$\mathfrak{sl}_n$-link homologies using $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-highest weight theory

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The d is not a typo!

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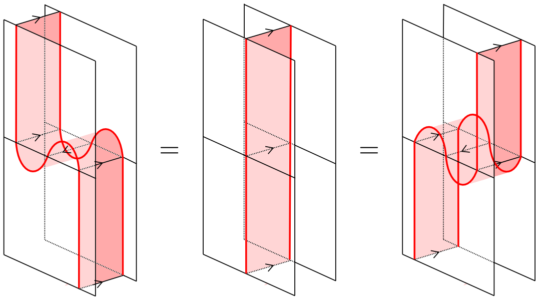


1. What is categorification?
   - From the viewpoint of “natural” constructions
   - From the viewpoint of topology
   - From the viewpoint of algebra

2. The uncategorified story
   - $\mathfrak{sl}_2$-webs
   - Connection to $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$
   - Connection to the $\mathfrak{sl}_2$-link-polynomials

3. Its categorification!
   - $\mathfrak{sl}_2$-foams
   - “Higher” q-skew Howe duality
   - Connection to the $\mathfrak{sl}_n$-link homologies

What is categorification?

Forced to reduce this presentation to one sentence, the author would choose:

Interesting integers are shadows of richer structures in categories.

The basic idea can be seen as follows. Take a "set-based" structure S and try to find a "category-based" structure $\mathcal{C}$ such that S is just a shadow of $\mathcal{C}$.

Categorification, which can be seen as "remembering" or "inventing" information, comes with an "inverse" process called decategorification, which is more like "forgetting" or "identifying".

Note that decategorification should be easy.

The underlying basic example

Take $\mathcal{C} = K$-**FinVec** for a fixed field K, i.e. objects are finite dimensional K-vector spaces $V, V', \dots$ and morphisms are K-linear maps $f\colon V \to V'$ between them. $\mathcal{C}$ categorifies $\mathbb{N}$: We can go back by taking the dimension $\dim V \in \mathbb{N}$.

What is the upshot? Note the following:

- Much information is lost if we only consider $\mathbb{N}$, i.e.

$$n = n' \Leftrightarrow V \cong V'.$$

- We have the power of linear algebra between V and V', i.e. $\hom_K(V, V')$.
- A vector space can carry additional structure.

Never forget the original structure

The structure of $\mathbb{N}$ is reflected on a "higher" level!

- The direct sum $\oplus$ and the tensor product $\otimes_K$ categorify $+$ and $\cdot$, i.e.

$$\dim(V \oplus V') = \dim V + \dim V' \text{ and } \dim(V \otimes_K V') = \dim V \cdot \dim V'.$$

- The zero vector space 0 and the field K categorify the identities, i.e.

$$V \oplus 0 \cong V \cong 0 \oplus V \text{ and } V \otimes_K K \cong V \cong K \otimes_K V.$$

- The injections and surjections categorify the order relation, i.e.

$$\exists f \colon V \hookrightarrow V' \Leftrightarrow \dim V \leq \dim V' \text{ and } \exists f \colon V \twoheadrightarrow V' \Leftrightarrow \dim V \geq \dim V'.$$

One can write down the categorified statements of other properties as "Addition and multiplication are associative and commutative" etc.

Integer based invariants

A more topological flavoured example goes back to Riemann (1857), Betti (1871) and Poincaré (1895): The Betti numbers $b_k(X)$ and Euler characteristic $\chi(X)$ of a reasonable topological space X. Noether, Hopf and Alexandroff (1925) "categorified" these invariants as follows.

If we lift $n, n' \in \mathbb{N}$ to the two K-vector spaces V, V' with dimensions $\dim V = n, \dim V' = n'$, then the difference $n - n'$ lifts to the complex

$$0 \longrightarrow V \xrightarrow{d} V' \longrightarrow 0,$$

for any linear map d and V in even homology degree. As before, some of the basic properties of the integers $\mathbb{Z}$ can be lifted to the category $\mathbf{Kom}_b(\mathcal{C})$.

Conclusion (Noether): The homology groups $H_k(X, \bar{\mathbb{Q}})$ categorify $b_k(X)$ and chain complexes $(C(X), c_*)$ categorify $\chi(X)$.

Well-known upshots

We note the following observations.

- The homology extends to a functor and provides information about continuous maps as well.
- Again, homomorphisms between the $\bar{\mathbb{Q}}$-vector spaces tell how some $\bar{\mathbb{Q}}$-vector spaces are related.
- The space $H_i(X, \bar{\mathbb{Q}})$ is a $\bar{\mathbb{Q}}$-vector space: More information of X is encoded.
- Singular homology works for all topological spaces and the homological Euler characteristic can be defined for a huge class of spaces.
- More sophisticated constructions like multiplication in cohomology provide even more information.
- Although it is not the main point: The $H_i(X, \bar{\mathbb{Q}})$ are better invariants.

Categorified symmetries

Another viewpoint comes from representation theory. Let A be some algebra, M be a A-module and $\mathcal{C}$ be a suitable category.

$$\text{“Usual”} \rightsquigarrow \text{“Higher”}$$

$$a \mapsto f_a \in \mathrm{End}(M) \rightsquigarrow a \mapsto \mathcal{F}_a \in \mathrm{End}(\mathcal{C})$$

$$(f_{a_1} \cdot f_{a_2})(m) = f_{a_1 a_2}(m) \rightsquigarrow (\mathcal{F}_{a_1} \circ \mathcal{F}_{a_2})\binom{X}{\varphi} \cong \mathcal{F}_{a_1 a_2}\binom{X}{\varphi}$$

A (weak) categorification of the A-module M should be though of a categorical action of A on a suitable category $\mathcal{C}$ with an isomorphism ψ such that

$$\begin{array}{ccc} K_0(\mathcal{C}) \otimes A & \xrightarrow{[\mathcal{F}_a]} & K_0(\mathcal{C}) \otimes A \\ \psi \downarrow & \circlearrowleft & \downarrow \psi \\ M & \xrightarrow{\cdot a} & M. \end{array}$$

There is no direct minus

We have several upshots again.

- The natural transformations between functors give information invisible in "classical" representation theory. This gives a hint that we can go even "higher", e.g. actions of 2-categories on 2-categories.
- If $\mathcal{C}$ is suitable, e.g. module categories over an algebra, then its indecomposable objects X gives a basis $[X]$ of M with positivity properties.
- In particular, consider A as a A-module. Then $[X]$ gives a basis of A with positive structure coefficients c_k^{ij} via

$$X_{a_i} \otimes X_{a_j} \cong \bigoplus_k X_{a_k}^{\oplus c_k^{ij}} \rightsquigarrow a_i a_j = \sum_k c_k^{ij} a_k,\ c_k^{ij} \in \mathbb{N}.$$

An old story: Rumer, Teller and Weyl (1932)

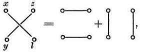
2. Fundamentalsatz: Alle linearen Abhängigkeiten zwischen den Monomen ergeben sich (in einem algebraisch genauer präzisierten Sinne) aus der einen Identität (2).

Wir werden uns hier auf den ersten, nicht aber auf den zweiten Fundamentalsatz stützen; vielmehr wird durch unsere Überlegungen ein neuer Beweis des 2. Fundamentalsatzes erbracht.

In der Quantenmechanik bedeuten die Zeichen $x, y, \ldots, z$ Atome, die sich zu einem Molekül zusammensetzen, $a, b, \ldots, c$ deren Valenzen. Jede Invariante der geforderten Ordnung stellt einen Spinzustand des Moleküls dar. Die durch die Monome repräsentierten „reinen Valenzzustände" veranschaulicht sich der Chemiker durch ein Valenzschema, in dem die Atome als Punkte erscheinen und jeder Klammerfaktor $[xy]$ durch einen die beiden Atome x und y verbindenden gerichteten Strich zum Ausdruck gebracht wird. $a, b, \ldots, c$ sind dann die Anzahlen der Valenzstriche, die von den einzelnen Atomen $x, y, \ldots, z$ im Valenzschema des Monoms ausgehen. Man zeichne die Punkte $x, y, \ldots, z$ auf einem Kreise auf. Die zu beweisende Regel lautet dann:

Jede Invariante J ist eine lineare Kombination solcher Monome, deren Valenzschema keine sich kreuzenden Valenzstriche enthält. Die Monome mit kreuzungslosem Valenzschema sind aber linear unabhängig von einander.

Beim Beweise des ersten Teils kann man nach dem 1. Fundamentalsatz annehmen, daß die Invariante J ein Monom ist, welches wir durch sein Valenzschema S abbilden. Es bestehe aus N Strichen zwischen den n Punkten $x, y, \ldots, z$. Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

(3) 

Kreuzungen auflösen kann [1]). Natürlich ist mit dieser Bemerkung nicht alles getan; denn wenn man in einem komplizierten Schema die Kreuzung zweier Valenzstriche auflöst, werden dadurch im allgemeinen andere Kreuzungen teils mit aufgelöst, teils neu entstehen. Dennoch kommt man durch ein geeignetes rekursives Arrangement zum Ziel, wie folgt.

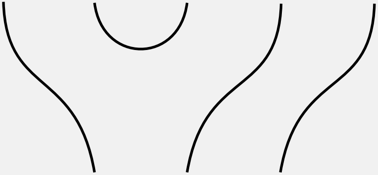
1) In der Figur wurde der Richtungssinn der Valenzstriche weggelassen.

The $\mathfrak{sl}_2$-webs

Definition(Rumar, Teller, Weyl 1932)

Fix two numbers $b, t \in \mathbb{N}$ with $b + t = 2\ell$. A $\mathfrak{sl}_2$-web w with b bottom points and t top points is an embedding (non-intersecting!) of a finite number of lines and circles in a rectangle with b fixed points at the bottom and t at the top such that the two boundary points of the lines are some of the fixed points. The set of all $\mathfrak{sl}_2$-webs w between b bottom points and t top points in denoted by $\tilde{W}_2(b, t)$.

Example($b = 3$ and $t = 5$)

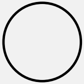


The $\mathfrak{sl}_2$-web space

Definition

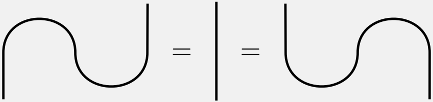
Fix two numbers $b, t \in \mathbb{N}$ with $b + t = 2\ell$. The $\mathfrak{sl}_2$-web space $W_2(b,t)$ is the free $\bar{\mathbb{Q}}(q)$-vector space generated by elements of $\tilde{W}_2(b,t)$ modulo

- The circle removal



$$= \quad [2] \quad = \quad q + q^{-1}$$

- The isotopy relations



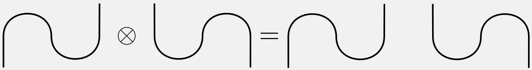
Note that $W_2(b,t)$ is a finite dimensional $\bar{\mathbb{Q}}(q)$-vector space!

The $\mathfrak{sl}_2$-web category

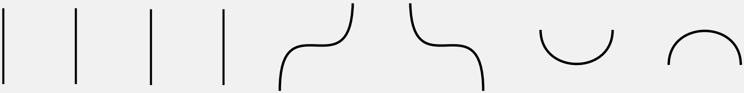
Definition(Kuperberg 1997)

The $\mathfrak{sl}_2$-web category or web spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of

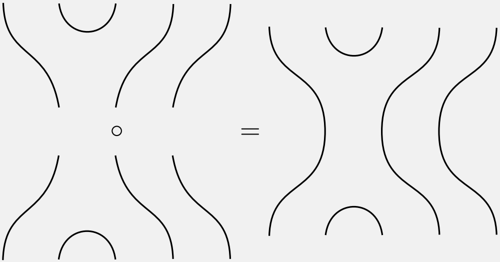
- The objects are the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The 1-cells $w\colon b \to t$ are the elements of $W_2(b, t)$.
- The $\bar{\mathbb{Q}}(q)$-linear composition is stacking.
- The monoidal structure $\otimes$ is given by juxtaposition, i.e. $b \otimes b' = b + b'$ and



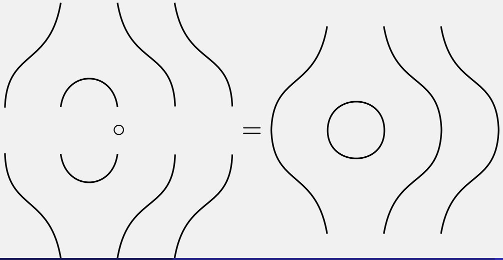
- As generators suffices the identities, shifts, cups and caps



Example



and

 $= [2] \cdot \mathbf{id}(3,3)$

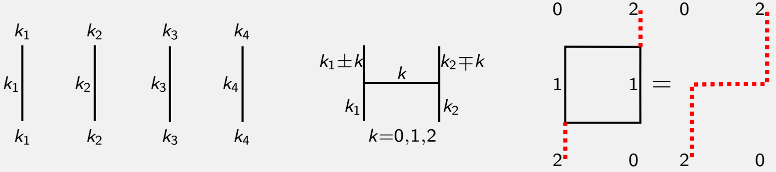
Rigidity of $\mathfrak{sl}_2$-webs

A seemingly very small point turned out to be a crucial step if we want to consider bigger n: Topology is continuous and Algebra is rigid.

Definition, second try - rigid version

The $\mathfrak{sl}_2$-web category or web spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of

- The objects are ordered partitions $\vec{k}$ of $2\ell \in \mathbb{N}$ with only $0, 1, 2$ as entries.
- The 1-cells $w\colon \vec{k} \to \vec{k}'$ are labeled ladders (we use the convention and do not draw edges labeled 0 and use a dotted line for those labeled 2) generated by juxtaposition and vertical composition of (plus relations and rest as before)



What is the upshot? "Easy" to generalize to $\mathfrak{sl}_n$: Take labels $0, 1, \ldots, n-1, n$ and "directly" connected to the algebra (which I explain in a second!).

The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_d)$

Definition

For $d \in \mathbb{N}_{>1}$ the quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_d)$ is the associative, unital $\bar{\mathbb{Q}}(q)$-algebra generated by $K_i^{\pm 1}$ and E_i and F_i, for $i = 1, \ldots, d-1$, subject to some relations (that we do not need today).

Definition(Beilinson-Lusztig-MacPherson)

For each $\vec{k} \in \mathbb{Z}^{d-1}$ adjoin an idempotent $1_{\vec{k}}$ (think: projection to the $\vec{k}$-weight space!) to $\mathbf{U}_q(\mathfrak{sl}_d)$ and add some relations, e.g.

$$1_{\vec{k}} 1_{\vec{k}'} = \delta_{\vec{k},\vec{k}'} 1_{\vec{k}} \text{ and } K_{\pm i} 1_{\vec{k}} = q^{\pm \vec{k}_i} 1_{\vec{k}} \text{ (no } K's \text{ anymore!)}.$$

The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_d) = \bigoplus_{\vec{k},\vec{k}' \in \mathbb{Z}^{d-1}} 1_{\vec{k}} \mathbf{U}_q(\mathfrak{sl}_d) 1_{\vec{k}'}.$$

The category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$

Definition

The representation category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of

- The objects are finite tensor products of the $\mathbf{U}_q(\mathfrak{sl}_2)$-representations $\Lambda^k\bar{\mathbb{Q}}^2$. Denote them by $\vec{k} = (k_1, \dots, k_m)$ with $k_i \in \{0, 1, 2\}$.
- The 1-cells $w\colon \vec{k} \to \vec{k}'$ are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Composition of 1-cells is composition of intertwiners and $\otimes$ is the ordered tensor product.

It is worth noting that $\Lambda^0\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}$ is the trivial $\mathbf{U}_q(\mathfrak{sl}_2)$-representation, $\Lambda^2\bar{\mathbb{Q}}^2 \cong \bar{\mathbb{Q}}$ its dual and $\Lambda^1\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}^2$ is the (self-dual) $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation.

Theorem(Kuperberg 1997, $n > 3$: Cautis-Kamnitzer-Morrison 2012)

The 1-categories $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ and $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ are equivalent.

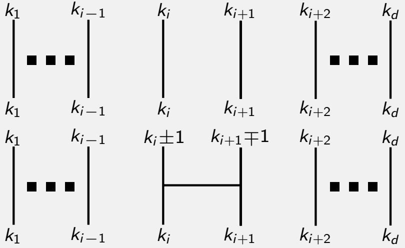
I am lying a little bit: One has to be a little more careful with objects and duals, but we ignore this for today.

How to prove it? Quantum skew Howe duality!

Theorem

There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-action on $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^d$ (objects of length d)!

$1_{\vec{k}} \mapsto$; $E_i 1_{\vec{k}},\ F_i 1_{\vec{k}} \mapsto$

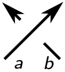


Thus, $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^d$ is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-module and not just a $\mathbf{U}_q(\mathfrak{sl}_2)$-module.

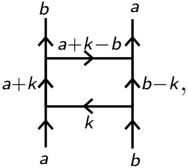
Even better: Since, we only need "left-minus-ladders", aka F's, it can be realized as a $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-module of a certain highest weight: We can use $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-highest weight theory to prove statements about $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner!

Consider a diagram of an oriented tangle. Its components can be colored with colors $k \in \{0, \dots, n\}$. These colors correspond to the fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representations $\Lambda^k \bar{\mathbb{Q}}^n$. Straightening it into a Morse position.

Let $b \leq a$. Define an $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner $\Lambda^a \bar{\mathbb{Q}}^n \otimes \Lambda^b \bar{\mathbb{Q}}^n \to \Lambda^b \bar{\mathbb{Q}}^n \otimes \Lambda^a \bar{\mathbb{Q}}^n$ as follows.



$$= \sum_{k=0}^{b} (-1)^{k+(a+1)b} q^{-b+k}$$

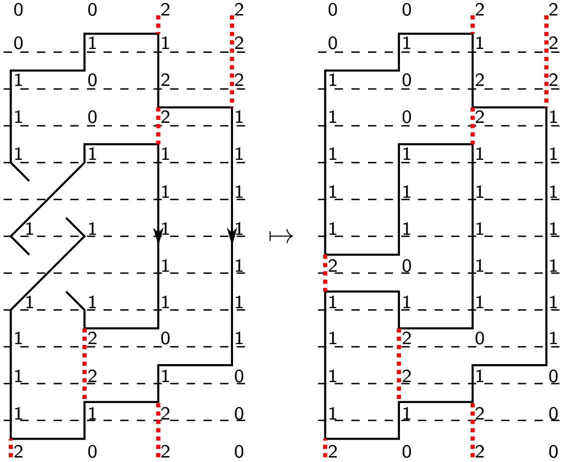


"Morally" (up to some signs, shifts, re-orientations) the same for $a < b$ and .

The polynomial $P_2(T_D)$ is the sum of the local replacements f_s of the crossings.

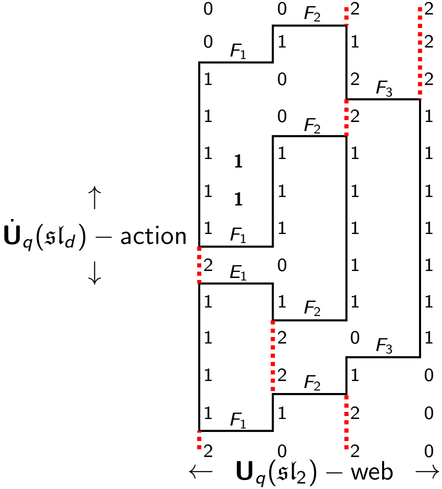
Thus, since closed $\mathfrak{sl}_2$-webs are intertwiner $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$, aka polynomials in $\mathbb{Z}[q, q^{-1}]$, the tangle invariant is a polynomial $P_2(\cdot) \in \mathbb{Z}[q, q^{-1}]$.

Exempli gratia: Hopf link for $\mathfrak{sl}_2$



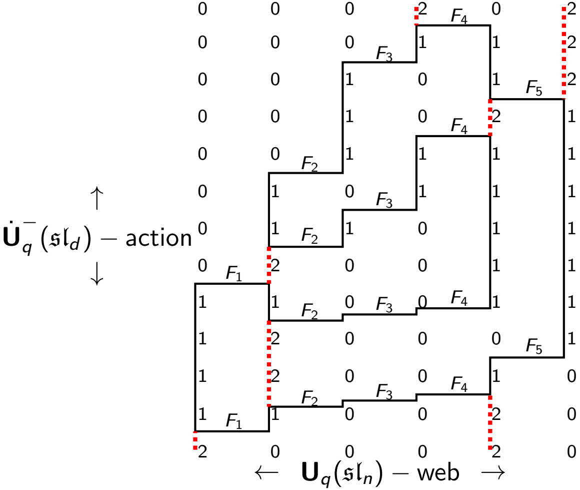
$f_{10}(\textbf{Hopf})\colon \Lambda^2\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}} \otimes \Lambda^2\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}} \to \bar{\mathbb{Q}} \otimes \bar{\mathbb{Q}} \otimes \Lambda^2\bar{\mathbb{Q}}^2 \otimes \Lambda^2\bar{\mathbb{Q}}^2$ is an intertwiner.

Quantum skew Howe duality helps



Recall that we have an $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-action on $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^d$. In the example above

$$f_{10}(\textbf{Hopf}) = F_2F_1F_3F_2F_1E_1F_2F_3F_2F_1F_2^{(2)}v_{2200}.$$



A crucial observation: We need only F's!

$$f_{10}(\mathbf{Hopf}) = F_4^{(2)} F_4 F_3 F_5 F_4 F_2 F_3 F_2 F_1 F_4 F_3 F_2 F_5 F_4 F_3 F_2 F_1 F_4^{(2)} F_3^{(2)} F_2^{(2)} v_{220000}.$$

The $\mathfrak{sl}_n$-polynomials using $\mathfrak{sl}_d$-symmetries

Let us summarize the connection between (colored) $\mathfrak{sl}_n$-polynomials and the $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-$\mathbf{U}_q(\mathfrak{sl}_n)$-skew Howe duality.

- $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner are vectors in $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight spaces.
- Only F's: The space of invariant $\mathbf{U}_q(\mathfrak{sl}_n)$-tensors is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-representation of some highest weight v_h and $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_d)$ suffices.
- Conclusion: The (colored) $\mathfrak{sl}_n$-polynomials $P_n(\cdot)$ are instances of highest $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight representation theory!
- If L_D is a link diagram, then $P_n(L_D)$ is obtained by jumping via F's from a highest $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight v_h to a lowest $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-weight v_l!

Please, fasten your seat belts!

Let's categorify everything!

$\mathfrak{sl}_2$-foams: Natural transformations between $\mathfrak{sl}_2$-webs

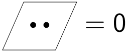
A $\mathfrak{sl}_2$-pre-foam is a cobordism between two $\mathfrak{sl}_2$-webs. Composition consists of placing one $\mathfrak{sl}_2$-pre-foam on top of the other. The following are called the saddle up and down respectively.

They have dots that can move freely about the facet on which they belong. Define the $q-$degree of a $\mathfrak{sl}_2$-foam F with d dots and b boundary components as

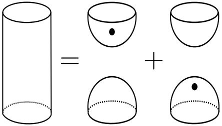
$$\mathrm{qdeg}(F) = -\chi(F) + 2d + \frac{b}{2}.$$

A $\mathfrak{sl}_2$-foam is a formal $\bar{\mathbb{Q}}$-linear combination of isotopy classes of $\mathfrak{sl}_2$-pre-foams modulo the following (degree preserving!) relations.

The $\mathfrak{sl}_2$-foam relations $\ell = (2D, NC, S)$



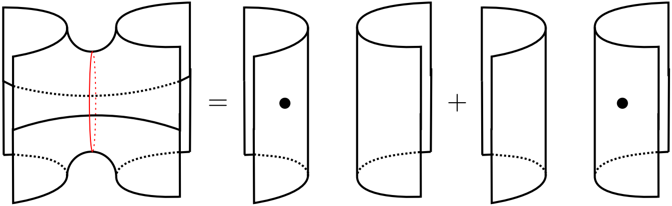
(2D)



(NC)

(S)

The relations $\ell = (2D, NC, S)$ suffice to evaluate $\mathfrak{sl}_2$-foam without boundary!



The $\mathfrak{sl}_2$-foam category

Foam$_2$ is the $\mathbb{Z}$-graded 2-category of $\mathfrak{sl}_2$-foams consisting of:

- The objects are sequences of points in the interval $[0, 1]$.
- The 1-cells are formal direct sums of $\mathbb{Z}$-graded $\mathfrak{sl}_2$-webs with boundary corresponding to the sequences of points for the source and target.
- The 2-cells are formal matrices of $\bar{\mathbb{Q}}$-linear combinations of degree-zero dotted $\mathfrak{sl}_2$-foams modulo isotopy and $\mathfrak{sl}_2$-foam relations.
- Vertical composition $\circ_v$ is stacking on top of each other and horizontal composition $\circ_h$ is stacking next to each other. We write $\mathrm{hom}_{\mathbf{Foam}_2}(u, v) = \mathrm{hom}(u, v)$.

The $\mathfrak{sl}_2$-foam homology of a closed $\mathfrak{sl}_2$-web $w\colon \emptyset \to \emptyset$ is defined by

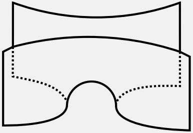
$$\mathcal{F}(w) = \mathrm{hom}_{\mathbf{Foam}_2}(\emptyset, w) = \mathrm{hom}(\emptyset, w).$$

$\mathcal{F}(w)$ is a $\mathbb{Z}$-graded, $\bar{\mathbb{Q}}$-vector space.

Exempli gratia

Example

A saddles are 2-morphisms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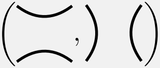
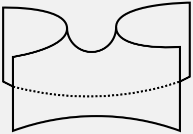
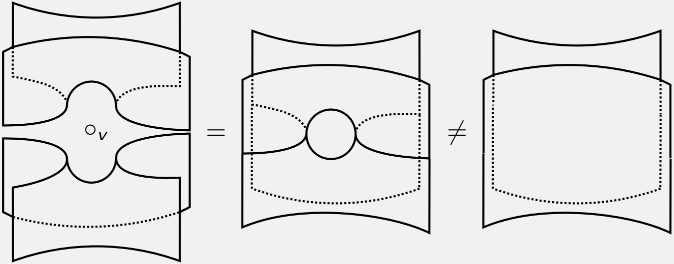


∈ hom

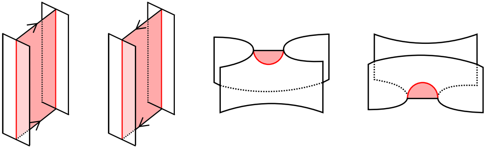

Vertical composition gives a non-trivial "natural transformation" in hom($\asymp$, $\asymp$)!



Rigid $\mathfrak{sl}_2$-foams: Sloppy version

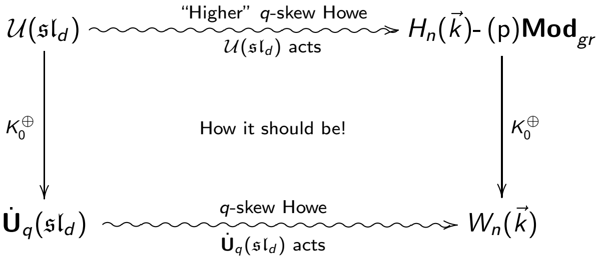
Instead of giving the formal definition of the rigid $\mathfrak{sl}_2$-foam category $\mathbf{Foam}_2$ let me just give some examples.

- The rigid versions of the $\mathfrak{sl}_2$-foams are locally generated by



  where facet get the numbers of their incident edges. Facets labeled 0 are removed, facets labeled 1 really exists and facet labeled 2 are pictured using leashes as boundary (but they exist). Thus, these will be singular surfaces!
- The singular surfaces above are called identities and singular saddles.
- Facets with label 1 are allowed to carry dots. Dots move freely on a facet but are not allowed to cross singular lines.
- There are some relations and the 2-category is graded by a slight rearrangement of the geometrical Euler characteristic.

The overview



This is how it should be: There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-action on the $\mathfrak{sl}_n$-web spaces (for us it was mostly the case $n = 2$). Moreover, suitable module categories over diagrammatic algebras called the $\mathfrak{sl}_n$-web algebras $H_n(\vec{k})$ categorify these spaces.

On the left side: There is Khovanov-Lauda's categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$ denoted by $\mathcal{U}(\mathfrak{sl}_d)$ (which I very shortly recall here).

Conclusion: There should be a 2-action of $\mathcal{U}(\mathfrak{sl}_d)$ on the top right!

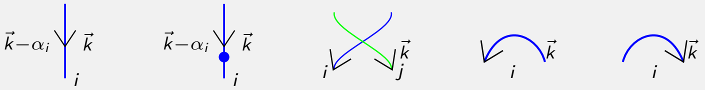
Idea(Khovanov-Lauda)

The algebra $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$ has a basis with surprisingly nice behaviour, e.g. positive structure coefficients. Thus, there should be a categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$ pulling the strings from the background!

Definition(Khovanov-Lauda 2008)

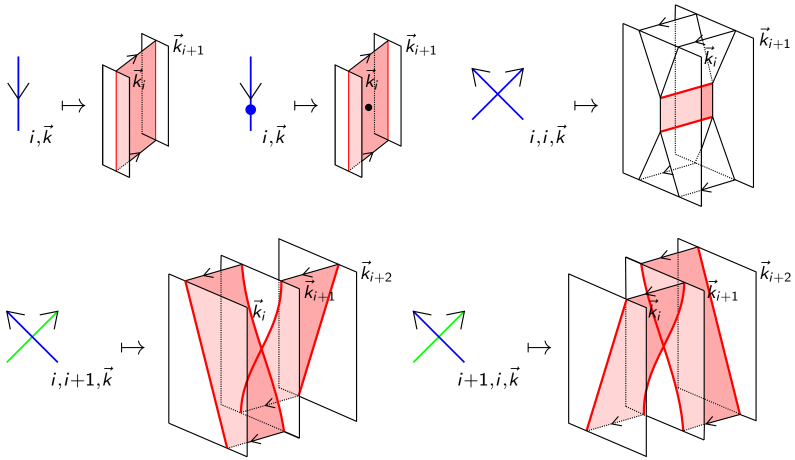
The 2-category $\mathcal{U}(\mathfrak{sl}_d)$ is defined by (everything suitably $\mathbb{Z}$-graded and $\bar{\mathbb{Q}}$-linear):

- The objects in $\mathcal{U}(\mathfrak{sl}_d)$ are the weights $\vec{k} \in \mathbb{Z}^{d-1}$.
- The 1-morphisms are finite formal sums of the form $\mathcal{E}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$ and $\mathcal{F}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$.
- 2-cells are graded, $\bar{\mathbb{Q}}$-vector spaces generated by compositions of diagrams (additional ones with reversed arrows) as illustrated below plus relations.



$\mathfrak{sl}_2$-foamation (0-cells and 1-cells as before)

On 2-**cells:** We define



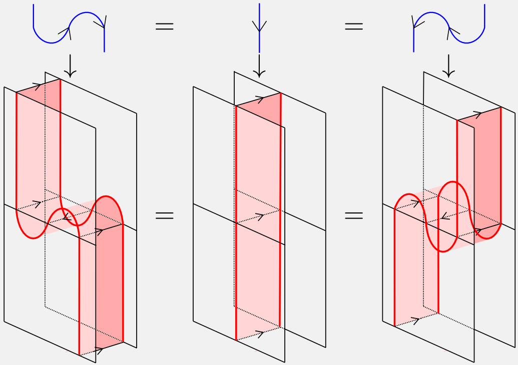
And some others (that are not important today).

Everything fits

Theorem

The 2-functor $\Psi\colon \mathcal{U}(\mathfrak{sl}_d) \to \mathcal{W}^{(p)}_{(2^\ell)}$ categorifies q-skew Howe duality. Thus the $\mathfrak{sl}_n$-link homology as an instance of categorified highest weight theory of $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$.

Example without labels (One has to check well-definedness!)



Khovanov's categorification of the Jones polynomial

Recall the rules for the Jones polynomial.

- $\langle \emptyset \rangle = 1$ (normalization).
- $\langle \times \rangle = \langle)\,(\rangle - q\langle \asymp \rangle$ (recursion step 1).
- $\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$ (recursion step 2).
- $[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$ (Re-normalization).

Definition/Theorem(Khovanov 1999)

Let L_D be a diagram of an oriented link. Denote by $A = \bar{\mathbb{Q}}[X]/X^2$ the dual numbers with $\mathrm{qdeg}(1) = 1$ and $\mathrm{qdeg}(X) = -1$ - this is a Frobenius algebra with a given comultiplication Δ. We assign to it a chain complex $[\![L_D]\!]$ of $\mathbb{Z}$-graded $\bar{\mathbb{Q}}$-vector spaces using the categorified rules:

- $[\![\emptyset]\!] = 0 \to \bar{\mathbb{Q}} \to 0$ (normalization).
- $[\![\times]\!] = \Gamma\left(0 \to [\![)\,(]\!] \xrightarrow{d} [\![\asymp]\!] \to 0\right)$ with $d = m, \Delta$ (recursion step 1).
- $[\![\bigcirc \amalg L_D]\!] = A \otimes_{\bar{\mathbb{Q}}} [\![L_D]\!]$ (recursion step 2).
- $\mathbf{Kh}(L_D) = [\![L_D]\!][-n_-]\{n_+ - 2n_-\}$ (Re-normalization).

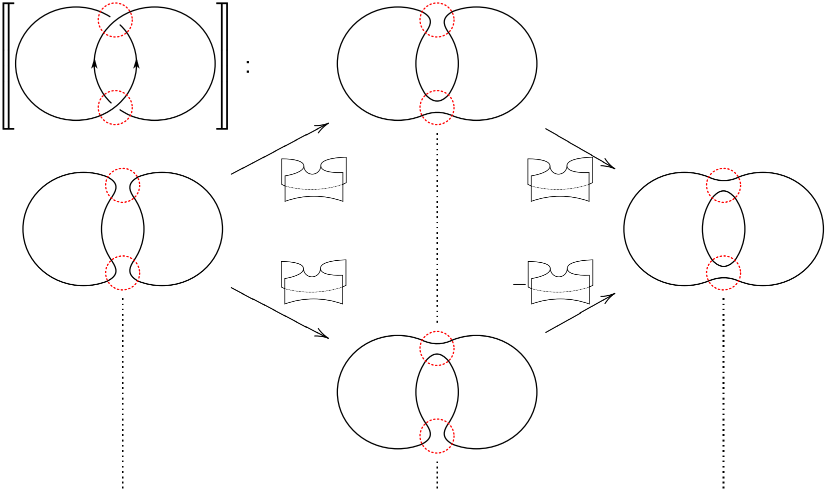
Then $\mathbf{Kh}(\cdot)$ is an invariant of oriented links whose graded Euler characteristic gives $\chi_q(\mathbf{Kh}(L_D)) = [2]J(L_D)$.

This is better than the Jones polynomial

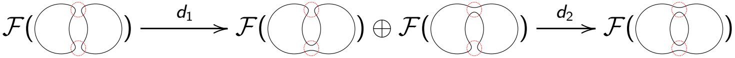
- Khovanov's construction can be extended to a categorification of the HOMFLY-PT polynomial.
- It is functorial (in this formulation only up to a sign).
- Kronheimer and Mrowka showed that Khovanov homology detects the unknot. This is still an open question for the Jones polynomial.
- Rasmussen obtained from the homology an invariant that "knows" the slice genus and used it to give a combinatorial proof of the Milnor conjecture.
- Rasmussen also gives a way to combinatorial construct exotic $\mathbb{R}^4$.
- The categorification is not unique, e.g. the so-called "odd Khovanov homology" differs over $\bar{\mathbb{Q}}$.
- Before I forget: It is a strictly stronger invariant.

History repeats itself: After Jones lots of other link polynomials were discovered and after Khovanov lots of other homologies of "Khovanov-type" were discovered. So we need to understand this better.

Exempli gratia - Khovanov homology using $\mathfrak{sl}_2$-foams

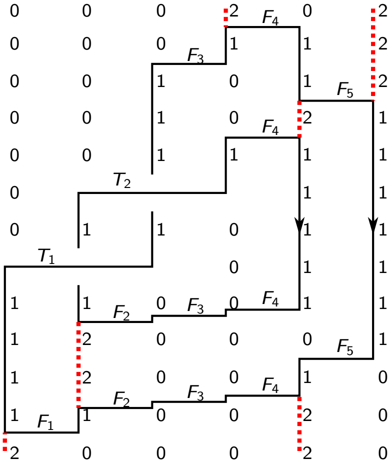


$$\mathcal{F}(\cdot) \xrightarrow{d_1} \mathcal{F}(\cdot) \oplus \mathcal{F}(\cdot) \xrightarrow{d_2} \mathcal{F}(\cdot)$$



Recall: Only F's suffices!

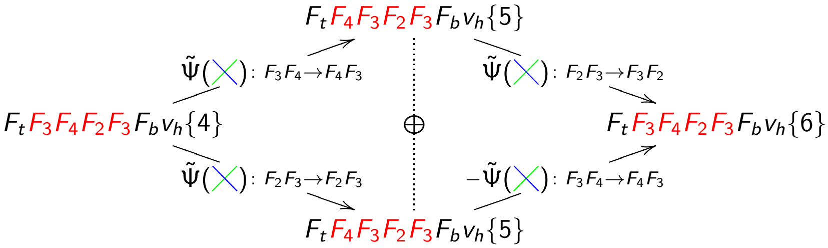
Hopf :



$$F_4^{(2)} F_4 F_3 F_5 F_4 T_2 T_1 F_4 F_3 F_2 F_5 F_4 F_3 F_2 F_1 F_4^{(2)} F_3^{(2)} F_2^{(2)} v_{220000} = F_t T_2 T_1 F_b v_{220000}$$

Exempli gratia (The Hopf link - part two)

The Hopf link example from before will give a complex



that, up to some degree conventions, agrees with the $\mathfrak{sl}_2$-link homology of **Hopf**, because the  "are" the saddles.

Observation - a more "down to earth" point of view

One can use the Hu-Mathas basis for the cyclotomic KL-R algebra to write down a basis for each of the $\mathfrak{sl}_2$-web algebra modules. The  are homomorphisms: Calculating the homology reduces to linear algebra because we only need to track the image of the basis elements!

The $\mathfrak{sl}_n$-homologies using $\mathfrak{sl}_d$-symmetries

Let us summarize the connection between $\mathfrak{sl}_n$-homologies and the higher q-skew Howe duality.

- Khovanov, Khovanov-Rozansky and others: The $\mathfrak{sl}_n$-link homology can be obtained using the $\mathfrak{sl}_n$-"foams".
- Only "F's": The $\mathfrak{sl}_n$-foams are part of the (Karoubian) of the KL-R algebra.
- Conclusion: The $\mathfrak{sl}_n$-homologies are instances of highest $\mathcal{U}(\mathfrak{sl}_d)$-weight representation theory!
- If L_D is a link diagram, then its homology is obtained by "jumping via higher F's" from a highest $\mathcal{U}(\mathfrak{sl}_d)$-object v_h to a lowest $\mathcal{U}(\mathfrak{sl}_d)$-object v_l!
- Missing: Connection to Webster's categorification of the RT-polynomials!
- Missing: Is the module category of the cyclotomic KL-R algebra braided?
- Missing: Details about colored $\mathfrak{sl}_n$-homologies has to be worked out!

There is still much to do...

Thanks for your attention!