02.03.2020

Representation theory of quivers
Talk 1: The basics

General motivation: two questions as a starter

• Q1: What is representation theory?

We want to study a set with some algebraic structure, e.g.

G group
A algebra
$\mathfrak{g}$ Lie algebra
⋮
(complicated)

—Representation theory→

Study its representations!

$\rho: G \to GL(V)$, V v. space over $\mathbb{K}$

$g \in G \mapsto \rho(g): V \to V$ s.t. $\rho(e) = id_V$

and for $g, h \in G$ $\rho(g) \circ \rho(h) = \rho(gh)$

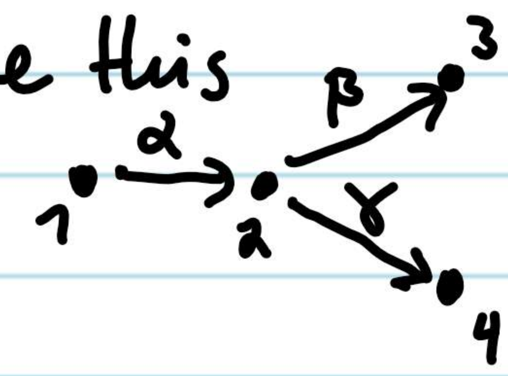
It's linear algebra!

More abstract: study the representations via group algebras $\mathbb{K}G$.

$$\left\{\begin{matrix}\text{representations}\\ \text{of } G\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{representations of}\\ \mathbb{K}G\end{matrix}\right\}$$
→ general approach

• Q2: Why quivers?

Give a beautiful way to visualize the modules of a given algebra via certain diagrams. → like this



Quiver $Q \longmapsto$ path algebra $\mathbb{K}Q$

$$\left\{\begin{matrix}\text{quiver}\\ \text{reps.}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\mathbb{K}Q\text{-modules}\right\}$$

Advantage of quiver: Study the rep. theory of <u>any</u> finite-dimensional algebra.

Plan for today:

- quivers, representations and morphisms
- the building blocks of representations
- a first glimpse of category theory
- Examples, examples, examples ...

1. First definitions

<u>def:</u> A quiver Q is a quadruple (Q_0, Q_1, s, t) consisting of the following data:

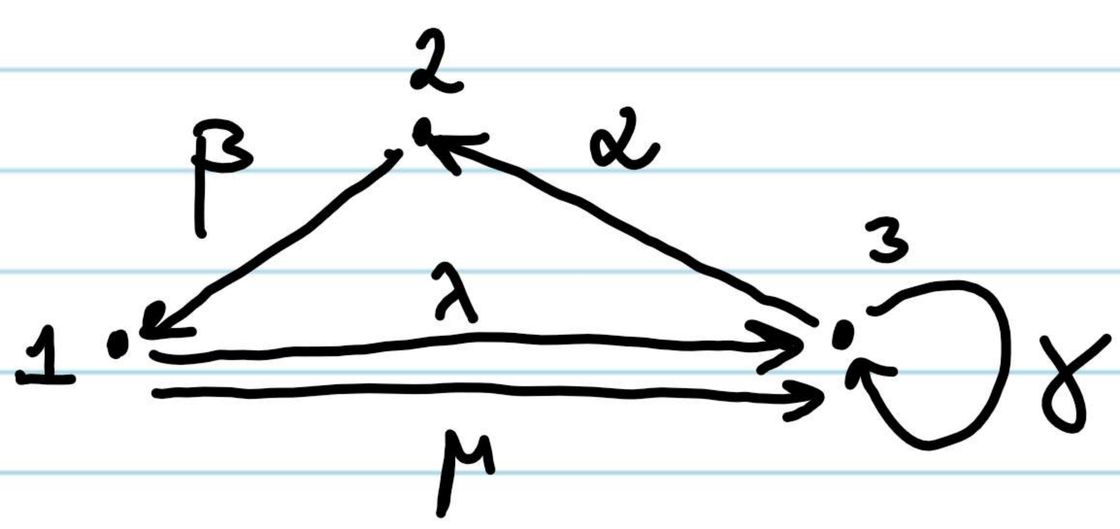
- Q_0 a set of vertices, $\{1,2,3,\dots\}$
- Q_1 a set of arrows, $\{\alpha,\beta,\gamma,\dots\}$
- "source" $s: Q_1 \to Q_0$ a map sending an arrow to its starting point,
- "target" $t: Q_1 \to Q_0$ a map sending an arrow to its ending point.

Represent an element $\alpha \in Q_1$ by drawing an arrow from its starting point $s(\alpha)$ to its endpoint $t(\alpha)$ as follows:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha).$$

Examples:

①



$Q_0 = \{1,2,3\}$, $Q_1 = \{\alpha,\beta,\gamma,\lambda,\mu\}$

$s(\alpha)=3$, $s(\beta)=2$, $s(\gamma)=t(\gamma)=3$, $s(\lambda)=s(\mu)=1$,
$t(\alpha)=2$, $t(\beta)=1$, $t(\lambda)=t(\mu)=3$

②

$Q_0=\{1\}$, $Q_1=\{\alpha\}$
$s(\alpha)=t(\alpha)=1$

Jordan quiver

③ $\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \xrightarrow{\beta} \overset{3}{\bullet}$ $\qquad Q_0 = \{1,2,3\}, \quad Q_1 = \{\alpha, \beta\}$

A_3-quiver $\qquad s(\alpha) = 1, \quad s(\beta) = t(\alpha) = 2, \quad t(\beta) = 3$

Conventions:

- Our quivers will be finite, i.e. Q_0 and Q_1 will be finite
- $\mathbb{K}$ field, algebraically closed (e.g. $\mathbb{C}$).

<u>def:</u> A representation $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q is a collection of $\mathbb{K}$-v.spaces M_i $(i \in Q_0)$, and a collection of $\mathbb{K}$-linear maps $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ for each $\alpha \in Q_1$.

M is a finite-dimensional rep. if each M_i is finite-dimensional.

Dimension vector $\underline{\dim}\, M = (\dim M_i)_{i \in Q_0}$. An element of a rep. M is a tuple $(m_i)_{i \in Q_0}$, $m_i \in M_i$.

<u>Example:</u>

$$\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet}$$

Representations:

$M: \quad \mathbb{K} \xrightarrow{1} \mathbb{K}$

$M': \quad \mathbb{K} \xrightarrow{0} \mathbb{K}$

$M'': \quad \mathbb{K} \xrightarrow{0} 0$

$M''': \quad \mathbb{K}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}} \mathbb{K}^3$

Dimension vectors:

$\underline{\dim}\, M = \underline{\dim}\, M' = (1,1)$

$\underline{\dim}\, M'' = (1,0)$

$\underline{\dim}\, M''' = (2,3)$

(M and M'' indecomposable)

(M', M''' decomposable)

<u>def:</u> Q quiver, $M = (M_i, \varphi_\alpha)$, $M' = (M_i', \varphi_\alpha')$ be two representations of Q. A morphism of reps. $f: M \to M'$ is a collection $(f_i)_{i \in Q_0}$ of linear maps $f_i : M_i \to M_i'$ s.t. for each arrow $i \xrightarrow{\alpha} j$ in Q_1 the diagram below commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ M_i' & \xrightarrow[\varphi_\alpha']{} & M_j' \end{array} \qquad\qquad f_j \circ \varphi_\alpha(m) = \varphi_\alpha' \circ f_i(m), \ \forall m \in M_i.$$

We call $f = (f_i): M \to N$ an <u>isomorphism</u>, if each f_i is bijective.
The class of all isomorphic reps. to M is called an <u>isoclass</u> of M.

<u>Examples:</u>

① Consider $\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet}$ $\qquad M: \mathbb{K} \xrightarrow{1} \mathbb{K}$

$\qquad M'': \mathbb{K} \xrightarrow{0} 0$

$$\begin{array}{c} M \\ {\scriptstyle f}\downarrow \\ M'' \end{array} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{1} & \mathbb{K} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{K} & \xrightarrow[0]{} & 0 \end{array}$$

$f = (f_1, f_2)$

$f_1 : \mathbb{K} \xrightarrow{a} \mathbb{K}$ multiplication by $a \in \mathbb{K}$

$f_2 : \mathbb{K} \xrightarrow{0} 0$ zero map

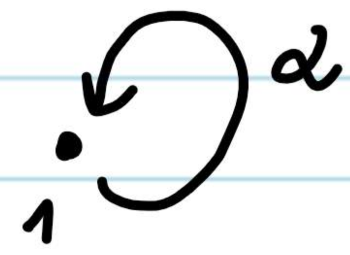
Want to check if $\exists\, g: M'' \to M$. Assume we have:

$$\begin{array}{c} M'' \\ {\scriptstyle g}\downarrow \\ M \end{array} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{0} & 0 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ \mathbb{K} & \xrightarrow[1]{} & \mathbb{K} \end{array}$$

$id \circ g_1 = g_1 = g_2 \circ 0 = 0$ (zero map)

$\Rightarrow g = (g_1, g_2) = (0, 0)$

$\Rightarrow f$ is not an iso.

②  Let $\mathbb{K} = \mathbb{C}$.

$$M: \mathbb{C}^2 \circlearrowleft \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$f \downarrow \quad f = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$M': \mathbb{C}^2 \circlearrowleft \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$M': \mathbb{C}^2 \circlearrowleft \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \det \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} = -3$$

$$g \downarrow \quad g = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M: \mathbb{C}^2 \circlearrowleft \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$\Rightarrow$ f is an isomorphism.

__Remark:__ The trivial representation, i.e. M_i is a trivial vector space for each $i \in Q_0$, always exists.

__Remark:__ We can compose morphisms.

__Prop:__ Let M and M' be two representations of Q. Then the set of all morphisms $\mathrm{Hom}(M, M')$ is a $\mathbb{K}$-vector space with respect to addition and scalar multiplication.

Exercise 1.1 __Proof:__ Let $f, g \in \mathrm{Hom}(M, M')$, where $f = (f_i)_{i \in Q_0}$, $g = (g_i)_{i \in Q_0}$.

For each $i \in Q_0$ we have that $\mathrm{Hom}(M_i, M_i')$ is a vector space.

We set $f + g = (f_i + g_i)_{i \in Q_0}$, $af = (af_i)_{i \in Q_0}$ for $a \in \mathbb{K}$.

Check all the vector space properties, having that:

$\varphi'_\alpha \circ (f_i + g_i) = (f_j + g_j) \circ \varphi_\alpha$

$\varphi'_\alpha \circ af_i = af_j \circ \varphi_\alpha$.

The neutral element w.r.t. addition is the zero morphism $0 \to 0$.

□

Examples:

① Consider $\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet}$

$M : \mathbb{K} \xrightarrow{1} \mathbb{K}$

$M'' : \mathbb{K} \xrightarrow{0} 0$

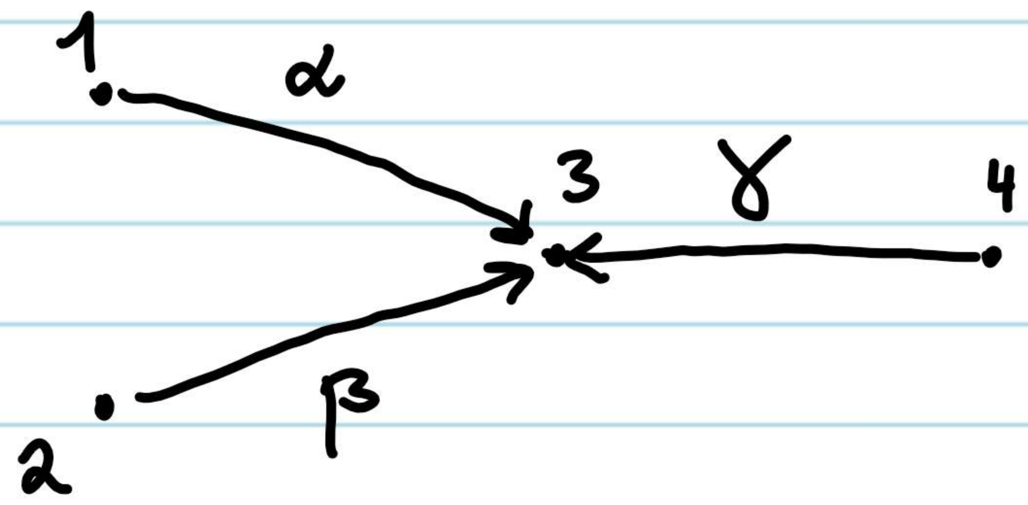
We can identify $\mathrm{Hom}(M, M'') \cong \{(a,0) \mid a \in \mathbb{K}\} \cong \mathbb{K}$

↳ 1-dim space

$\mathrm{Hom}(M'', M) \cong 0$, since the only morphism is the zero morphism.

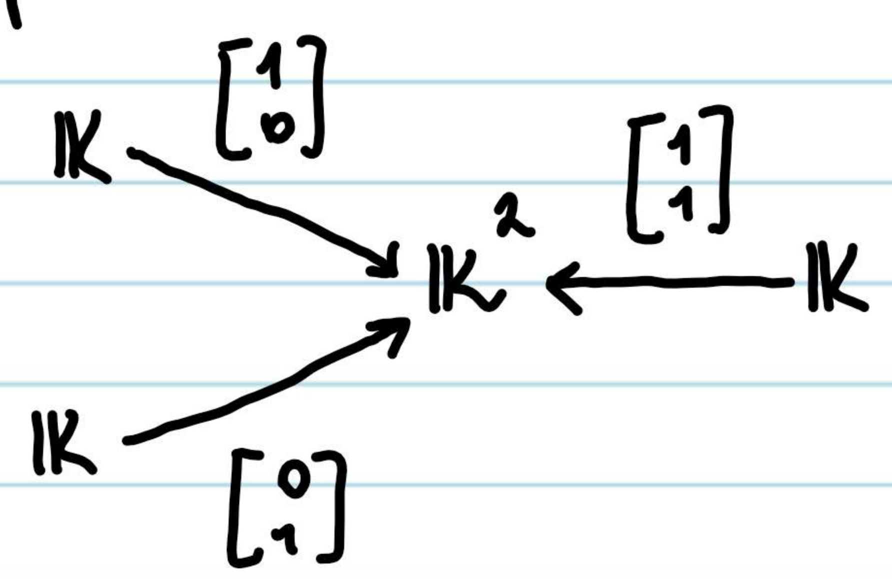
② Consider

Q:

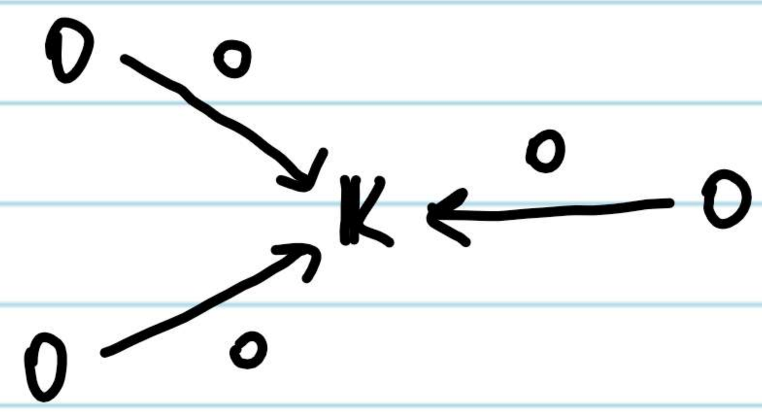


The representations:

M:

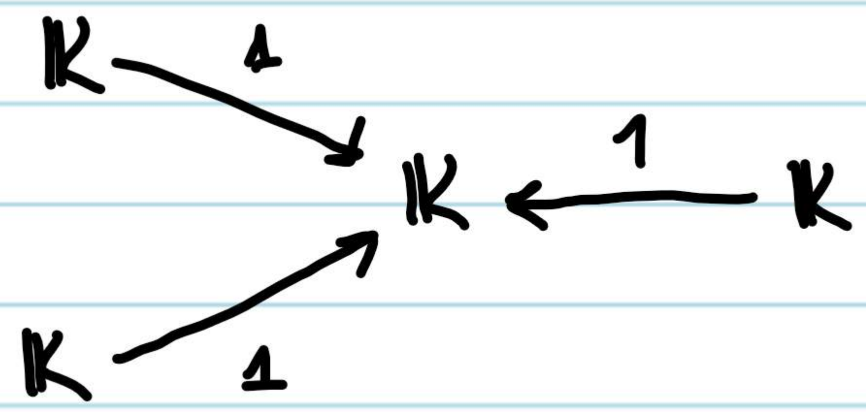


M':



(M, M', M'' indecomposable)

M'':

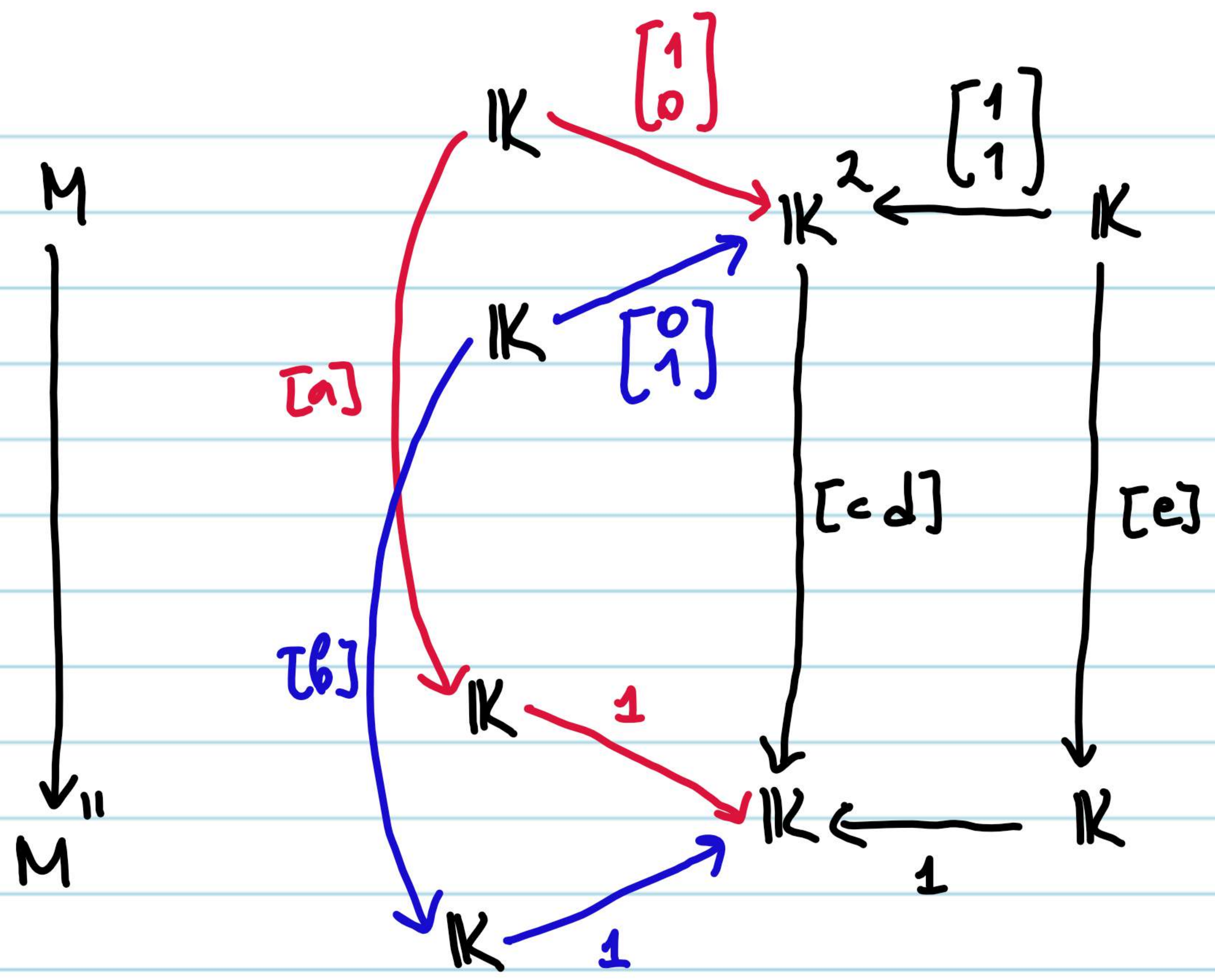


We have $\mathrm{Hom}(M, M') = 0$, $\mathrm{Hom}(M, M'') \cong \mathbb{K}^2$,

$\mathrm{Hom}(M', M) \cong \mathbb{K}^2$, $\mathrm{Hom}(M'', M) = 0$.

We show only one case, the rest follows the same strategy.

Let $a, b, c, d, e \in \mathbb{K}$. Consider $\mathrm{Hom}(M, M'')$, show as a commutative diagram:



$[c\ d]\begin{bmatrix}1\\1\end{bmatrix} = 1.[e] \Leftrightarrow c+d=e$

$1.b = [c\ d]\begin{bmatrix}0\\1\end{bmatrix} \Leftrightarrow b=d$

$1.a = [c\ d]\begin{bmatrix}1\\0\end{bmatrix} \Leftrightarrow a=c$

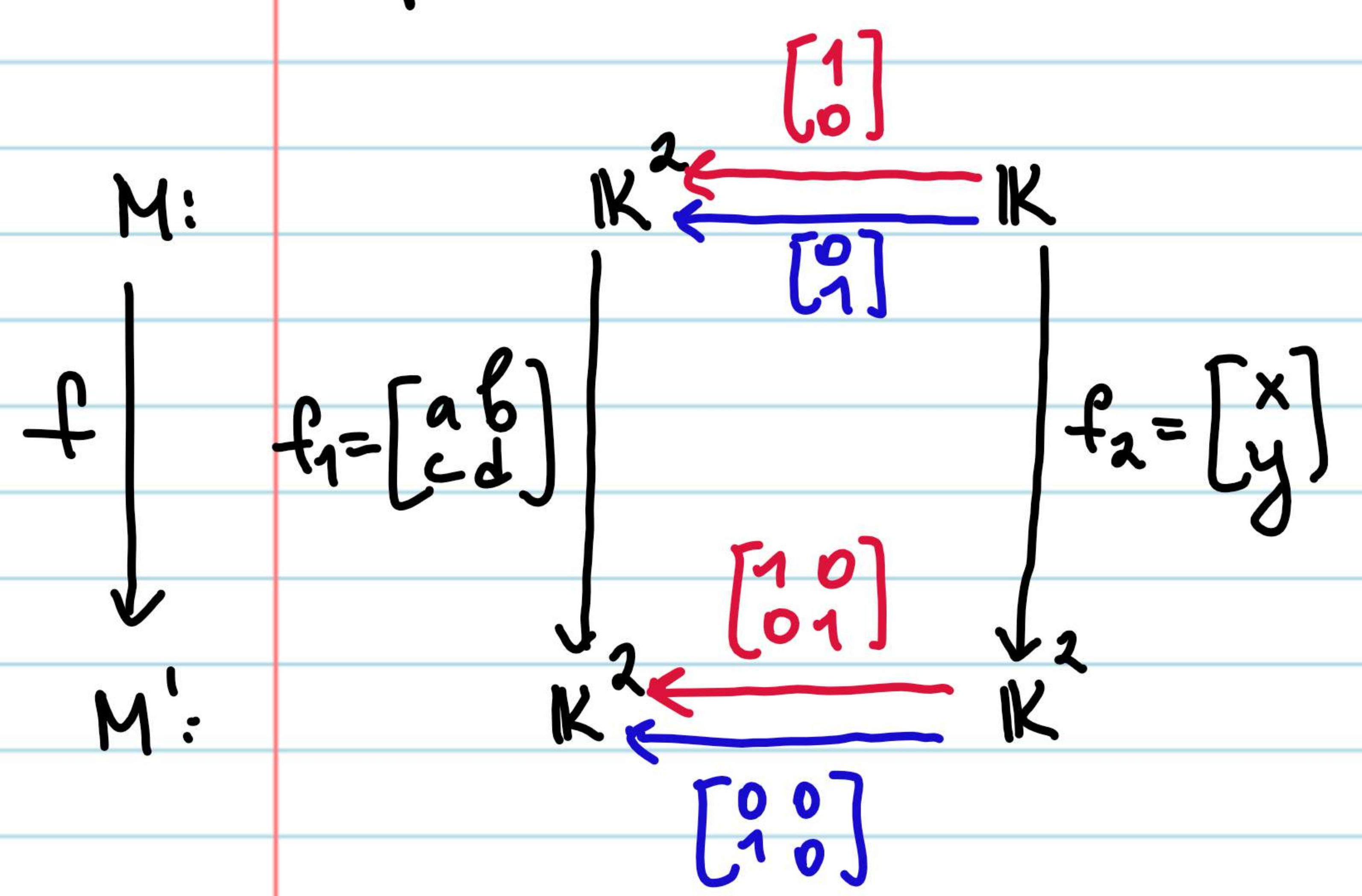
It is enough to know a and b to know the morphism.

But $a, b \in \mathbb{K}$ and $a \neq b$ $\Rightarrow$ $Hom(M, M'') \cong \mathbb{K}^2$.

(3) $1 \bullet \overset{\alpha}{\underset{\beta}{\leftleftarrows}} \bullet\, 2$

Representations:

Let $a, b, c, d, x, y \in \mathbb{K}$.



(M, M' indecomposable)

From both commutative diagrams we get:

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ and $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ $\qquad \begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}0\\x\end{bmatrix}$ $\Rightarrow$ $a=d=x$, $b=0$, $c=y$

Thus, $f=(f_1,f_2) = \left(\begin{bmatrix}x & 0\\ y & x\end{bmatrix}, \begin{bmatrix}x\\y\end{bmatrix}\right)$.

$\Rightarrow$ $Hom(M, M') \cong \mathbb{K}^2$ with basis $\left\{\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right), \left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)\right\}$.

2. Indecomposables

If we know M and N representations, then it is easy to say what $M \oplus N$ is.

More interesting but harder question.

Q: Given a representation X, is it possible to decompose it into a direct sum $X = M_1 \oplus M_2 \oplus \dots \oplus M_t$, where each M_i is indecomposable?

<u>def:</u> Let $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ be two reps of Q.

Their direct sum is given by $M \oplus M' = \left(M_i \oplus M'_i, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix}\right)_{i \in Q_0, \alpha \in Q_1}$.

Recursively, we define $M_1 \oplus M_2 \oplus \dots \oplus M_t = (M_1 \oplus \dots \oplus M_{t-1}) \oplus M_t$.

<u>Example:</u> $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet}$

Consider the reps.

$M: \ K \xrightarrow{1} K \xleftarrow{0} 0$ (M indecomposable)

$M': \ K^2 \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} K^2 \xleftarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} K$ (M' decomposable)

The direct sum: $M \oplus M': \ K \oplus K^2 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} K \oplus K^2 \xleftarrow{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}} 0 \oplus K$.

This is isomorphic to $K^3 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} K^3 \xleftarrow{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} K$.

<u>def:</u> A representation M is called <u>indecomposable</u>, if $M \neq 0$ and can't be written as a direct sum of two nonzero reps, i.e. whenever $M \cong N \oplus L$, then $N = 0$ or $L = 0$.

<u>Examples:</u> ① The reps. marked with (indecomposable) in the previous examples are indecomposable.

② Consider the quiver $1 \bullet \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet}$

M': $\mathbb{K}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \mathbb{K}^2 \xleftarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathbb{K}$

$M' \cong N \oplus L$, namely $\underbrace{(\mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K})}_{N} \oplus \underbrace{(\mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0)}_{L}$

Easy to see: construct the matrices and check that $M \xrightarrow{\sim} N \oplus L$ is an isomorphism.

Important remark: How to study representations?

Goal of Representation theory

Classify all representations of a quiver Q and all morphisms between them up to an isomorphism.

The following theorem helps with this task.

Thm: (Krull-Schmidt Theorem)

Let Q be a quiver and M a representation. Then we decompose

$M \cong M_1 \oplus M_2 \oplus \dots \oplus M_t$, where each M_i is an indecomposable rep.

and unique up to order. More precisely, if we have a second decomposition of M, given by $M'_1 \oplus M'_2 \oplus \dots \oplus M'_s$, then $t = s$ and there exists a permutation $\sigma \in S_t$ s.t. $M_i \cong M'_{\sigma(i)}$ for any $1 \leq i \leq t$.

Proof: Existence follows easily by induction on t. ✓

Uniqueness is trickier, we need a bit more knowledge of quiver reps. to do it, so we omit it from these notes.

□

3. A first glimpse of category theory

From now on we'll speak about reps as objects of category Rep(Q) and morphisms between them as morphisms of the category.

<u>def:</u> A <u>category</u> $\mathcal{C}$ consists of the following data:

- A class of objects $Ob(\mathcal{C})$;
- A class of morphisms $Hom_{\mathcal{C}}$, s.t. each morphism $f \in Hom_{\mathcal{C}}$ has a unique source X and a unique target Y in $Ob(\mathcal{C})$.
  Write $f: X \to Y$. The class of morphisms from X to Y is denoted by $Hom_{\mathcal{C}}(X,Y)$.
- A binary operation
$$\circ: Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \to Hom_{\mathcal{C}}(X,Z)$$
$$(f, g) \mapsto g \circ f$$

for every three objects $X, Y, Z \in Ob(\mathcal{C})$.

This composition satisfies the following axioms:

1. For $f: W \to X$, $g: X \to Y$ and $h: Y \to Z$ we have:
$$h \circ (g \circ f) = (h \circ g) \circ f \qquad \text{(associativity)}$$

2. For every object $X \in Ob(\mathcal{C})$ there exists a morphism $1_X \in Hom_{\mathcal{C}}(X,X)$ called the identity morphism, s.t. for every $f \in Hom_{\mathcal{C}}(X,Y)$ and $g \in Hom_{\mathcal{C}}(Z,X)$ we have:
$$f \circ 1_X = f \quad \text{and} \quad 1_X \circ g = g \qquad \text{(identity morphism)}$$

Thus, all that we defined until now about Rep(Q) constitutes its structure as a category.

<u>Remark:</u> In fact, Rep(Q) admits more structure than just a category.

It is an <u>abelian</u> category.

Bonus:

Exercise 1.2: Given is the quiver $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet}$ and

$$M: \quad \mathbb{K} \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{K}^2 \xleftarrow{\begin{bmatrix}0\\1\end{bmatrix}} \mathbb{K}$$

$$M': \quad \mathbb{K} \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{K}^2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{K}$$

① Show that M and M' are not indecomposable.

Solution:

$$M_1: \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0$$
$$M_2: \quad 0 \xrightarrow{0} \mathbb{K} \xleftarrow{1} \mathbb{K}$$
indecomposable reps.

Then,

$$M_1 \oplus M_2: \quad \mathbb{K} \oplus 0 \xrightarrow{\begin{bmatrix}1&0\\0&0\end{bmatrix}} \mathbb{K} \oplus \mathbb{K} \xleftarrow{\begin{bmatrix}0&0\\0&1\end{bmatrix}} 0 \oplus \mathbb{K}$$

$$\cong M: \quad \mathbb{K} \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{K}^2 \xleftarrow{\begin{bmatrix}0\\1\end{bmatrix}} \mathbb{K}$$

Similarly, $M_3: \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} 0$

$$M_4: \quad 0 \xrightarrow{0} \mathbb{K} \xleftarrow{0} 0$$

$$M_3 \oplus M_4: \quad \mathbb{K} \oplus 0 \xrightarrow{\begin{bmatrix}1&0\\0&0\end{bmatrix}} \mathbb{K} \oplus \mathbb{K} \xleftarrow{\begin{bmatrix}1&0\\0&0\end{bmatrix}} \mathbb{K} \oplus 0$$

$$\cong M': \quad \mathbb{K} \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{K}^2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{K}$$

② Show that M and M' are not isomorphic.

Solution:

$$\begin{array}{ccccccc}
M: & \mathbb{K} & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & \mathbb{K}^2 & \xleftarrow{\begin{bmatrix}0\\1\end{bmatrix}} & \mathbb{K} \\
f \downarrow & {\scriptstyle [a]}\downarrow{\scriptstyle f_1} & & {\scriptstyle f_2}\downarrow{\scriptstyle \begin{bmatrix}x&y\\z&w\end{bmatrix}} & & {\scriptstyle f_3}\downarrow{\scriptstyle [b]} \\
M': & \mathbb{K} & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & \mathbb{K}^2 & \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} & \mathbb{K}
\end{array}$$

$$\begin{bmatrix}a\\0\end{bmatrix} = \begin{bmatrix}x&y\\z&w\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x\\z\end{bmatrix}$$

$$\begin{bmatrix}b\\0\end{bmatrix} = \begin{bmatrix}x&y\\z&w\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}y\\w\end{bmatrix}$$

Assume $M \cong M'$, then $f_2 = \begin{bmatrix}x&y\\z&w\end{bmatrix}$ should be invertible, but

$\begin{bmatrix}x&y\\z&w\end{bmatrix} = \begin{bmatrix}a&b\\0&0\end{bmatrix}$ is not of full rank $\Rightarrow$ can't be invertible

$\Rightarrow M \ncong M'$.