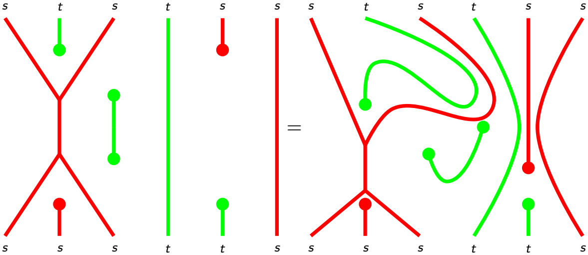
Diagram categories for $\mathbf{U}_{\mathrm{q}}$-tilting modules at $q^{\ell} = 1$

Or: fun with diagrams!

Daniel Tubbenhauer



Joint work with Henning Haahr Andersen

October 2014

1. The why of diagram categories
   - String calculus
   - Biadjoint functors

2. Categorification of Hecke algebras
   - Hecke algebras and Soergel bimodules
   - Soergel's categorification

3. Let us use diagrams!
   - The F_i are selfadjoint functors
   - Diagrammatic categorification

4. What about $\mathbf{U}_{\mathrm{q}}$-modules at roots of unity?

String calculus for 2-categories - Part 1

Question: Can we interpret $\mathbf{Cat}^2$ using diagrams? Let us start with $\mathbf{Cat}^1$:

Instead of

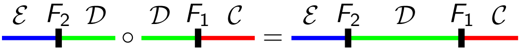
$$\mathcal{C} \xrightarrow{F_1} \mathcal{D}$$

use the Poincaré dual

Composition

$$\mathcal{D} \xrightarrow{F_2} \mathcal{E} \circ \mathcal{C} \xrightarrow{F_1} \mathcal{D} = \mathcal{C} \xrightarrow{F_2 \circ F_1} \mathcal{E}$$

becomes

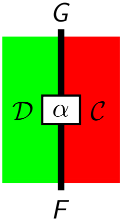


Not really spectacular...

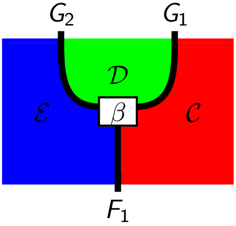
String calculus for 2-categories - Part 2

Let us go to $\mathbf{Cat}^2$ now:

Think of a natural transformations $\alpha, \beta, \cdots$ as a proceeding in time:

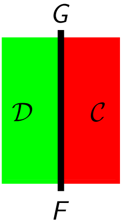


$= \alpha \colon F \Rightarrow G$

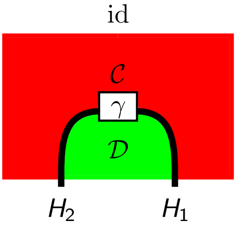


$= \beta \colon F_1 \Rightarrow G_2 \circ G_1$

We do not draw identities



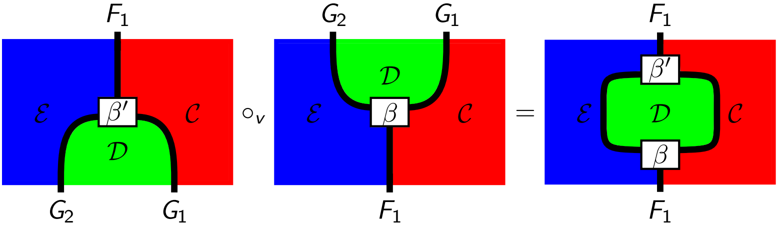
$= \mathrm{id} \colon F \Rightarrow G$



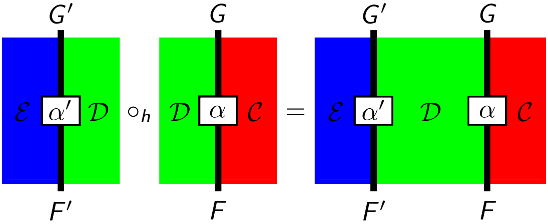
$= \gamma \colon H_2 \circ H_1 \Rightarrow \mathrm{id}$

String calculus for 2-categories - Part 3

Compositions? Sure! Vertical:



and horizontal



That looks promising: 2-categories are like 2-dimensional spaces.

Adjoint functors abstract

Definition(Dan Kan 1958)

Two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are adjoint iff there exist natural transformations called unit $\iota\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}_F \circ \iota"] \arrow[rr, bend right, "\mathrm{id}_F"'] & FGF \arrow[r, "\varepsilon \circ \mathrm{id}_F"] & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\iota \circ \mathrm{id}_G"] \arrow[rr, bend right, "\mathrm{id}_G"'] & GFG \arrow[r, "\mathrm{id}_G \circ \varepsilon"] & G \end{tikzcd}$$

commute. Here F is the left adjoint of G.

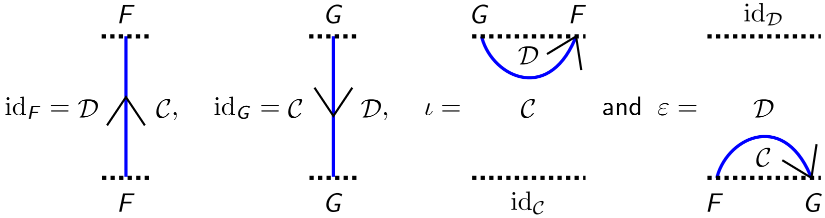
Example

$\mathrm{forget}\colon \mathbb{Q}$-**Vect** $\to$ **Set** has a left adjoint $\mathrm{free}\colon$ **Set** $\to \mathbb{Q}$-**Vect**.

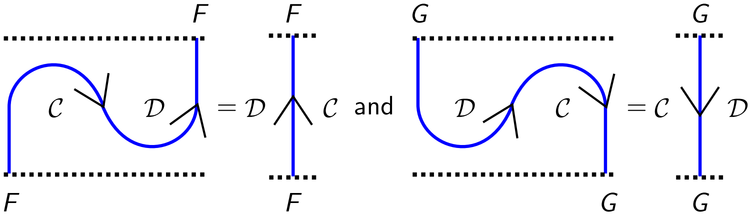
In words: If you have lost your key, then the only guaranteed solution is to search everywhere.

Adjoint functors such that I understand

Let us draw sting pictures!

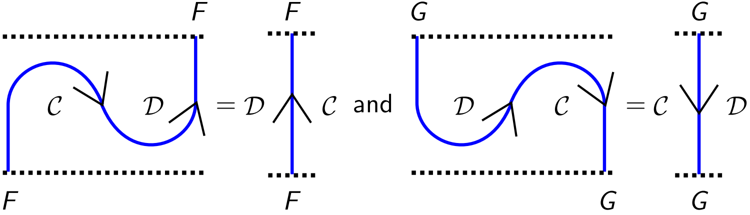


Adjointness is just straightening of the strings

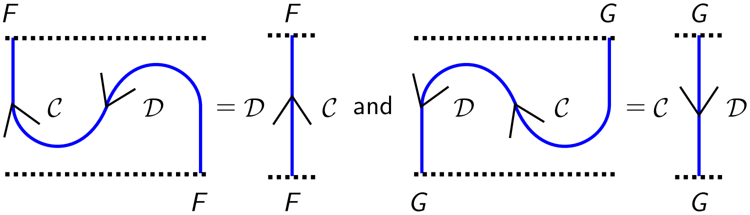


Biadjoint functors = Isotopies

If F is also the right adjoint of G, then the picture gets topological. Biadjointness is just straightening of the strings! First left



then right



Biadjoint functors in “nature” (do not ask me for details...)

- Categories of modules over finite dimensional symmetric algebras and their derived counterparts have plenty of built-in biadjoint functors (tensoring with certain bimodules).
- Prominent examples are finite groups and induction and restriction functors between them.
- Various categories arising in representation theory of Hecke algebras and category $\mathcal{O}$ admit lots of biadjoint functor. For example translation functors and Zuckerman functors.
- Every (extended) TQFT $\mathfrak{F}\colon \mathbf{Cob}^{n+2} \to \mathbf{Vec}^2$ gives a bunch of biadjoint functors: $(\mathfrak{F}(M), \mathfrak{F}(\tau(M)))$ for any $n+1$ manifold M where τ flips M.
- Prominent examples come from commutative Frobenius algebras for $n = 2$, Witten-Reshetikhin-Turaev TQFT's for $n = 3$, Donaldson-Floer for $n = 4$, and way more...
- Other fancy stuff like Fukaya-Floer categories, derived categories of constructible sheaves on flag varieties...

The Iwahori-Hecke algebra (for me $n = 3$ is enough)

Let us fix $n = 3$. Then the group ring of the symmetric group $\mathbb{Q}[S_3]$ has two generators s_1, s_2. They satisfy

$$s_1^2 = 1 = s_2^2 \quad \text{and} \quad s_1 s_2 s_1 = s_2 s_1 s_2.$$

Iwahori: The Hecke algebra $H_3 = H[S_3]$ is a q-deformation of $\mathbb{Q}[S_3]$.

Definition/Theorem(Iwahori 1965)

The Hecke algebra H_3 has generators T_1, T_2 and relations

$$T_1^2 = (q-1)T_{1,2} + q = T_2^2 \quad \text{and} \quad T_1 T_2 T_1 = T_2 T_1 T_2.$$

The classical limit $q \to 1$ gives $\mathbb{Q}[S_3]$.

Nowadays Hecke algebras à la Iwahori appear “everywhere”, e.g. low dimensional topology, combinatorics, representation theory of $\mathfrak{gl}_n$ etc.

Idempotents are better!

Recall that primitive idempotents $e_j \in A$ in any finite dimensional $\mathbb{Q}$-algebra A give rise to Ae_j which is indecomposable.

The group algebra $\mathbb{Q}[S_3]$ admits "idempotents": $i_1 = 1 + s_1$ and $i_2 = 1 + s_2$, because they satisfy

$$i_1^2 = 2i_{1,2} = i_2^2 \ \text{ and } \ i_1 i_2 i_1 + i_2 = i_2 i_1 i_2 + i_1.$$

For the Hecke algebra: Set $t = \sqrt{q}$ and define $b_{1,2} = t^{-1}(1 + T_{1,2})$ (we see the Hecke algebra over $\mathbb{Q}[t, t^{-1}]$ now).

The b_1, b_2 satisfy

$$b_1^2 = (t + t^{-1})b_{1,2} = b_2^2 \ \text{ and } \ b_1 b_2 b_1 + b_2 = b_2 b_1 b_2 + b_1.$$

Only positive coefficients? Suspicious...

Bimodules do the job?

Take $R = \mathbb{Q}[X_1, X_2, X_3]$ (with degree of $X_i = 2$) and define the $s_{1,2}$-invariants as

$$R^{s_1} = \{p(X_1, X_2, X_3) \in R \mid p(X_1, X_2, X_3) = p(X_2, X_1, X_3)\}$$

and

$$R^{s_2} = \{p(X_1, X_2, X_3) \in R \mid p(X_1, X_2, X_3) = p(X_1, X_3, X_2)\}.$$

For example $X_1 + X_2 \in R^{s_1}$, but $X_1 + X_2 \notin R^{s_2}$.

The algebra R is a (left and right) $R^{s_{1,2}}$-module. Thus,

$$B_1 = R \otimes_{R^{s_1}} R\{-1\} \quad \text{and} \quad B_2 = R \otimes_{R^{s_2}} R\{-1\}$$

are R-bimodules. Write short $B_{\underline{ij}}$ for $B_i \otimes_R B_j$. Funny observation ($i = 1, 2$):

$$B_{\underline{ii}} \cong B_{\underline{i}}\{+1\} \oplus B_{\underline{i}}\{-1\} \quad \text{and} \quad B_{\underline{121}} \oplus B_{\underline{2}} \cong B_{\underline{212}} \oplus B_{\underline{1}}.$$

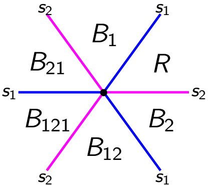
We have seen this before...

The combinatoric of S_3

The bimodule world: Take tensor products $B_{\underline{\mathbf{i}}}$ of the B_i's.
The atoms of the bimodules world are the indecomposables: All M such that $M \cong M_1 \oplus M_2$ implies $M_{1,2} \cong 0$.

We have $B_\emptyset = R$, $B_1 = B_{\underline{1}}$, $B_2 = B_{\underline{2}}$, $B_{\underline{12}} = B_{12}$ and $B_{\underline{21}} = B_{21}$ as atoms, but

$$B_{\underline{121}} \cong B_1 \oplus R \otimes_{R^{S_3}} R\{-3\} \quad \text{and} \quad B_{\underline{212}} \cong B_2 \oplus R \otimes_{R^{S_3}} R\{-3\}$$



and $B_{121} = B_{212} = R \otimes_{R^{S_3}} R\{-3\}$ is indecomposable.

There are exactly as many indecomposables as elements in S_3. Suspicious...

What is a morphism of elements of H_3?

Definition(Soergel 1992)

Define $\mathcal{SC}(3)$ to be the category with the following data:

- Objects are (shifted) direct sums $\oplus$ of tensor products $B_{\underline{\mathbf{i}}}$ of B_i's.
- Morphisms are matrices of (graded) bimodule maps.

Theorem(Soergel 1992)

$\mathcal{SC}(3)$ categorifies H_3. The indecomposables categorify the Kazhdan-Lusztig basis elements of H_3.

Morally: $\mathcal{SC}(3)$ is the categorical analogon of H_3. The morphisms in $\mathcal{SC}(3)$ are invisible in H_3.

Wait: What do you mean by categorify?

(Split) Grothendieck group

If you have a suitable category $\mathcal{C}$, then we can easily collapse structure by totally forgetting the morphisms:

The (split) Grothendieck group $K_0^{\oplus}(\mathcal{C})$ of $\mathcal{C}$ has isomorphism classes $[M]$ of objects $M \in \mathrm{Ob}(\mathcal{C})$ as elements together with

$[M_0] = [M_1] + [M_2] \Leftrightarrow M_0 \cong M_1 \oplus M_2, [M_1][M_2] = [M_1 \otimes M_2]$ and $[M\{s\}] = t^s[M]$.

This is a $\mathbb{Z}[t, t^{-1}]$-module.

Example

We have

$$K_0^{\oplus}(\mathbb{Q}\text{-}\mathbf{Vect}_{\mathrm{gr}}) \xrightarrow{\cong} \mathbb{Z}[t, t^{-1}], \qquad [\mathbb{Q}\{s\}] \mapsto t^s \cdot 1.$$

The whole power of linear algebra is forgotten by going to $K_0(\mathbb{Q}\text{-}\mathbf{Vect})_{\mathrm{gr}}$.

Categorification?

We have two functors $F_1 = B_1 \otimes_R \cdot$ and $F_2 = B_2 \otimes_R \cdot$. These are additive endofunctors of $\mathcal{SC}(3)$. Thus, the introduce an action $[F_i]$ on $K_0^{\oplus}(\dot{\mathcal{SC}}(3))$. We have a commuting diagram (we ignore to tensor with $\mathbb{Q}(t)$)

$$\begin{tikzcd}
K_0^{\oplus}(\dot{\mathcal{SC}}(3)) \arrow[r, "{[F_i]}"] \arrow[d, "\phi"', "\cong"] & K_0^{\oplus}(\dot{\mathcal{SC}}(3)) \arrow[d, "\cong"', "\phi"] \\
H_3 \arrow[r, "\cdot b_i"'] & H_3.
\end{tikzcd}$$

Thus, the functors F_1, F_2 categorify the multiplication in H_3! Said otherwise: They categorify the action of H_3 on itself.

Moreover, the indecomposables give a good basis of H_3.

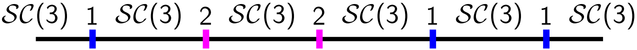
The speaker is lost...

The speaker is lost: That was too abstract. Can we understand this topological?

Observation(Elias-Khovanov 2009)

The functors F_1 and F_2 are selfadjoint! Thus, there is a stringy calculus for $\mathcal{SC}(3)$.

As before: Well denote compositions like $F_1F_2F_2F_1F_1$ by



or simplified

Think: Apply $F_1F_2F_2F_1F_1$ to R on the right.

Generators

We have the following one color generators:

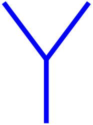
$\deg = +1$

$F_1 \Rightarrow \mathrm{id}$

$\deg = +1$

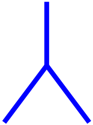
$\mathrm{id} \Rightarrow F_1$

$\deg = -1$



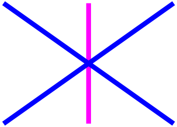
$F_1 \Rightarrow F_1 F_1$

$\deg = -1$



$F_1 F_1 \Rightarrow F_1$

$\deg = 0$



$F_1 F_2 F_1 \Rightarrow F_1 F_2 F_1$

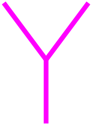
$\deg = +1$

$F_2 \Rightarrow \mathrm{id}$

$\deg = +1$

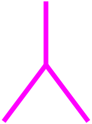
$\mathrm{id} \Rightarrow F_2$

$\deg = -1$



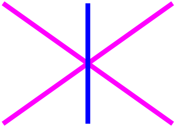
$F_2 \Rightarrow F_2 F_2$

$\deg = -1$



$F_2 F_2 \Rightarrow F_2$

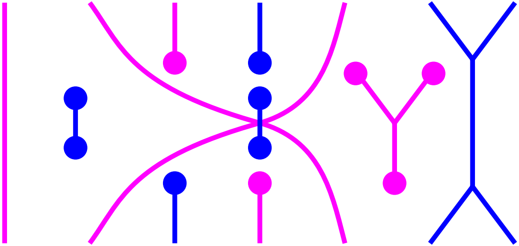
$\deg = 0$



$F_2 F_1 F_2 \Rightarrow F_2 F_1 F_2$

Exempli gratia

$$\deg = +8$$

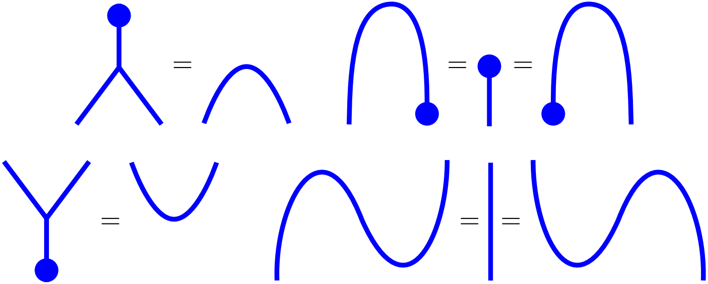


$$F_2 F_2 F_1 F_2 F_2 F_1 F_1 \Rightarrow F_2 F_2 F_2 F_1 F_2 F_1 F_1$$

These Soergel diagrams can get very complicated, but this is an information completely invisible in H_3.

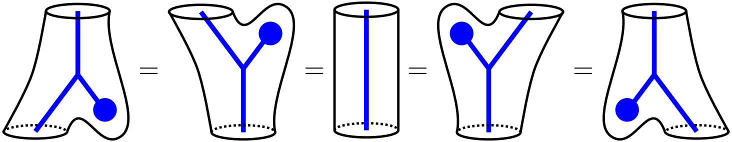
Some relations - Part 1

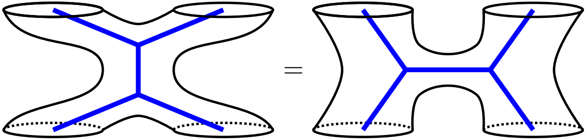
We need some additional relations to make the story work. Some are combinatorial (which we do not recall), but, due to biadjointness, some are topological.



Some relations - Part 2

Some are really topological: There is more than planar isotopies. The functors F_1 and F_2 are Frobenius. This gives





The diagram category suffices

Definition(Elias-Khovanov 2009)

Define $\mathcal{DC}(3)$ to be the category with the following data:

- Objects are (shifted) formal direct sums $\oplus$ of sequences of the form $F_2 F_2 F_2 F_1 F_2 F_1 F_1$.
- Morphisms are matrices of (graded) Soergel diagrams module the local relations.

Theorem(Elias-Khovanov 2009)

There is an equivalence of graded, monoidal, $\mathbb{Q}$-linear categories

$$\mathcal{DC}(3) \cong \mathcal{SC}(3).$$

Conclusion: The (seemingly very rigid) Hecke algebra H_3 has an overlying topological counterpart!

Some upshots of Elias-Khovanov's approach

- No restriction to S_3: Any Coxeter system works.
- Diagrammatic categorification is "low tech". Playing with diagrams is fun, easy and the topological flavour gives new insights. For example, Elias and Williamson's algebraic proof that the Kazhdan-Lusztig polynomials have positive coefficients for arbitrary Coxeter systems was discovered using the diagrammatic framework.
- New insights into topology:
  - Elias used the topological behaviour to give a new categorification of the Temperley-Lieb algebra.
  - Rouquier produced a braid group action on (chain complexes of) Soergel diagrams. This is functorial: It also talks about braid cobordisms (these live in dimension 4!).
  - Rouquier's results can be extended to give HOMFLY-PT homology. This still mysterious homology is related to knot Floer homology.
- More is to be expected!

Non-associative=bad

Recall that $\mathfrak{sl}_2$ is $[\cdot,\cdot]$-spanned by $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Non-associative: Take $\mathbf{U}(\cdot)\colon \mathbf{LieAlg} \to \mathbf{Asso}\mathbb{Q}\textbf{-}\mathbf{Alg}$ which is the left adjoint of $[\cdot,\cdot]\colon \mathbf{Asso}\mathbb{Q}\textbf{-}\mathbf{Alg} \to \mathbf{LieAlg}$. Thus, the universal envelope $\mathbf{U}(\mathfrak{sl}_2)$ is the free, associative $\mathbb{Q}$-algebra spanned by symbols E, F, H, H^{-1} modulo

$$HH^{-1} = H^{-1}H = 1, \quad HE = EH \text{ and } HF = FH.$$
$$EF - FE = H.$$

By magic: $\mathfrak{sl}_2$-**Mod** $\cong$ $\mathbf{U}(\mathfrak{sl}_2)$-**Mod**.

Naively quantize: $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_2) = \mathbf{U}_{\mathrm{q}}$ is the free, associative $\mathbb{Q}(q)$-algebra spanned by symbols E, F and K, K^{-1} (think: $K = q^H, K^{-1} = q^{-H}$) modulo

$$KK^{-1} = K^{-1}K = 1, \quad EK = q^2 KE \text{ and } KF = q^{-2} FK.$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \quad \left(\text{think: } \frac{q^H - q^{-H}}{q - q^{-1}} \xrightarrow{q \to 1} H\right).$$

What are the atoms?

Fact of life

If q is an indeterminate, then $\mathbf{U}_q$ has the "same" representation theory as $\mathfrak{sl}_2$. In particular, $\mathbf{U}_q$-$\mathbf{Mod}_{\mathrm{fin}}$ is semisimple: Atoms are the irreducibles.

If $q^{\ell} = 1$, then this totally fails: $\mathbf{U}_q$-$\mathbf{Mod}_{\mathrm{fin}}$ is far away to be semisimple.

Why do we want to study something so nasty?

- Magic: Many similarities to the representation theory of a corresponding almost simple, simply connected algebraic group G modulo p.
- Many similarities to the representation theory of a corresponding affine Kac-Moody algebra.
- It provides ribbon categories (link invariants) which can be "semisimplified" to provide modular categories ($2+1$-dimensional TQFT's).

It turns out that the "right" atoms are the so-called indecomposable $\mathbf{U}_q$-tilting modules. The corresponding category $\mathfrak{T}$ is what we want to understand.

Translation functors

Principle(Bernstein-Gelfand-Gelfand 1970)

Do not study representations explicitly: That is too hard. Study the combinatorial and functorial behaviour of their module categories!

So let us adopt the BGG principle from category $\mathcal{O}$!

In particular, there are two endofunctors Θ_s, Θ_t of $\mathfrak{T}_\lambda$ (there is a decomposition of $\mathfrak{T}$ into blocks $\mathfrak{T}_\lambda$) called translation through the s, t-wall. These are selfadjoint Frobenius functors with combinatorial behaviour governed by the ∞-dihedral group $D_\infty = \{s, t \mid s^2 = 1 = t^2\}$:

$$\Theta_s\Theta_s \cong \Theta_s \oplus \Theta_s \quad \text{and} \quad \Theta_t\Theta_t \cong \Theta_t \oplus \Theta_t.$$

We have seen something similar before: There should be a diagram category (inspired by the corresponding one for $H(D_\infty)$) that governs $\mathfrak{T}$ and p**End**$(\mathfrak{T})$.

Sorry: No tenure means I have to stress my own results

Definition/Theorem(Elias 2013)

There is a diagram category $\mathfrak{D}(\infty)$ that categorifies $H(D_\infty)$ (that is what we are looking for!). The indecomposables categorify the Kazhdan-Lusztig basis elements of $H(D_\infty)$.

Definition/Theorem

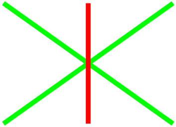
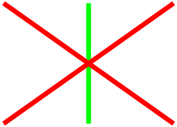
There is are diagram categories $\mathfrak{Q}\mathfrak{D}(\infty)$ and $\mathbf{Mat}^{\mathrm{fs}}_{\infty}(\widehat{\mathfrak{Q}\mathfrak{D}}(\infty))_c$ for $\mathfrak{T}$ and p$\mathbf{End}(\mathfrak{T})$. The diagram categories are naturally graded which introduce a non-trivial grading on $\mathfrak{T}$ and p$\mathbf{End}(\mathfrak{T})$.

We have $K_0^{\oplus}(\mathfrak{T}^{\mathrm{gr}}_{\lambda}) \cong \mathcal{B}_\infty$: Thus, $\mathfrak{T}^{\mathrm{gr}}_{\lambda}$ categorifies the Burau representation $\mathcal{B}_\infty$ of the braid group B_∞ in ∞-strands (cut-offs are possible). The action of B_∞ is categorified using certain chain complexes of truncations of Θ_s, Θ_t.

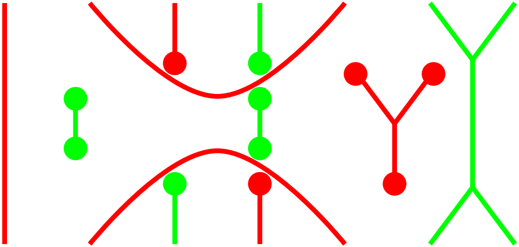
We have $K_0^{\oplus}(\mathrm{p}\mathbf{End}(\mathfrak{T}^{\mathrm{gr}}_{\lambda})) \cong \overline{TL}^q_\infty$: Thus, p$\mathbf{End}(\mathfrak{T}^{\mathrm{gr}}_{\lambda})$ categorifies (a certain summand of) the Temperley-Lieb algebra in ∞-strands (cut-offs are possible).

Elias' dihedral cathedral

The category $\mathfrak{D}(\infty)$ is almost as before, but easier: No relations among the "colors" red s and green t:

Neither  nor 

Pictures look like



Our $\mathfrak{Q}\mathfrak{D}(\infty)$ looks similar plus some extra relations.

Open connections

- Question: What is the non-trivial grading (purely a root of unity phenomena) trying to tell us about the link and 3-manifold invariants deduced from $\mathfrak{T}$?
- Question: Similarly, what is the non-trivial grading (purely a root of unity phenomena) trying to tell us about algebraic groups modulo p?
- We argue that each block $\mathfrak{T}_{\lambda}^{\mathrm{gr}}$ separately can be used to obtain invariants of links and tangles - there are very explicit relations to (sutured) Khovanov homology and bordered Floer homology.
- Hence, each block $\mathfrak{T}_{\lambda}^{\mathrm{gr}}$ separately yields information about link and tangle invariants in the non-root of unity case, while the ribbon/modular structure of $\mathfrak{T}$ yields the Witten-Reshetikhin-Turaev invariants. Question: What is going on here?
- As in the $H(S_n)$ case: Question: Is there a "cobordism" theory that explains the grading and the Frobenius structure topological?

There is still much to do...

Thanks for your attention!