THE CLASSICAL THEORY V

SOERGEL BIMODULES, THE FEAST

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Twisted actions

- Let A be a commutative graded algebra over a base ring $\mathbb{k}$
- An A-bimodule on a $\mathbb{k}$ – module M with actions

$$A \times M \to M \text{ and } M \times A \to M$$

eqiuvalent to a left $A \otimes_{\mathbb{k}} A$ – module with action

$$A \otimes_{\mathbb{k}} A \times M \to M$$

- Twist a right A-module with $\mathbb{k}$ - algebra automorphism $\eta: A \to A$
  - For $a \in A, m \in M$ $m \cdot_\eta a := m \cdot \eta(a)$
- If we have an A-bimodule with structure on M encoded by

$$\rho: A \otimes_{\mathbb{k}} A \to End_{\mathbb{k}}(M)$$

then the composition $\rho \circ (id \otimes \eta)$ defines a new A-bimodule with same left action and twisted right action, denoted by $\boldsymbol{M_\eta}$

- If we had two automorphisms η and ψ, then

$$id \otimes (\eta \circ \psi) = (id \otimes \eta) \circ (id \otimes \psi)$$

  from which follows, that $M_{\eta \circ \psi} = (M_\eta)_\psi$

- The bimodule M_η can be naturally identified with $M \otimes_A A_\eta$

- We deduce that

$$A_{\eta \circ \psi} \simeq (A_\eta)_\psi \simeq A_\eta \otimes_A A_\psi$$

Standard bimodules

- Consider automorphisms of R of the form

$$\eta_x : R \to R, a \longmapsto xa \quad \text{for } x \in W$$

- DEFINITION: The **standard bimodules** are the R-bimodules of the form $R_x \coloneqq R_{\eta_x}$ obtained by twisting the regular bimodule R on the right side by η_x for some $x \in W$.
- DEFINITION: The $\boldsymbol{Std}\mathbf{Bim}$ is the smallest strictly full subcategory of R-gibm which contains $R_x\ \forall\ x \in W$ and is closed under finite direct sums and grading shifts

- From $A_{\eta\circ\psi} \simeq (A_\eta)_\psi \simeq A_\eta \otimes_A A_\psi$ we see that

$$R_x \otimes R_y \backsimeq R_{xy}$$

- $\Rightarrow$ $Std\mathrm{Bim}$ is monoidal
- EXAMPLE: Let $R = \mathbb{R}[x_1, x_2, x_3]$ and $W = S_3$, for $f(x_1, x_2, x_3) \in R$ and $s = (2,3) \in W$ then the left action of $\boldsymbol{R_s}$ is simple multiplication with $f(x_1, x_2, x_3)$, the right action is multiplication with $f(x_1, x_3, x_2)$

- DEFINITION: For $M, N \in R - gbim$ the **graded Hom space** is

  $Hom^{\bullet}(M, N) \coloneqq \bigoplus_{i \in \mathbb{Z}} Hom\big(M, N(i)\big)$

  We say morphism which send M^i to N^{i+k} for some $k \in \mathbb{Z}$ are homogeneous of degree k

- LEMMA: For any $x, y \in W$ we have

  $$Hom^{\bullet}(R_x, R_y) = \begin{cases} R & , if\ x = y \\ 0, & otherwise \end{cases}$$

  as a graded vector space

- We follow that R_x is indecomposable $\forall\, x \in W$, since the Lemma implies that $End(R_x) = Hom(R_x, R_x) = R$ so it has no non-trivial idempotents.

Split Grothendieck group

- DEFINITION: The **split Grothendieck group** $[\boldsymbol{Std}\mathbf{Bim}]_{\oplus}$ is an abelian group generated by symbols $[B]$ for each object B in $Std\mathrm{Bim}$
  - with the relations $[B] = [B'] + [B'']$, whenever $B \simeq B' \oplus B''$
  - $[Std\mathrm{Bim}]_{\otimes}$ is a ring, via $[B][B'] = [BB']$
- We can make $[Std\mathrm{Bim}]_{\oplus}$ into a $\mathbb{Z}[v^{\pm 1}]$-algebra, via $v[B] \coloneqq [B(1)]$

- REMARK: The split Grothendieck group $[Std\mathrm{Bim}]_{\oplus}$ is isomorphic to the group algebra $\mathbb{Z}[v^{\pm 1}][W]$ with an isomorphism sending $[R_x]$ to x
- $Std\mathrm{Bim}$ is a **categorification** of this group algebra
- Soergel Bimodules categorify the Hecke algebra

$$\begin{tikzcd} \mathit{Std}\text{Bim} \arrow[r, "\text{categorify}"] & \text{group algebra} \\ \mathbb{S}\text{Bim} \arrow[r, "\text{categorify}"] & \text{Hecke algebra} \arrow[u, bend right, "\text{Deformation}"'] \end{tikzcd}$$

- Recall: $B_s = R \otimes_{R^s} R(1)$ and the Bott-Samelson bimodule $BS(x)$ associated with the expression $\underline{x = (s_1, s_2, \dots, s_n)}$ is the bimodule $BS(x) = B_{s_1} B_{s_2} \dots B_{s_d} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_d}} R(d)$
- Consider $c_s \coloneqq \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$ and

$$d_s \coloneqq \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$$

- LEMMA: For any $f \in R$,

$$f \cdot c_{id} = c_{id} \cdot f + d_s \cdot \delta_s(f)$$

$$f \cdot d_s = d_s \cdot s(f)$$

  - Proof: First let f be s-symmetric, where $\delta_s(f) = 0$, $s(f) = f$
    - $f \cdot c_{id} = c_{id} \cdot f = c_{id} \cdot f + d_s \cdot 0$
    - $f \cdot d_s = d_s \cdot f = d_s \cdot \delta_s(f)$

Then let $f = \alpha_s, \delta_s(f) = 2\ and\ s(f) = -f$

- $c_{id} \cdot f + d_s \cdot \delta_s(f) = (1 \otimes 1)\alpha_s + 2d_s = 1 \otimes \alpha_s + \alpha_s \otimes 1 - 1 \otimes \alpha_s = \alpha_s \otimes 1$

  which is equal to $\alpha_s(1 \otimes 1)$

- $\alpha_s \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s) = \frac{1}{2}(\alpha_s^2 \otimes 1 - \alpha_s \otimes \alpha_s) = \frac{1}{2}(\alpha_s \otimes (-\alpha_s) - 1 \otimes \alpha_s \cdot (-\alpha_s))$

$$= \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)(-\alpha_s)$$

- Split $f = \delta_s\left(f \frac{\alpha_s}{2}\right) + \frac{\alpha_s}{2}\delta_s(f)$ symmetric and antisymmetric part
- Combine previous results

Filtrations

- c_s generates a copy of $R(-1)$ inside B_s
- d_s generates a copy of $R_s(-1)$ inside B_s
- Short exact sequences:
  - $0 \longrightarrow R_s(-1) \xrightarrow{1 \mapsto d_s} B_s \xrightarrow{\mu_{id}} \mathrm{R}(1) \longrightarrow 0$ with $\mu_{id}(f \otimes g) = fg$ (Δ)
  - $0 \longrightarrow R(-1) \xrightarrow{1 \mapsto c_s} B_s \xrightarrow{\mu_s} \mathrm{R_s}(1) \longrightarrow 0$ with $\mu_s(f \otimes g) = f \cdot s(g)$ (∇)

- For an expression $\underline{w} = (s_1, s_2, \dots, s_d)$ we can tensor (Δ) together to get a filtration of the Bott-Samelson bimodule $BS(w)$
- For $B_s B_s$ we get $0 \longrightarrow R_s B_s(-1) \longrightarrow B_s B_s \longrightarrow B_s(1) \longrightarrow 0$

- Enumeration of W such that $\mathrm{x_i} \leq x_j$ in Bruhat order implies $i \leq j$
- EXAMPLE: For A_2 we would have $id < s < t < st < ts < sts$
- DEFINITION: For an enumeration as above, a $\boldsymbol{\Delta - filtration}$ of a Soergel bimodule B is a filtration $B^k \subset B^{k-1} \subset \cdots \subset B^0 = B$ with subquotients $B^i/B^{i+1} \simeq {R_{x_i}}^{\oplus h_{x_i}}, \quad$ where $h_{x_i} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$
- Even if W is infinite, this filtration needs to be of finite length.

- EXAMPLE: If $B^i/B^{i+1} \simeq R_{x_i} \otimes R_{x_i}(3) \otimes R_{x_i}(-5)$, then

$$h_{x_i} = 1 + v^3 + v^{-5}$$

- THEOREM: For a fixed enumeration of W, any Soergel bimodule B has a unique Δ-filtration. Moreover, for any $x \in W$ the graded multiplicity h_x of R_x in the Δ-filtration depends only on B and x, not the choice of enumeration on W.

- DEFINITION: The $\boldsymbol{\Delta - character}$ of a Soergel bimodule B is the element $ch_\Delta(B) \coloneqq \sum_{x\in W} v^{\ell(x)} h_x(B)\delta_x$, of H, where δ_x are the standard basis elements.

- EXAMPLE: We have $h_{id}(B_s) = v^1$ and $h_s(B_s) = v^{-1}$, therefore $ch_\Delta(B_s) = v\delta_{id} + v \cdot v^{-1}\delta_s = v + \delta_s$,
  hence $\boldsymbol{ch_\Delta(B_s)} = \mathbf{b_s}$ for any $\mathrm{s} \in S$

EXAMPLE: Let $W = S_2 = < 1, s >$

- Standard basis $\{\delta_1, \delta_s\}$
- Kazhdan-Lusztig basis $\{b_1, b_s\}$
- Change of basis matrix is $\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$, indeed $(\delta_1, \delta_s)\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = (b_1, b_s)$

  since $\mathrm{b}_1 = \delta_1$ and $b_2 = \delta_1 v_1 + \delta_s$

- DEFINITION: The $\nabla - \boldsymbol{character}$ of a Soergel bimodule B is the element $ch_{\nabla}(B) \coloneqq \sum_{x\in W} v^{\ell(x)}\overline{h'_x(B)}\delta_x \in H$
- EXAMPLE: We have $h'_{id}(B_s) = v^{-1}$ and $h'_s(B_s) = v^1$, therefore $ch_{\nabla}(B_s) = \overline{v^{-1}}\delta_{id} + v \cdot \overline{v}\delta_s = v + \delta_s$,
  hence $\boldsymbol{ch_{\Delta}(B_s)} = \mathbf{b_s} = \mathbf{ch_{\nabla}(B_s)}$ for any $s \in S$

- Properties: $$ch_\Delta(B \oplus B') = ch_\Delta(B) + ch_\Delta(B')$$
$$ch_\nabla(B \oplus B') = ch_\nabla(B) + ch_\nabla(B')$$
and
$$ch_\Delta\big(B(1)\big) = v\, ch_\Delta(B)$$
$$ch_\nabla\big(B(1)\big) = v^{-1}\, ch_\nabla(B)$$
for all Soergel bimodules B and B'
- ⇒ We have $\mathbb{Z}$-linear maps $ch_\Delta\ , ch_\nabla \colon [\mathbb{S}Bim]_\oplus \longrightarrow H$
from the split Grothendieck group of $\mathbb{S}Bim$

Soergel‘s Categorification Theorem

1. There is a $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphism $c: H \rightarrow [\mathbb{S}Bim]_{\otimes}$ sending b_s to $[B_s]$ $for\ all\ s \in S$
2. There is a bijection between W and the set of indecomposable objects of $\mathbb{S}Bim$ up to shift and isomorphism:

$$W \leftrightarrow \{indec. objects\ in\ SBim\}/\simeq, (1)$$

$$w \leftrightarrow B_w$$

The indecomposable object B_w appears as direct summand of the Bott-Samelson bimodule $BS(w)$ for a reduced expression of w. Moreover, all other summands of $BS(w)$ are shifts of B_x for $x < w$ in the Bruhat order .

3. The character function $ch = ch_{\Delta}$ defined above descends to a $\mathbb{Z}[v^{\pm 1}]$ -module homomorphism

$$ch\colon\ [\mathbb{S}Bim]_{\otimes} \to H$$

Which is the inverse to c. Thus, both are isomorphisms.

$$[SBim]_{\otimes} \simeq Hecke\ algebra$$

Soergel's Conjecture

For any $x \in W, ch(B_x) = b_x$. In other words, the Kazhdan-Lusztig polynomial $h_{x,y}$ is equal to $h_x(B_y)$.