From Frobenius Algebras to TQFTs

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The goal of this talk is to prove and illustrate the main result of [Kock, 2003]:

Theorem *There is an equivalence of categories:* $\mathbf{2TQFT}_k \cong \mathbf{cFA}_k$

The illustrations are shamelessly stolen from [Kock, 2003].

1 Review of TQFTs and Frobenius Algebras

1.1 The categories $\mathbf{FA}_k$ and $\mathbf{cFA}_k$

The objects of the category $\mathbf{FA}_k$ are exactly the Frobenius algebras, i.e. k-algebras A in the usual sense equipped with a linear form, the Frobenius form $\varepsilon : A \to k$, satisfying a certain number of axioms. They have the following properties:

Proposition 1.1 *Let A be a vector space equipped with a multiplication map $\mu : A \otimes A \to A$, a unit $\eta : k \to A$, a comultiplication $\delta : A \to A \otimes A$ and a counit $\varepsilon : A \to k$, such that the Frobenius relation holds. Then μ is associative, δ co-associative, A finite dimensional and ε a Frobenius form on the k-algebra A.*

Conversely, if (A, ε) is a Frobenius algebra, we can construct all the preceding maps in a unique way such that the Frobenius relation is satisfied.

Hence FAs are exactly the algebras that admit a compatible co-algebra structure.

An FA homomorphism is an algebra homomorphism between FAs which is also a coalgebra homomorphism. We call the category of all Frobenius algebras together with Frobenius algebra homomorphisms by $\mathbf{FA}_k$.

A FA is **commutative**, if the multiplication is. The full subcategory of commutative Frobenius Algebras will be denoted by $\mathbf{cFA}_k$. The categories $\mathbf{FA}_k$ and $\mathbf{cFA}_k$ are "rigid" in the sense that:

Proposition 1.2 *The categories $\mathbf{FA}_k$ and $\mathbf{cFA}_k$ are groupoid categories.*

1.2 The category $\mathbf{2TQFT}_k$

By definition, a **linear representation** of a symmetric monoidal category $(\mathbf{V}, \square, I, \tau)$ is a symmetric monoidal functor

$$\mathscr{A} : (\mathbf{V}, \square, I, \tau) \to (\mathbf{Vect}_k, \otimes, k, \sigma).$$

We will denote the category of all such representations together with monoidal natural transformations for morphisms as $\mathbf{Repr}_k(\mathbf{V})$.

We can now succinctly define the category of n-dimensional TQFTs:

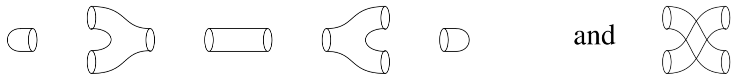
$$\mathbf{nTQFT}_k := \mathbf{Repr}_k(\mathbf{nCob})$$

In our particular case, we get that

$$\boxed{\mathbf{2TQFT}_k = \mathbf{Repr}_k(\mathbf{2Cob}, \sqcup, \emptyset, T)}$$

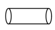
2 Main Result

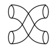
By the previous talks, we have complete control over **2Cob** in terms of generators and relations. The objects of **2Cob** are exactly given by $\mathbb{N} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, ...\}$ and the morphisms are generated by the following arrows:



Hence to define a functor $\mathscr{A} : \mathbf{2Cob} \to \mathbf{Vect}_k$, we only need to assign vector spaces to the generating objects $\mathbb{N}$, and a linear map to every generating arrow. Now since we are talking about symmetric monoidal functors, a lot is prescribed, once we fix a vector space A as the image of $\mathbf{1}$.

$$\begin{aligned} \textit{\textbf{2Cob}} &\longrightarrow \textit{\textbf{Vect}}_{\Bbbk} \\ \mathbf{1} &\longmapsto A \\ \mathbf{n} &\longmapsto A^n \end{aligned}$$

 $\longmapsto [\mathrm{id}_A : A \to A]$

 $\longmapsto [\sigma : A^2 \to A^2]$.

By monoidicity, we have $\mathscr{A}(\mathbf{n}) = A \otimes \cdots \otimes A =: A^n$ where the dots signify an n-fold tensor product. By symmetry, we have $\mathscr{A}(T) = \sigma$.

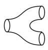
What we still have to choose in a manner satisfying the relations are the following:

$$\textit{\textbf{2Cob}} \longrightarrow \textit{\textbf{Vect}}_{\Bbbk}$$

 $\longmapsto [\eta : \Bbbk \to A]$

 $\longmapsto [\mu : A^2 \to A]$

 $\longmapsto [\varepsilon : A \to \Bbbk]$

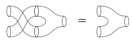
 $\longmapsto [\delta : A \to A^2]$.

Here is now where our main result comes in, which tells us how $\eta, \mu, \varepsilon, \delta$ can be chosen.

Theorem *There is an equivalence of categories* $\mathbf{2TQFT}_k \cong \mathbf{cFA}_k$

Proof Need to check that:

- We can assign a TQFT to each Frobenius Algebra and vice-versa.
- We can assign to each monoidal natural transformation between TQFTs a Frobenius Algebra homomorphism and vice-versa.

Considering the first point, if we have TQFT $\mathscr{A} : \mathbf{2Cob} \to \mathbf{2TQFT}_k$, then consider the vector space $A : \mathscr{A}(\mathbf{1})$. The relations **2Cob** imposes on cobordisms translates through $\mathscr{A}$ on relations bewteen the maps $\eta, \mu, \varepsilon, \delta$. Now, these turn out to be translate exactly the axioms of commutative Frobenius algebras for (A, ε) together with the multiplication μ, comultiplication δ and so forth! For instance the Frobenius relationship between cobordisms translates into the algebraic Frobenius relationship, and the fact that multiplication μ on A is commutative can be traced back to  in **2Cob**. From the graphical calculus developed last week it is easy to accept that also the rest of the commutativ Frobenius algebra axioms are satisfied.

On the other hand, given a Frobenius algebra (A, ε), we get unique maps η, μ, δ by proposition 1.1. Define a TQFT $\mathscr{A} : \mathbf{2Cob} \to \mathbf{2TQFT}_k$ by sending $\mathbf{n} \mapsto A^n$ and the generating cobordisms to the maps they correspond to in the above table. One needs to check that this is well defined, i.e. does not lead to contradictions, however this is not the case, since again, the relations in **2Cob** correspond exactly to the axioms of a $\mathbf{2TQFT}_k$.

Furthermore, the two correspondances are inverse to each other, since when you start with a Frobenius Algebra (A, ε), construct the corresponding TQFT, an look at the induced Frobenius algebra $(\mathscr{A}(1), \mathscr{A}(\,))$, we get (A, ε) back.

What happens to morphisms? Suppose we have two TQFTs $\mathscr{A}$ and $\mathscr{B}$ with a monoidal natural transformation $\Theta : \mathscr{A} \to \mathscr{B}$ between them. In other words, we are given a collection of maps $\theta_n : A^n \to B^n$. In fact, by monoidality, we have $\theta_n = (\theta_1)^{\otimes n}$, so we only have one map, $\theta = \theta_1 : A \to B$ to choose. Naturality reduces to naturality for generating arrows, and since it is automatic for the identity and twist arrow, four commutative diagrams for the unit, multiplication, counit and comultipication remain. The diagrams for unit and multiplication are given by:

$$\begin{tikzcd} A^2 \arrow[r] \arrow[d, "\mu_A"'] & B^2 \arrow[d, "\mu_B"] \\ A \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] & B \\ \Bbbk \arrow[u, "\eta_A"] \arrow[r, equal] & \Bbbk \arrow[u, "\eta_B"'] \end{tikzcd}$$

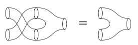
The first one means that it θ respects multiplications and the second one that the unit gets mapped to the unit, hence that θ is an algebra homomorphism. The two other diagrams, for counit and comultiplication mean that θ is also a co-algebra homomorphism. So θ is in fact a Frobenius algebra homomorphis between A and B. Doing these steps in reverse, we can also produce a monoidal natural transformation between $\mathscr{A}$ and $\mathscr{B}$, given a Frobenius homomorphism. Hence there is also a correspondance between morphisms, which concludes the proof. $\square$

3 Symmetry and Commutativity

Why do we restrict ourselves to symmetric monoidal functors and commutative Frobenius algebras?

Consider the following example. Let M be a compact manifold, and let $H = H^*(M)$ be the cohomology ring of M. Then H is an example of a **graded-commutative** Frobenius algebra. It is in general not commutative, but we can still construct a non-symmetric monoidal functor $\mathscr{H} : \mathbf{2Cob} \to \mathbf{Vect}_k$ from it. We do so by setting $\mathbf{1} \mapsto H$, the unit, counit, multiplication and comultiplication get sent to their respective counterparts in H, but the twist is sent to the so called **Koszul twist** :

$$\kappa : a \otimes b \mapsto (-1)^{pq} b \otimes a$$

where p and q are the degrees of a and b respectively. This functor is well defined, since the relation  translates into graded commutativity by the choice of the twist, for it implies $\kappa \circ \mu = \mu$. A note here, in the previous section this relation translates into commutativity. This is because by symmetric functors the twist gets sent to σ, which then implies $\sigma \circ \mu = \mu$, commutativity. This functor $\mathscr{H}$, although not commutative, is still interesting to look at. But it unfortunately doesn't fit our definition of TQFT. What should we do?

One possibility would be to drop the requirement that a TQFT should be symmetric. However, the class of FA's corresponding to such not necessarily symmetric TQFT's might be difficult to describe. It would include graded-symmetric examples such as H, but maybe there are other even more general Frobenius Algebras that it still couldn't encompass? Furthermore, there are symmetries exists on both sides of the TQFT-FA correspondence, so we should strive to respect them!

The more elegant solution in our case, is to consider different target categories, with different symmetries! The case we considered so far was of functors into $(\mathbf{Vect}_k, \sigma)$, but if we consider $\mathscr{H}$ as a functor into $(\mathbf{grVect}_k, \kappa)$, then it is a symmetric functor. In this sense the choice of the target category is crucial for symmetry and commutativity.

4 Examples

By the above euivalence of categories we can now construct a number of TQFTs from their corresponding Frobenius algebras. Generally, one can distinguish two major classes of TQFTs, the **nilpotent** and the **semi-simple**. Let's do one example each.

Example (Nilpotent) *Consider the Frobenius algebra:*

$$A = k[t]/t^n \qquad \varepsilon(t^i) = \begin{cases} 1, \textit{ if } i = n-1 \\ 0, \textit{ else} \end{cases}$$

The TQFT corresponding to this Frobenius algebra assigns to the torus the invariant n, and all other surfaces have invariant 0*. Let's calculate step by step in the basis* $\{t^i : 0 \leq i < n\}$.

Example (Semi-simple) *Consider the Frobenius algebra, which corresponds to the groups algebra of the cyclic group of order n.*

$$A = k[t]/(t^n - 1) \qquad \varepsilon(t^i) = \begin{cases} 1, \textit{ if } i = 0 \\ 0, \textit{ else} \end{cases}$$

Then, assuming that the characteristic of k does not divide n, the TQFT that corresponds to this Frobenius algebra assigns to a closed surface of genus g the invariant n^g.

Two further nice applications:

Example (Sphere-counting) *Consider the Frobenius algebra:*

$$A = k \qquad \varepsilon(\lambda) = 2\lambda$$

The TQFT corresponding to this Frobenius algebra assigns to a collection of k spheres the invariant 2^k*, and to every other surface* 0.

Example (Sphere counting) *Consider the Frobenius algebra, which corresponds to the groups algebra of the cyclic group of order p and a field of characteristic p, say* $\mathbb{Z}/p\mathbb{Z}$*, where p is a prime. Let furthermore q be a primitive root* $\mod p$.

$$A = k[t]/(t^p - 1) \qquad \varepsilon(t^i) = \begin{cases} q, \text{ if } i = 0 \\ 0, \text{ else} \end{cases}$$

Then, this TQFT can distinguish the number of spheres $\bmod p$*, and gives zero if a surface of higher genus appears.*

The calculations can be found in the appendix.

5 Further results

Theorem *There is a monoidal category* $\Delta \subset \mathbf{2Cob}$*, such that a monoidal functor from* Δ *to* $\mathbf{Vect}_k$ *is the same as an algebra:*

$$\mathbf{MonCat}(\Delta, \mathbf{Vect}_k) \cong \mathbf{Alg}_k$$

Notice that **2Cob** has more relations than Δ by the fact that it is a bigger category. Hence, the functors from it are more restricted, leading to the more general notion of an algebra instead of the restrictive Frobenius algebra. This however can be extended even further to

Theorem *There is a monoidal category* $\Delta \subset \mathbf{2Cob}$*, such that a monoidal functor from* Δ *to a monoidal category* $\mathbf{V}$ *corresponds to a monoid in* $\mathbf{V}$*. In other words,* Δ *is the free monoidal category on a monoid.*

References

[Kock, 2003] Kock, J. (2003). *Frobenius Algebras and 2D topological Quantum Field Theories*.

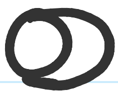
Nilpotent

05 December 2018 18:06

$$[P] := \begin{cases} 1, & \text{if } P \text{ holds} \\ 0, & \text{else} \end{cases}$$

$$\boxed{A = k[t]/t^n}$$

Basis: $\{t^i : 0 \leq i < n\}$.

- 

$$\varepsilon : A \longrightarrow k$$
$$\varepsilon(t^i) = [i = n-1]$$

- 

$$\beta : A^2 \longrightarrow k$$
$$\beta(t^i, t^j) = \varepsilon(t^{i+j})$$
$$= [i+j+1 \equiv 0 \bmod n]$$
$$= \sum_{i=0}^{n-1} dt^i \otimes dt^{n-i-1}$$

Where

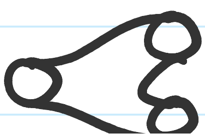
$$dt^j : A \longrightarrow k$$
$$t^i \longmapsto \delta_j^i$$

- 

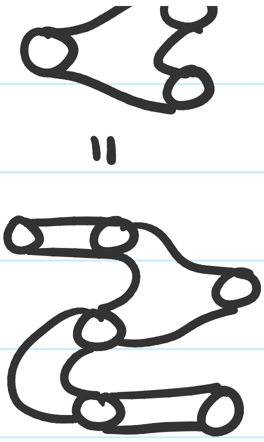
$$\gamma : k \longrightarrow A^2$$

Matrix of γ inverse to matrix of β!

$$\Rightarrow \gamma(\lambda) = \lambda \sum_{i=0}^{n-1} t^i \otimes t^{n-i-1}$$

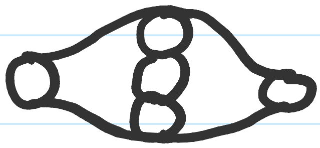
- 

$$\delta : A \to A^2$$



- $\delta: A \to A^2$

$$t^k \mapsto \sum_{i=0}^{n-1-k} t^{k+i} \otimes t^{n-i-1}$$



- $H: A \longrightarrow A$

$$t^k \mapsto \mu(\delta(t^k))$$

$$\mu(\delta(t^k)) = \mu\Big(\sum_{i=0}^{n-1-k} t^{k+i} \otimes t^{n-i-1}\Big)$$

$$= \sum_{i=0}^{n-1-k} t^{k+n-1}$$

$$= \begin{cases} n t^{n-1}, & k = 0 \\ 0, & \text{else} \end{cases}$$

- $\eta: k \longrightarrow A$

$$\eta(\lambda) = \sum_{i=0}^{n-1} t^i \otimes \varepsilon(t^{n-i-1})$$

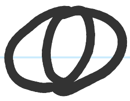
$$= \underline{1}$$

Consider a closed 2-mfld M.
Then the TQFT A_g defined by A

Then the TQFT $\mathcal{A}_g$ defined by A associates $\mathcal{A}_g(M): k \to k$ to M.

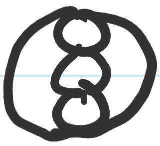
In other words it assigns to M an invariant : $\boxed{\mathrm{inv}(M) := \mathcal{A}_g(M)(1)}$

Invariants of closed surfaces:

$\boxed{S^2}$ = 

$$k \xrightarrow{\eta} A \xrightarrow{\varepsilon} k$$
$$\lambda \mapsto \lambda \mapsto 0$$

so $\boxed{\mathrm{inv}(S^2) = 0}$

$\boxed{\mathbb{T}^2}$ 

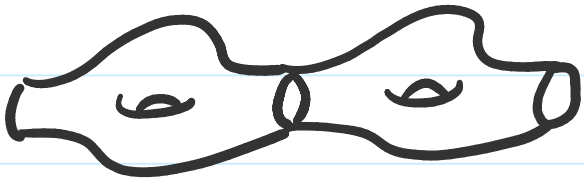
$$k \to A \otimes A \to k$$
$$1 \mapsto \sum_{i=0}^{n-1} t^i \otimes t^{n-i-1}$$
$$\downarrow$$
$$\sum_{i=0}^{n-1} \varepsilon(t^{n-1}) = n \cdot 1$$
$$= n$$

Surface of genus $g > 1$: M_g

Notice that the handle operator is nilpotent:

is nilpotent:



$$A \xrightarrow{H} A \xrightarrow{H} A$$

$$t^i \longmapsto n t^{n-1}[i=0] \longmapsto 0$$

But:

$$\boxed{\mathrm{inv}(M_g)} = \eta H^g \varepsilon(1) = \eta H^2 H^{g-2} \varepsilon(1) = \eta\, 0\, H^{g-2} \varepsilon(1) = \boxed{0}$$

Nonconnected surfaces

Consider $M = \bigsqcup_i M_i$

then: $A_G(M) = \bigotimes_i A_G(M_i)$

so $$\boxed{\mathrm{inv}(M) = \prod_i \mathrm{inv}(M_i) = \begin{cases} n^t, & \text{if } M \text{ consists of } t \text{ Tori.} \\ 0, & \text{else} \end{cases}}$$

Semi-Simple

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$$\boxed{A = k[t]/(t^n - 1)} \qquad \text{Basis: } \{t^i : 0 \leq i < n\}$$

- 

$$\varepsilon : A \longrightarrow k$$
$$\varepsilon(t^i) = [i = 0]$$

- 

$$\beta : A^2 \longrightarrow k$$
$$\beta(t^i, t^j) = \varepsilon(t^{i+j})$$
$$= [i+j \equiv 0 \bmod n]$$

$$\text{so } \beta = \sum_{i+j \equiv 0 \bmod n} dt^i \otimes dt^j$$

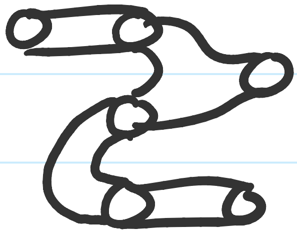
- 

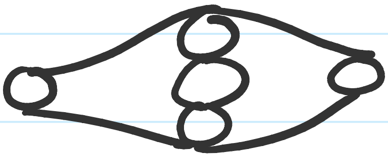
$$\gamma : k \longrightarrow A^2$$

$$\gamma(1) = \sum_{i+j = 0 \bmod n} t^i \otimes t^j$$
$$= \sum_{i=0}^{n-1} t^i \otimes t^{n-i}$$

- 

$$\delta : A \longrightarrow A^2$$
$$t^k \longmapsto \sum_{i=0}^{n-1} t^{i+k} \otimes t^{n-i}$$

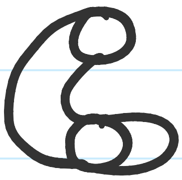


-  $H : A \to A$

$$t^k \mapsto \mu\left(\sum_{i=0}^{n-1} t^{i+k} \otimes t^{n-i}\right)$$

$$= \sum_{i=0}^{n-1} t^{k+n} = n \cdot t^k$$

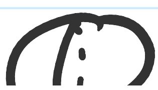
so $a \mapsto na$ given that the characteristic of k does not divide n.

-  $\eta : k \to A$

$$\eta(1) = \sum_{i=0}^{n-1} t^i \otimes \varepsilon(t^{n-i})$$

$$= 1$$

Invariant of connected surfaces

$\boxed{S.}$ 

$\boxed{S_2}$ 

$$\mathrm{inv}(S_2) = \eta\,\varepsilon\,(1) = 1$$

$\boxed{\text{Surface of genus } g:\ M_g}$

$\eta \quad H \quad H \quad \cdots \quad H \quad \varepsilon$

$$\boxed{\begin{aligned}\mathrm{inv}(M_g) &= \eta H^g \varepsilon\,(1)\\ &= n^g\end{aligned}}$$

Disconnected surfaces

$$M = \bigsqcup_i M_{g_i}$$

$$\begin{aligned}\mathrm{inv}(M) &= \prod_i \mathrm{inv}(M_{g_i})\\ &= n^{\sum_i g_i}\end{aligned}$$

$$= x^{\sum_i y_i}$$

Sphere Counting

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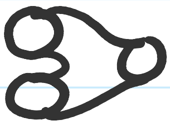
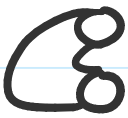
$A : k \to k$

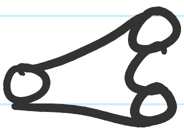
 : $\varepsilon(\lambda) = 2\lambda$

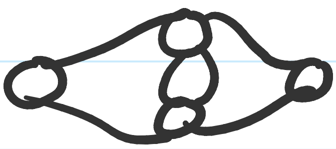
 : $\mu(a,b) = a \cdot b$

 : $\beta(a,b) = 2ab$

 : $\gamma(1) = \frac{1}{2}(a \otimes 1)$

 : $\eta(1) = 1$

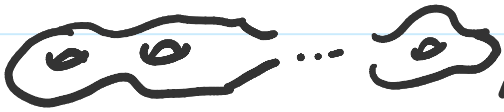
 : $\delta(a) = \frac{1}{2}(a \otimes 1)$

 : $H(a) = \frac{1}{2}a$

$\mathrm{inv}($  $) = 2$

$\mathrm{inv}\Big(\bigsqcup_{k=1}^{n}$  $\Big) = \prod_{k=1}^{n} \mathrm{inv}($  $) = 2^n$

$\mathrm{inv}($  $) = 1$

$\mathrm{inv}($  $) = 2^{1-g}$