(Singular) TQFTs, link homologies and Lie theory 1

Or: a story of foams and $\mathcal{O}$

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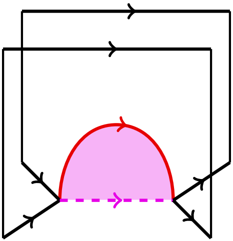

Joint work (in progress) with Michael Ehrig and Catharina Stroppel

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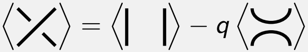
Let L_D be a diagram of an oriented link $L \subset S^3$. Let $[2] = q + q^{-1}$.

Definition/Theorem(Jones 1984, Kauffman 1987)

Define $\langle L_D \rangle \in \mathbb{Z}[q, q^{-1}]$ recursively (and locally):

-  (recursion rule 1).
- $\left\langle L'_D \dot{\cup} \bigcirc \right\rangle = [2] \langle L'_D \rangle$ (recursion rule 2).
- $\langle \text{empty link diagram} \rangle = 1$ (normalization).

Then set

The polynomial $J(L)$ is an invariant of links.

The Jones revolution

Jones discovery revolutionized low dimensional topology:

Before Jones: Lack of link polynomials;

After Jones: (too) many link polynomials.

Quantum topology was born! Some mentionable developments:

- The Jones polynomial single-handed solved open problems in knot theory.
- Shortly after Jones several "friends" of the Jones polynomial were found. In particular, one for each semisimple Lie algebra $\mathfrak{g}$ and each "coloring" with representations of $\mathfrak{g}$ (Reshetikhin-Turaev).
- Connections to 3-dimensional quantum Chern-Simons theory and $2+1$-dimensional TQFTs were discovered (Witten-Reshetikhin-Turaev). There are also connections to the volume of hyperbolic 3-manifolds (Kashaev). Thus, the Jones polynomial "knows" 3-dimensional topology.
- Many more connections (beyond QFT's and topology).

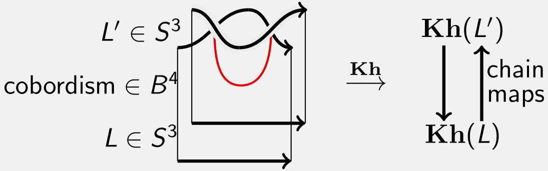
Its categorification

Theorem(Khovanov 1999)

There is a chain complex $\mathbf{Kh}(\cdot)$ of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces whose homotopy type is a link invariant. Its graded Euler characteristic gives the Jones polynomial.

Theorem(Khovanov 1999, Bar-Natan 2004)

Up to a sign: the $\mathbf{Kh}(\cdot)$ can be extended to a functor from the category of links in S^3 to the category chain complexes of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces.



Morally: $J(L)/\mathbf{Kh}(L) \leftrightarrow \chi(X)/H_*(X)$.

Khovanov homology "should know" 4-dimensional topology/qCS.

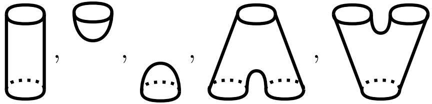
History repeats itself

- Shortly after Khovanov several "friends" of $\mathbf{Kh}$ were discovered.
- Rasmussen obtained from the homology an invariant that "knows" the slice genus and used it to give a combinatorial proof of the Milnor conjecture. In particular, Rasmussen/Gompf give a way to combinatorial construct exotic $\mathbb{R}^4$. This is a big hint that 4-dimensional smooth topology is encoded in $\mathbf{Kh}$.
- Kronheimer and Mrowka showed that Khovanov homology detects the unknot by relating $\mathbf{Kh}$ to knot Floer homology (this is still an open question for the Jones polynomial). Hence, $\mathbf{Kh}$ relates to symplectic geometry.
- Several other connections of Khovanov homology are known nowadays.
- Before I forget: $\mathbf{Kh}$ is a strictly stronger link invariant.

One of our goals is to understand Khovanov homology algebraically. Similarly for its relation to 4-dimensional topology (the first step here is to understand functoriality, but we come back to this later on).

Topological quantum field theories

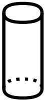
Roughly: let 2-$\mathbf{Cob}$ be the category of 2-dimensional cobordisms:

Objects : $\coprod_{\text{finite}} \bigcirc$; Morphisms : 

Composition in 2-$\mathbf{Cob}$ is gluing. A Witten-type 2-TQFT $\mathcal{T}$ is a functor

$$\mathcal{T}\colon 2\text{-}\mathbf{Cob} \to \mathbb{C}\text{-}\mathbf{Vect},$$

$$\emptyset \mapsto \mathbb{C}, \quad \bigcirc \mapsto V, \quad \bigcirc \,\dot{\cup}\, \bigcirc \mapsto V \otimes V, \quad \text{etc.}$$

 $\mapsto \mathrm{id}\colon V \to V$,  $\mapsto \mathrm{unit}\colon \mathbb{C} \to V$, etc.

which satisfies the Atiyah-Segal axioms (we do not really need them and do not recall them here, but they "take care that gluing etc. is well-behaved").

TQFTs "are" Frobenius algebras

Recall that a finite-dimensional Frobenius algebra A is a $\mathbb{C}$-vector space with a multiplication m, a comultiplication Δ, a unit ι and a counit ε plus some relations.

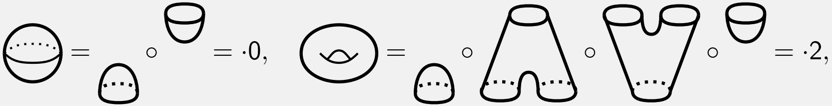
Theorem(Folklore, Dijkgraaf 1989, Abrams 1996)

There is a $1:1$ correspondence (with sets regarded up to isomorphisms)

$$\{\text{2-dimensional TQFTs}\} \leftrightarrow \{\text{finite-dimensional, commutative Frobenius algebras}\}.$$

Example

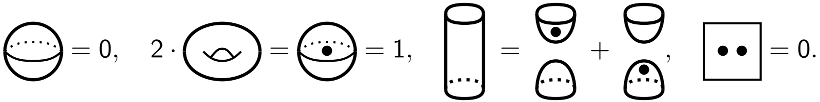
Take $A = \mathbb{C}[X]/(X^2)$ with $\Delta(1) = 1 \otimes X + X \otimes 1, \Delta(X) = X \otimes X$ and $\varepsilon(1) = 0, \varepsilon(X) = 1$. Then the associated 2-TQFT $\mathcal{T}_A$ satisfies some "relations", e.g. (dropping $\mathcal{T}(\cdot)$ everywhere) the sphere and torus relation



From TQFTs to $\mathbb{C}$-linear cobordism categories

Let $\mathcal{C} = 2\text{-}\mathbf{Cob}_{\mathbb{C}}$ be the $\mathbb{C}$-linear category whose objects are $\coprod_{\text{finite}} \bigcirc$ and:

- The hom spaces $\mathrm{Hom}_{\mathcal{C}}(\text{circles}, \text{circles})$ is the $\mathbb{C}$-vector whose basis are all (embedded) cobordisms between these circles modulo relations.
- The relations are isotopies and the (local) relations: sphere, torus, neck cutting and the cyclotomic relation

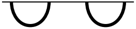
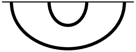
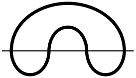


Example

We have $\mathbb{C}$-bases  of $\mathrm{Hom}_{\mathbb{C}}(\emptyset, \bigcirc)$ and  of $\mathrm{Hom}_{\mathbb{C}}(\bigcirc, \emptyset)$.

A "cobordism algebra" - the $\mathbb{C}$-vector space structure

Fix some $m \in \mathbb{Z}_{\geq 0}$. Let u and v be cup diagrams with m top boundary points, and denote by * the horizontal flip and by uv^* the stacked diagram:

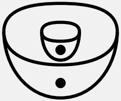
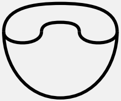
$u =$  , $v =$  , $uv^* =$ 

(we also allow internal closed circles, but we ignore them today). Let

$$_u(\mathbf{W}_m)_v = \{\text{all } \mathbb{C}\text{-linear combinations of cobordisms in } \mathcal{C} \text{ from } \emptyset \text{ to } uv^*\}.$$

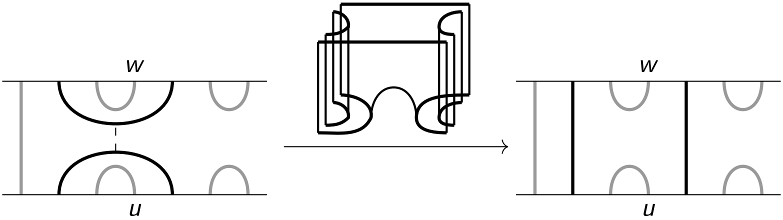
Example

The following are elements of $_u(\mathbf{W}_2)_u$ respectively of $_v(\mathbf{W}_2)_v$ and $_u(\mathbf{W}_2)_v$:

  $\in {}_u(\mathbf{W}_2)_u$,  $\in {}_v(\mathbf{W}_2)_v$,  $\in {}_u(\mathbf{W}_2)_v$.

A "cobordism algebra" - the multiplication

Define a multiplication iteratively ${}_u(\mathbf{W}_m)_v \otimes {}_v(\mathbf{W}_m)_w \to {}_u(\mathbf{W}_m)_w$ via "surgery":



(the multiplication is defined to be zero if the middle pictures do not match). This gives $\tilde{\mathbf{W}}_m = \bigoplus_{u,v} {}_u(\mathbf{W}_m)_v$ the structure of an associative, unital, finite-dimensional Frobenius algebra (this is not obvious!).

Example

We have $\tilde{\mathbf{W}}_1 \cong \mathbb{C}[X]/(X^2)$. The isomorphism is

 $\mapsto 1,$  $\mapsto X.$

A “cobordism algebra” - the grading

$\tilde{\mathbf{W}}_m$ has a natural grading: the degree of its elements (cobordisms) is given by (minus) the topological Euler characteristic $\chi(\cdot)$. Define the graded version:

$$\mathbf{W}_m = \tilde{\mathbf{W}}_m\{m\}.$$

Example

If $m = 1$, then we have to shift by 1. Thus,

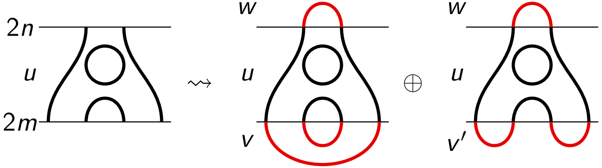
$\deg\Big($  $\Big) = -\chi\Big($  $\Big) + 1 = 0,$

$\deg\Big($  $\Big) = -\chi\Big($  $\Big) + 1 = -\chi\Big(\frac{1}{2}$  $\Big) + 1 = 2.$

Thus, the algebra $\mathbf{W}_1 \cong \mathbb{C}[X]/(X^2)$ is graded with X being of degree 2.

The representation theory is also topological

Fix $m, n \in \mathbb{Z}_{\geq 0}$ and a planar matching u with $2m$ bottom/$2n$ top boundary points:



The $\mathbf{W}_m$-$\mathbf{W}_m$-bimodule $\mathbf{W}(u)$ is the $\mathbb{C}$-vector space obtained from u by closing the bottom and top in all possible planar ways (denote these by vuw), and then consider $\bigoplus_{v,w} \mathrm{Hom}_{\mathcal{C}}(\emptyset, vuw)$ with the induced action (saddles!).

Surprisingly there are no other bimodules:

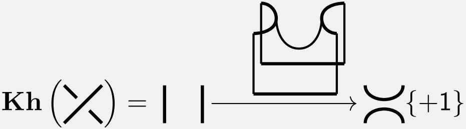
Theorem(Brundan-Stroppel 2008)

All finite-dimensional, graded, bi-projective $\mathbf{W}_m$-$\mathbf{W}_n$-bimodules are (up to isomorphism) of the form $\mathbf{W}(u)$.

Its origin: tangle/link homology

Definition/Theorem(Khovanov 2001)

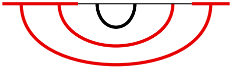
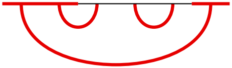
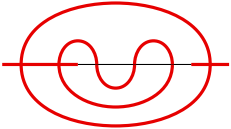
Given a tangle T_m^n with $2m$ bottom and $2n$ top boundary components, we can associate to it a chain complex $\mathbf{Kh}(T_m^n)$ of $\mathbf{W}_m$-$\mathbf{W}_n$-bimodules via the local rule (the whole complex is obtained via tensoring)



The chain homotopy equivalence class of $\mathbf{Kh}(T_m^n)$ is an invariant of the tangle T_m^n. This can be extended (up to a sign) to a functor from the category of tangles to the category of chain complexes of $\mathbf{W}_m$-$\mathbf{W}_n$-bimodules.

Marking certain cobordisms

We mark diagrams with "platforms" (the colors are only for illustration):

$u =$  , $v =$  , $uv^* =$ 

Let $\overline{\mathbf{W}}^k_{m-k}$ be the subalgebra of $\mathbf{W}_m$ with $m-k$-marked first points and k-marked right points. Define $\mathbf{K}^k_{m-k} = \overline{\mathbf{W}}^k_{m-k}/$ideal with the ideal generated by

 (and similar turnbacks), dotted cobordisms touching the marked parts.

Remark

Everything from before works for $\mathbf{K}^k_{m-k}$ as well and is still topological in nature.

Categorification of tensor products

Theorem(Chen-Khovanov 2006)

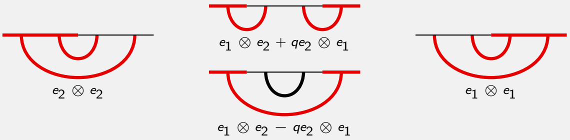
Set $\mathbf{K}_m = \bigoplus_{k=0}^{m} \mathbf{K}_{m-k}^{k}$. Let $\mathbf{K}_m\text{-pMod}$ be the category of finite-dimensional, graded, bi-projective $\mathbf{K}_m$-bimodules. Then $\mathbf{K}_m\text{-pMod}$ categorifies the m-fold tensor product $(\mathbb{C}_q^2)^{\otimes m}$ of the vector representation $\mathbb{C}_q^2 = \langle e_1, e_2 \rangle_{\mathbb{C}(q)}$ of quantum $\mathfrak{sl}_2$. Here $\mathbf{K}_{m-k}^{k}$ categorifies the $(m-2k)$-th weight space of $(\mathbb{C}_q^2)^{\otimes m}$.

This categorification is based: certain indecomposable bi-projective modules attached to marked arc diagrams categorify the canonical basis of $(\mathbb{C}_q^2)^{\otimes m}$.

Example

Let $m = 2$. Then $k = 0, 1, 2$ and we have:

$\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$:



Category $\mathcal{O}$ can do the same

Take the following Cartan, Borel and parabolic in $\mathfrak{gl}_m$:

$$\mathfrak{h} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \mathfrak{b} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad \mathfrak{p}^k_{m-k} = \mathfrak{b} + \begin{pmatrix} \mathfrak{gl}_{m-k} & 0 \\ 0 & \mathfrak{gl}_k \end{pmatrix}$$

Denote by $\mathcal{O}_0^{m-k,k}$ the corresponding full subcategory of $\mathcal{O}_0$ for $\mathfrak{gl}_m$.

Theorem(Bernstein-Frenkel-Khovanov 1999)

$\mathcal{O}_0^m = \bigoplus_{k=0}^m \mathcal{O}_0^{m-k,k}$ categorifies the m-fold tensor product $(\mathbb{C}^2)^{\otimes m}$ of the vector representation $\mathbb{C}^2 = \langle e_1, e_2 \rangle_{\mathbb{C}}$ of $\mathfrak{sl}_2$. Here $\mathcal{O}_0^{m-k,k}$ categorifies the $(m-2k)$-th weight space of $(\mathbb{C}^2)^{\otimes m}$.

Theorem(Frenkel-Khovanov-Stroppel 2005)

Similarly for graded category $\mathcal{O}$ and the quantum set-up.

So what is the connection to $\mathbf{K}_m$?

A topologically version of category $\mathcal{O}$

The following are based on work of Braden:

Theorem(Brundan-Stroppel 2008)

We have

$$\mathbf{K}_m\text{-pMod} \cong \mathcal{O}_0^m, \quad \mathbf{K}_{m-k}^k\text{-pMod} \cong \mathcal{O}_0^{m-k,k}.$$

Let $\mathrm{pi}\mathcal{O}_0^m$ denote the subcategory of $\mathcal{O}_0^m$ for $\mathfrak{gl}_m$ consisting of projective-injective modules (similar for $\mathcal{O}_0^{m-k,k}$). Then

$$\mathbf{W}_m\text{-pMod} \cong \mathrm{pi}\mathcal{O}_0^m, \quad \mathbf{W}_{m-k}^k\text{-pMod} \cong \mathrm{pi}\mathcal{O}_0^{m-k,k}.$$

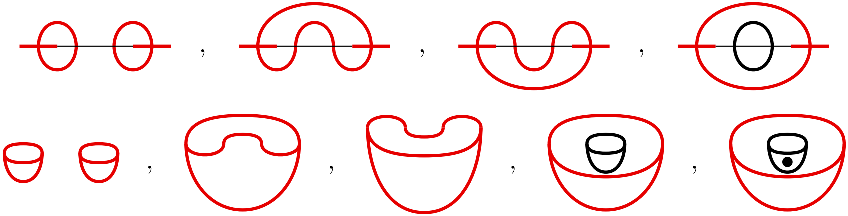
(These equivalences are explicit and they can also be done for all integral blocks).

Remark

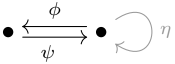
Brundan and Stroppel's equivalences give a way to topologically define graded category $\mathcal{O}$. The grading is the Euler characteristic of cobordisms.

Exampli gratia

The algebra $\mathbf{K}^1_{2-1}$ has diagrams and basis



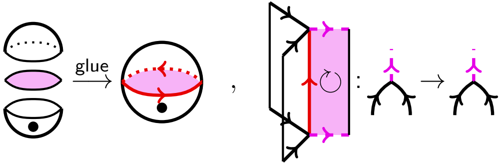
of degrees $0, 1, 1, 0, 2$. Thus, $\mathbf{K}^1_{2-1}$ is isomorphic to the quiver algebra



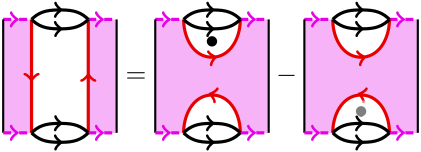
with $\phi\psi = 0$ (and $\eta = \psi\phi$). This quiver is the description of $\mathcal{O}_0$ for $\mathfrak{gl}_2$.

Singular TQFTs and foams

Instead of 2-dimensional cobordisms, one can (and should!) use a category $p\mathcal{F}$ of singular surfaces obtained via gluing of surfaces (called pre-foams):



Again, cook-up a singular functor TQFT $\mathcal{T}\colon p\mathcal{F} \to \mathbb{C}\text{-}\mathbf{Vect}$ and find "relations in its kernel", e.g. (finding these is the hard part):

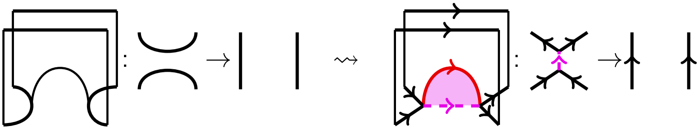


Then we can play the same game: form a $\mathbb{C}$-linear category $\mathfrak{F}$ of foams and an algebra $\mathbf{W}_{\vec{k}}$, called web algebra, and study its representation theory. Again, everything connected to $\mathbf{W}_{\vec{k}}$ (gradings, modules etc.) will be topological gadgets.

The state of the arts

What we know by now:

- There is a $\mathfrak{sl}_M/\mathfrak{gl}_M$-version of Khovanov's arc algebra (which is the case $M = 2$). Again, one uses "saddles" for the multiplication:



- The foamy $M = 2$ version gives functorial Khovanov homology.
- The foamy story caries a natural 2-action of the KL-R 2-category.

What needs to be done (and is partially work in progress):

- The foamy version should give functorial Khovanov $\mathfrak{sl}_M/\mathfrak{gl}_M$-homology.
- The right ideal needs to be identified such that the foamy $M = 3$ version categorifies the quantum $\mathfrak{sl}_M/\mathfrak{gl}_M$ tensor product $\mathbb{C}_q^M \otimes \cdots \otimes \mathbb{C}_q^M$.
- Relate everything to an M-block parabolic of category $\mathcal{O}$.
- Other open issues, e.g. foams in types B, C, D.

There is still much to do...

Thanks for your attention!