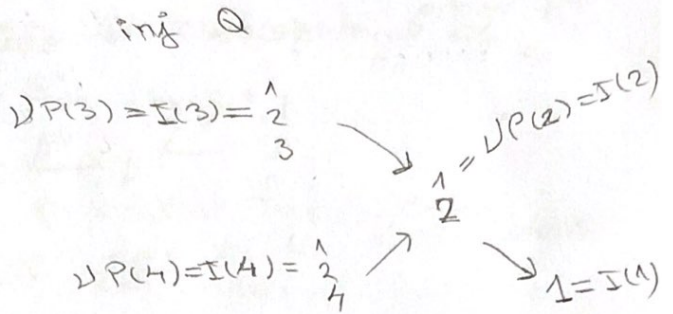
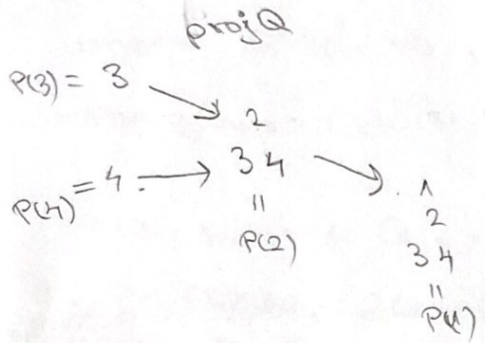
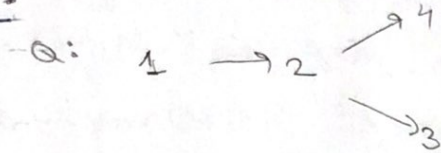
$$P(3) = 3, \quad P(2) = \frac{2}{3}, \quad P(1) = \frac{1}{3}$$

and



□

Ex 7.



Def 8. Let \mathcal{C} and \mathcal{D} be ab. categories. A covariant or contra-variant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called exact if it maps exact sequences in \mathcal{C} to exact sequences in \mathcal{D} .

We also have the weaker definitions:

$F: \mathcal{C} \rightarrow \mathcal{D}$ covariant is called left exact if for any exact seq

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N$$

the sequence

$$0 \rightarrow F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \text{ is exact.}$$

F is called right exact if for any exact seq.

$$L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0 \text{ the seq}$$

$$F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N) \rightarrow 0 \text{ is exact.}$$

similarly for contra-variant functors (see Sc[14]).

One of the important properties of the Nakayama functor is the following:

Prop 9. The Nakayama functor \mathcal{D} is right exact.

Prf: Clear since it is a composition of the left exact functor $\text{Hom}(-, A)$ and the exact (contra-variant) functor \mathcal{D} .

EX 10. From EX 8 we get a s.e.s

$$0 \rightarrow 3 \xrightarrow{f} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \xrightarrow{g} \begin{matrix} 1 \\ 2 \end{matrix} \rightarrow 0$$

Applying ν yields the exact seq

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \xrightarrow{\nu f} 1 \xrightarrow{\nu g} 0 \rightarrow 0$$

Note that ν is not exact, since νf is not injective.

Another exact sequence that we get from a minimal projective resolution

$$0 \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{p_0} M \rightarrow 0$$

of an indecom. $M \in \text{rep } Q$ is

$$0 \rightarrow \tau M \rightarrow \nu P_1 \xrightarrow{\nu f_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \rightarrow 0, \text{ (we applied } \nu)$$

where $\tau M = \ker \nu p_1$ is called the A-R translate of M .

Playing a similar game with a minimal injective resolution, we get the inverse A-R translate.

III Extensions

Let $M \in \text{rep } Q$. Take a projective resolution

$$0 \rightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \rightarrow 0 \text{ of } M$$

and let $N \in \text{rep } Q$ be arbitrary. By applying $\text{Hom}(-, N)$ we get the exact seq.:

$$0 \rightarrow \text{Hom}(M, N) \xrightarrow{g^*} \text{Hom}(P_0, N) \xrightarrow{f^*} \text{Hom}(P_1, N) \rightarrow \text{Ext}^1(M, N) \rightarrow 0,$$

where $\text{Ext}^1(M, N) = \text{coker } f^*$ is called the first group of extensions of M and N .

Qmk 11 In arbitrary categories projective resolutions might have the following form:

$$\dots \rightarrow P_n \xrightarrow{f_n} P_{n-1} \rightarrow \dots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$$

i.e., they don't stop on the left. Then by applying $\text{Hom}(-, N)$ we get a cochain complex:

$$0 \rightarrow \text{Hom}(M, N) \xrightarrow{f_0^*} \text{Hom}(P_0, N) \xrightarrow{f_1^*} \dots \xrightarrow{f_n^*} \text{Hom}(P_n, N) \rightarrow \dots$$

where $f_i^* f_{i-1}^* = 0, \forall i$. Then we can define the i^{th} extension group $\text{Ext}^i(M, N) = \text{Ker } f_{i+1}^* / \text{Im } f_i^*$.

Since in rep \mathcal{Q} , minimal projective resolutions end in two steps, we have that Ext^i -groups vanish for $i > 2$. However, finding Ext-gps is not easy. We next look at $\text{Ext}^1(M, N)$.

Def 12 An extension ζ of M by N is a s.e.s. $0 \rightarrow N \xrightarrow{f} E \xrightarrow{g} M \rightarrow 0$. Two extensions ζ and ζ' are called equivalent if there is a comm. diagram:

$$\begin{array}{ccccccc} \zeta: & 0 & \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & & \downarrow = & & \downarrow \cong & & \downarrow = & & \\ \zeta': & 0 & \longrightarrow & N & \xrightarrow{f'} & E' & \xrightarrow{g'} & M & \longrightarrow & 0 \end{array}$$

An extension is split if the s.e.s is split.

Given ζ, ζ' extensions of M by N we define:

$$E'' = \{ (x, x') \in E \times E' \mid g(x) = g'(x') \} \text{ and}$$

$$F = E'' / \{ (s(n), -s'(n)) \in E \oplus E' \mid n \in N \}. \text{ Then we define}$$

$\zeta + \zeta'$ as

$$0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$$

The set of eq. classes $\tilde{\mathcal{E}}(M, N)$ of extensions of M by N together with the sum of extensions is an abelian gp. and the class of split extensions is the zero el. of that gp.

We define an iso. $\tilde{\mathcal{E}}(M, N) \rightarrow \text{Ext}^1(M, N)$ in the following way:

Pick a representative of a class in $\tilde{\mathcal{E}}(M, N)$

$$\zeta: 0 \rightarrow N \xrightarrow{u'} E \xrightarrow{v'} M \rightarrow 0 \text{ and a proj. resolution}$$

$$0 \rightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \rightarrow 0$$

Then there exist maps u, s', s'' , the following commutes

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ \downarrow u & & \downarrow s' & & \downarrow = & & \\ \end{array}$$

$$\xi: 0 \longrightarrow N \xrightarrow{u'} E \xrightarrow{v'} M \longrightarrow 0 \quad \left(\begin{array}{l} \text{see [Sc14] for} \\ \text{more details} \end{array} \right)$$

The iso $\xi(M, N) \rightarrow \text{Ext}^1(M, N)$ is given by

$$\xi \longmapsto u$$

So, $\text{Ext}^1(M, N)$, as a vector space, is isomorphic to the vector space of extensions of M by N .

The reader is welcome to work on examples :).