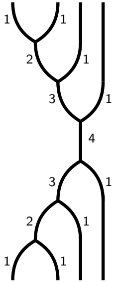
$\mathbf{U}_q(\mathfrak{sl}_n)$ diagram categories via q-Howe duality

Or: "Howe" to make diagrammatic categories work!

Daniel Tubbenhauer

$$\mathcal{JW}_4 = \frac{1}{[4]!}$$



Joint work with David Rose

February 2015

1. $\mathfrak{sl}_2$-spider and representation theory
   - Graphical calculus via Temperley-Lieb diagrams
   - The $\mathfrak{sl}_2$-spider is representation theory

2. Its cousins: The $\mathfrak{sl}_n$-spiders
   - The $\mathfrak{sl}_n$-spiders and representation theory
   - Proof? Quantum skew Howe duality!

3. More cousins: The symmetric $\mathfrak{sl}_2$-spider
   - The symmetric $\mathfrak{sl}_2$-spider and representation theory
   - Proof? Quantum symmetric Howe duality!

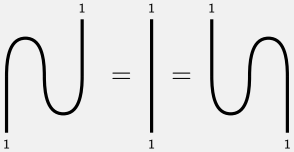
Definition(Rumer-Teller-Weyl 1932)

The $\mathfrak{sl}_2$-web space $W_2(b,t)$ is the free $\mathbb{C}(q)=\mathbb{C}_q$-vector space generated by non-intersecting arc diagrams with b bottom and t top boundary points modulo:

- The circle removal

$$= -q - q^{-1} = -[2]$$

- The isotopy relations



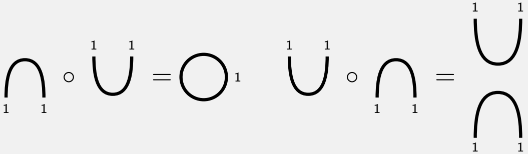
Note that $W_2(b,t)$ is a finite dimensional $\mathbb{C}_q$-vector space!

The $\mathfrak{sl}_2$-spider

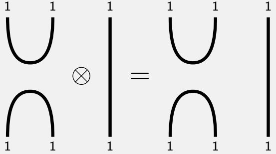
Definition(Kuperberg 1995)

The $\mathfrak{sl}_2$-spider $\mathbf{Sp}(\mathfrak{sl}_2)$ is the monoidal $\mathbb{C}_q$-linear category with:

- Objects are natural numbers and morphisms are $\mathrm{Hom}_{\mathbf{Sp}(\mathfrak{sl}_2)}(k, l) = W_2(k, l)$.
- Composition $\circ$:



- Tensoring $\otimes$:



Quantum enveloping algebras

Recall that $\mathfrak{sl}_2$ is generated by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The elements of its enveloping $\mathbb{C}$-algebra $\mathbf{U}(\mathfrak{sl}_2)$ are polynomials in the symbols $E, F, H^{\pm 1}$ modulo

$$HH^{-1} = H^{-1}H = 1,\ EF - FE = H,\ HE = EH + 2E,\ HF = FH + 2F.$$

The elements of its quantum cousin, the $\mathbb{C}_q$-algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ are polynomials in the symbols $E, F, K^{\pm 1}$ modulo

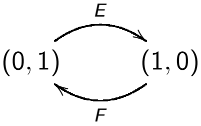
$$KK^{-1} = K^{-1}K = 1,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}},\ KE = q^2 EK,\ KF = q^{-2} FK.$$

Roughly: $K = q^H$ and $\lim_{q \to 1} \mathbf{U}_q(\mathfrak{sl}_2) = \mathbf{U}(\mathfrak{sl}_2)$.

Connection to representation theory

Recall that $\mathbf{U}_q(\mathfrak{sl}_2)$ is generated by $E, F, K^{\pm 1}$.
Let $V = \mathbb{C}_q^2$ the vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$. Morally:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Fact: All irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-modules are summands of $V^{\otimes k}$ for some $k \in \mathbb{N}$.

Let $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge$ be the monoidal, $\mathbb{C}_q$-linear category consisting of:

- Objects are tensor products $V^{\otimes k} = V \otimes \cdots \otimes V$ of finite length and morphisms are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners between these.
- Composition $\circ$ is composition of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Tensoring $\otimes$ is tensoring of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.

Diagrams for intertwiners

Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}\colon V \otimes V \to \mathbb{C}_q \quad \text{and} \quad \mathrm{cup}\colon \mathbb{C}_q \to V \otimes V.$$

Define a functor $\Gamma^2_\wedge\colon \mathbf{Sp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_\wedge$:

- On objects: k is send to $V^{\otimes k} = V \otimes \cdots \otimes V$.
- On morphisms:

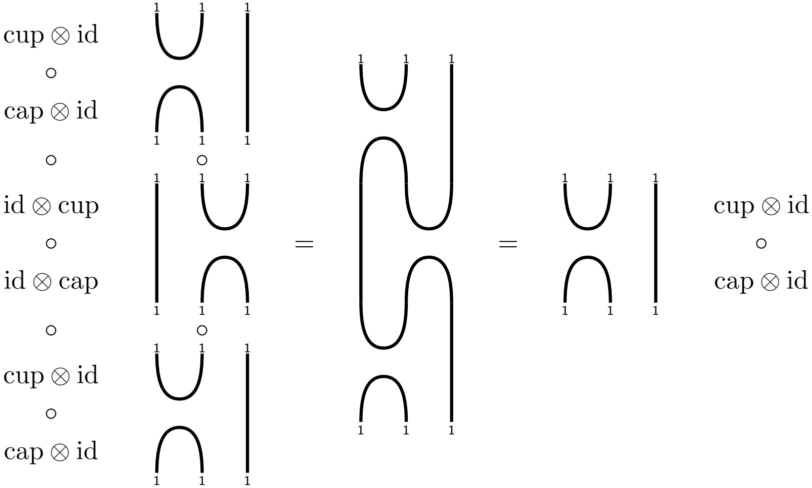
$\mapsto \mathrm{cap}$

$\mapsto \mathrm{cup}$

Theorem(Folklore)

The functor $\Gamma^2_\wedge\colon \mathbf{Sp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_\wedge$ is an equivalence of monoidal categories.

Diagrams are easier. At least for me...



For the $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners? Not obvious: one has to verify this by calculation!

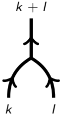
Kuperberg (1995): let us try the same for other $\mathfrak{g}$'s!

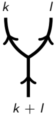
In 1995 Kuperberg rigorously defined "spiders" and introduced spiders for $\mathfrak{sl}_3$, B_2 and G_2. These spiders are diagrammatic categories for $\mathbf{U}_q(\mathfrak{g})$-module categories. His work was very influential: Spiders naturally appear in representation theory, combinatorics, low dimensional topology and algebraic geometry.

- Khovanov and Kuperberg gave a connection to dual canonical bases of $\mathbf{U}_q(\mathfrak{g})$.
- Fontaine, Kamnitzer and Kuperberg identified relations to the geometry of affine Grassmannians via the geometric Satake correspondence.
- Via this, there are relations to affine buildings over these Grassmannians.
- The Reshetikhin-Turaev's invariant of links "live" in spiders.
- Similarly from the Witten-Reshetikhin-Turaev invariants of 3-manifolds.
- $1+1$ or $2+1$-TQFT's and cobordism theories very often bound spiders.
- Via this connections to link homologies and related topics.
- More...

The main step beyond $\mathfrak{sl}_2$: trivalent vertices

A $\mathfrak{sl}_n$-web is an oriented, labeled trivalent graph locally made of

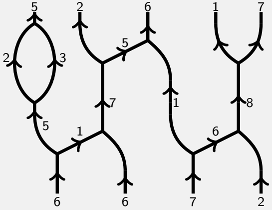
$\mathrm{m}_{k,l}^{k+l} =$ 

$\mathrm{s}_{k+l}^{k,l} =$ 

$k, l, k+l \in \{0, \ldots, n\}$

Plus mirrors and sign issues that we skip today. Ask an expert, aka not me.

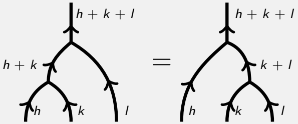
Example($n > 7$)



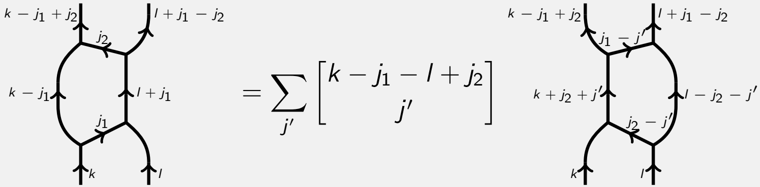
Definition(Cautis-Kamnitzer-Morrison 2012)

The $\mathfrak{sl}_n$-web space $W_n(\vec{k},\vec{l})$ is the free $\mathbb{C}_q$-vector space generated by $\mathfrak{sl}_n$-webs with $\vec{k}$ and $\vec{l}$ at the bottom and top modulo:

- Isotopy and associativity relations



- Others. Most notably the scary square switches:



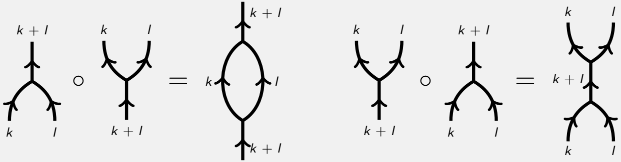
$$= \sum_{j'} \begin{bmatrix} k - j_1 - l + j_2 \\ j' \end{bmatrix}$$

The $\mathfrak{sl}_n$-spider

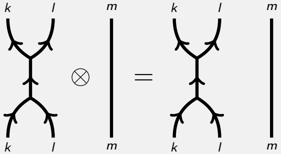
Definition(Cautis-Kamnitzer-Morrison 2012)

The $\mathfrak{sl}_n$-spider $\mathbf{Sp}(\mathfrak{sl}_n)$ is the monoidal $\mathbb{C}_q$-linear category with:

- Objects are $\vec{k} \in \mathbb{Z}^m_{\{0,\dots,n\}}$ and morphisms are $\mathrm{Hom}_{\mathbf{Sp}(\mathfrak{sl}_n)}(\vec{k},\vec{l}) = W_n(\vec{k},\vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



How to produce new representations from old ones

Recall that $\mathbf{U}_q(\mathfrak{sl}_n)$ is generated by $E_i, F_i, K_i^{\pm 1}$ for $i = 1, \dots, n-1$ (modulo some relations).

Note that $\mathbf{U}_q(\mathfrak{sl}_n)$ acts on $V = \mathbb{C}_q^n$ as "matrices". The representation V is called the vector representation of $\mathbf{U}_q(\mathfrak{sl}_n)$.

Question: how to produce new representations from old, known ones?

Taking tensor products produces new representations (usually not irreducible).

Taking alternating tensors $\bigwedge_q^k$, that is

$$\bigwedge\nolimits_q^k \mathbb{C}_q^n = V \otimes \cdots \otimes V / q\text{-symmetric tensors}$$

also works and gives the *k*-th fundamental representations of $\mathbf{U}_q(\mathfrak{sl}_n)$.

Connection to representation theory - yet again

Let $V = \mathbb{C}_q^n$ the vector representation of $\mathbf{U}_q(\mathfrak{sl}_n)$. For $k \in \{0, \dots, n\}$ let $\bigwedge_q^k \mathbb{C}_q^n$ denote the k-th fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representation.

Fact: All irreducible $\mathbf{U}_q(\mathfrak{sl}_n)$-modules are summands of

$$\textstyle\bigwedge_q^{\vec{k}} \mathbb{C}_q^n = \bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n$$

for some suitable vector $\vec{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m_{\{0,\dots,n\}}$.

Let $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ be the monoidal, $\mathbb{C}_q$-linear category consisting of:

- Objects are tensor products $\bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n$ of finite length and morphisms are $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners between these.
- Composition $\circ$ is composition of $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners.
- Tensoring $\otimes$ is tensoring of $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners.

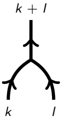
Diagrams for intertwiners - next try

Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners

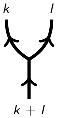
$$\mathrm{m}_{k,l}^{k+l}\colon \textstyle\bigwedge_q^k \mathbb{C}_q^n \otimes \bigwedge_q^l \mathbb{C}_q^n \to \bigwedge_q^{k+l}\mathbb{C}_q^n \quad \text{and} \quad \mathrm{s}_{k+l}^{k,l}\colon \bigwedge_q^{k+l}\mathbb{C}_q^n \to \bigwedge_q^k \mathbb{C}_q^n \otimes \bigwedge_q^l \mathbb{C}_q^n.$$

Define a functor $\Gamma_\wedge^n\colon \mathbf{Sp}(\mathfrak{sl}_n) \to \mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$:

- On objects: $\vec{k}$ is send to $\bigwedge_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m}\mathbb{C}_q^n$.
- On morphisms:



$\mapsto \mathrm{m}_{k,l}^{k+l}$



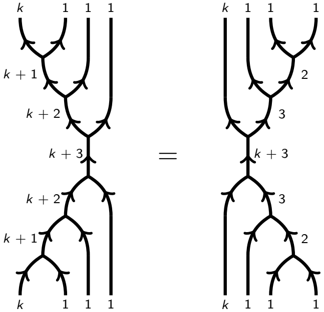
$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem(Cautis-Kamnitzer-Morrison 2012)

The functor $\Gamma_\wedge^n\colon \mathbf{Sp}(\mathfrak{sl}_n) \to \mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$ is an equivalence of monoidal categories.

Diagrams are still easier. At least for me...

$$\begin{array}{c} \mathrm{s}^{k,1}_{k+1}\otimes\mathrm{id}\otimes\mathrm{id} \\ \circ \\ \mathrm{s}^{k+1,1}_{k+2}\otimes\mathrm{id} \\ \circ \\ \mathrm{s}^{k+2,1}_{k+3} \\ \circ \\ \mathrm{m}^{k+3}_{k+2,1} \\ \circ \\ \mathrm{m}^{k+2}_{k+1,1}\otimes\mathrm{id} \\ \circ \\ \mathrm{m}^{k+1}_{k,1}\otimes\mathrm{id}\otimes\mathrm{id} \end{array}$$



$$\begin{array}{c} \mathrm{id}\otimes\mathrm{id}\otimes\mathrm{s}^{1,k}_{k+1} \\ \circ \\ \mathrm{id}\otimes\mathrm{s}^{1,k+1}_{k+2} \\ \circ \\ \mathrm{s}^{1,k+2}_{k+3} \\ \circ \\ \mathrm{m}^{k+3}_{1,k+2} \\ \circ \\ \mathrm{id}\otimes\mathrm{m}^{k+2}_{1,k+1} \\ \circ \\ \mathrm{id}\otimes\mathrm{id}\otimes\mathrm{m}^{k+1}_{1,k} \end{array}$$

For the $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners? Good luck...

Where do the relations come from?

In order to show that there is an equivalence of categories one has to show that one has exactly the right generators and relations.

To show that the generators are "ok" is a reasonably hard task and can be done "by hand" (if one likes to).

To show that the relations suffice is very hard: guessing them does not work for $n > 3$ anymore.

What was missing for a long time was a conceptual reason why some relations should appear.

The quantum algebra $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$

For each $\mathfrak{gl}_m$-weight $\vec{k} \in \mathbb{Z}^{m-1}$ adjoin an idempotent $1_{\vec{k}}$ (Think: projection to the $\vec{k}$-weight space!) to $\mathbf{U}_q(\mathfrak{gl}_m)$.

Definition(Beilinson-Lusztig-MacPherson 1990)

The idempotented quantum general linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_m) = \bigoplus_{\vec{k},\vec{k}' \in \mathbb{Z}^{m-1}} 1_{\vec{k}'} \mathbf{U}_q(\mathfrak{gl}_m) 1_{\vec{k}}.$$

It is generated by E_i, F_i for $i = 1, \dots, m-1$ suspect to some relations. These relations are just "cleaned-up" versions of the ones from $\mathfrak{gl}_m$. For instance,

$$E_i F_i 1_{\vec{k}} - F_i E_i 1_{\vec{k}} = [k_i - k_{i+1}] 1_{\vec{k}}$$

really just comes from

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

"Howe" to prove this?

Howe: The commuting actions of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ on

$$\textstyle\bigwedge_q^N(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) \cong \bigoplus_{k_1+\cdots+k_m=N} (\bigwedge_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m}\mathbb{C}_q^n)$$
$$\cong \bigoplus_{l_1+\cdots+l_n=N} (\textstyle\bigwedge_q^{l_1}\mathbb{C}_q^m \otimes \cdots \otimes \bigwedge_q^{l_n}\mathbb{C}_q^m)$$

introduce an $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$-action f on the first term and an $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-action on the second. Howe: Our $\bigwedge_q^{\vec{k}}\mathbb{C}_q^n$ is the $\vec{k}$-weight space of this.

In particular, there is a functorial action

$$\Phi_m^n\colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$$

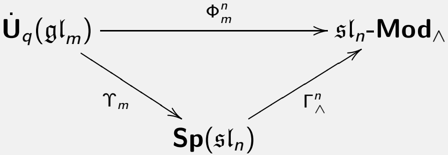
$$\vec{k} \mapsto \textstyle\bigwedge_q^{\vec{k}}\mathbb{C}_q^n, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge}(\bigwedge_q^{\vec{k}}\mathbb{C}_q^n, \bigwedge_q^{\vec{l}}\mathbb{C}_q^n)$$

Howe: Φ_m^n is full. Or in words: All relations in $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ follow from the (natural) ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and the ones in the kernel of Φ_m^n.

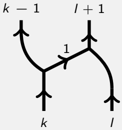
So how? “Howe”!

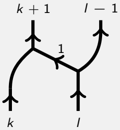
Theorem(Cautis-Kamnitzer-Morrison 2012)

There is a commutative diagram



with

$\Upsilon_m(F_i 1_{\vec{k}}) \mapsto$ 

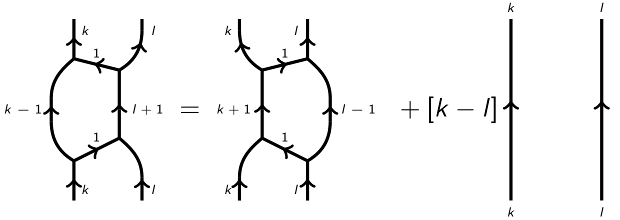
$\Upsilon_m(E_i 1_{\vec{k}}) \mapsto$ 

$\ker \Phi_m^n$ consists exactly of the $\mathfrak{gl}_m$-weights $\vec{k}$ with entries outside of $\{0, \ldots, n\}$.

In words: All the relations in $\mathbf{Sp}(\mathfrak{sl}_n)$ follow from the ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$.

Exempli gratia

The mysterious square switch



is just

$$EF1_{(k,l)} - FE1_{(k,l)} = [k-l]1_{(k,l)}$$

$$\approx$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This needs to be on one slide...

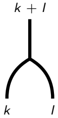
Some additional remarks.

- One can do slightly better: The $\mathfrak{sl}_n$-webs form a $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$-module of a certain highest weight. Thus, playing with $\mathfrak{sl}_n$-webs is doing highest weight representation theory of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$.
- Cautis, Kamnitzer and Morrison show that the R-matrix braiding on $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ and Lusztig's Weyl group braiding on $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ coincide.
- As a consequence, the Reshetikhin-Turaev polynomials of links obtained from $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ come (for all n) from highest weight representation theory of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ (for a suitable fixed m depending on the link L).
- Another consequence of this: For a fixed link L the whole family of all Reshetikhin-Turaev polynomials (for all possible n and colors) contains only a finite amount of information about L.
- Up to here: We can categorify everything in sight!

A symmetric $\mathfrak{sl}_2$-web is a labeled trivalent graph locally made of

$\mathrm{cap}_k =$ 

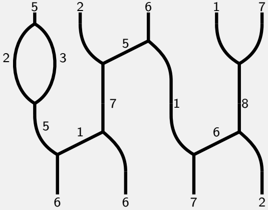
$\mathrm{cup}_k =$ 

$\mathrm{m}_{k,l}^{k+l} =$ 

$\mathrm{s}_{k+l}^{k,l} =$ 

No mirrors and sign issues but $k, l, k+l \in \{0, 1, \dots\}$.

Example

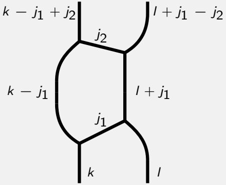


Never change a winning team: let us do the same again!

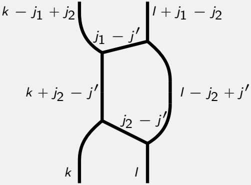
Definition

Given $\vec{k} \in \mathbb{Z}^k_{\geq 0}$ and $\vec{l} \in \mathbb{Z}^l_{\geq 0}$. The symmetric $\mathfrak{sl}_2$-web space $W_2^s(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by symmetric $\mathfrak{sl}_2$-webs between $\vec{k}$ and $\vec{l}$ modulo:

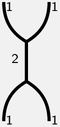
- Isotopy, associativity and "classical" relations, e.g. the scary square switches:



$$= \sum_{j'} \begin{bmatrix} k - j_1 - l + j_2 \\ j' \end{bmatrix}$$



- New, symmetric relations. For example dumbbells:



$$= [2]$$

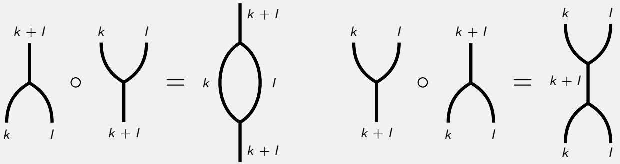
$$+$$

The symmetric $\mathfrak{sl}_2$-spider

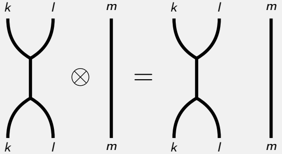
Definition

The symmetric $\mathfrak{sl}_2$-spider **SymSp**$(\mathfrak{sl}_2)$ is the monoidal $\mathbb{C}_q$-linear category with:

- Objects are $\vec{k} \in \mathbb{Z}^m_{\{0,1\ldots,\}}$ and morphisms are $\mathrm{Hom}_{\mathbf{Sp}(\mathfrak{sl}_n)}(\vec{k}, \vec{l}) = W_2^s(\vec{k}, \vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



Connection to representation theory - yet again

Let $V = \mathbb{C}_q^2$ the vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$. For $k \in \{0, 1 \dots\}$ let $\mathrm{Sym}_q^k \mathbb{C}_q^2$ denote the k-th symmetric $\mathbf{U}_q(\mathfrak{sl}_2)$-representation.

Let $\mathfrak{sl}_2$-fd**Mod** be the monoidal, $\mathbb{C}_q$-linear category consisting of:

- Objects are tensor products $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^2 \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^2$ of finite length and morphisms are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners between these.
- Composition $\circ$ is composition of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Tensoring $\otimes$ is tensoring of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.

Note that $\mathfrak{sl}_2$-$\mathbf{Mod}_{\wedge} \subsetneq \mathfrak{sl}_2$-$fd$**Mod**.

Fact: all irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-modules are of the form $\mathrm{Sym}_q^k \mathbb{C}_q^2$ for some k. Thus, $\mathfrak{sl}_2$-fd**Mod** contains all finite dimensional representations, aka: no splitting of tensor products is necessary.

Diagrams for intertwiners - I am not bored yet

Observe that there are (up to scalar) unique $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}_k\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \to \mathbb{C}_q \quad \mathrm{m}_{k,l}^{k+l}\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \to \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2$$

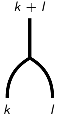
$$\mathrm{cup}_k\colon \mathbb{C}_q \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \quad \mathrm{s}_{k+l}^{k,l}\colon \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2 \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2$$

Define a functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}fd\mathbf{Mod}$:

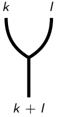
- On objects: $\vec{k}$ is send to $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^2 \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^2$.
- On morphisms:

$\mapsto \mathrm{cap}_k$

$\mapsto \mathrm{cup}_k$



$\mapsto \mathrm{m}_{k,l}^{k+l}$



$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem

Our $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}fd\mathbf{Mod}$ is an equivalence of monoidal categories.

“Howe” to prove this? You know “Howe”, right?

Howe: the commuting actions of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ on

$$\begin{aligned}\mathrm{Sym}_q^N(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) &\cong \bigoplus_{k_1+\cdots+k_m=N} (\mathrm{Sym}_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^n) \\ &\cong \bigoplus_{l_1+\cdots+l_m=N} (\mathrm{Sym}_q^{l_1}\mathbb{C}_q^m \otimes \cdots \otimes \mathrm{Sym}_q^{l_n}\mathbb{C}_q^m)\end{aligned}$$

introduce an $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$-action f on the first term and an $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-action on the second. Howe: our $\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^n$ is the $\vec{k}$-weight space of this.

In particular, there is a functorial action

$$\Phi_m^\infty \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_2\text{-}fd\mathbf{Mod}$$

$$\vec{k} \mapsto \mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{sl}_2\text{-}fd\mathbf{Mod}}(\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \mathrm{Sym}_q^{\vec{l}}\mathbb{C}_q^2)$$

Howe: Φ_m^∞ is full. Or in words: all relations in $\mathfrak{sl}_2$-$fd\mathbf{Mod}$ follow from the (natural) ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and the ones in the kernel of Φ_m^∞.

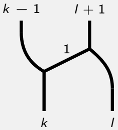
Let us copy-paste!

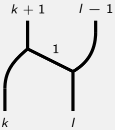
Theorem

There is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_m) & \xrightarrow{\quad\Phi_m^\infty\quad} & \mathfrak{sl}_2\text{-}fd\mathbf{Mod} \\ & {\scriptstyle\Upsilon_m}\searrow \quad \nearrow{\scriptstyle\Gamma_{\mathbf{sym}}} & \\ & \mathbf{SymSp}(\mathfrak{sl}_2) & \end{array}$$

with

$\Upsilon_m(F_i 1_{\vec{k}}) \mapsto$ 

$\Upsilon_m(E_i 1_{\vec{k}}) \mapsto$ 

$\ker \Phi_m^\infty$ consists of “throwing certain tableaux away”.

I do not have tenure. So I have to bore you a bit more.

Some additional remarks.

- The R-matrix braiding on $\mathfrak{sl}_2$-fd**Mod** and Lusztig's Weyl group braiding on $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ coincide again.
- As a consequence, on can obtain colored Jones polynomial without Jones-Wenzl projectors or infinite twists by a "MOY-like calculus".
- As a another consequence, the Reshetikhin-Turaev polynomials obtained from $\mathfrak{sl}_n$-**Mod**$_\wedge$ and the colored Jones polynomials are (almost) "dual" to each other. The only difference are the $\mathrm{End}_{\mathbb{C}_q}(V_m(\lambda))$ one has to kill.
- This give a hint: Categorify the colored Jones polynomial as Khovanov-Rozansky $\mathfrak{sl}_n$-homologies: Without infinite twists or categorified Jones-Wenzl projectors.
- As a possible upshot: Duality between Khovanov-Rozansky $\mathfrak{sl}_n$-homologies and colored Jones homologies (as predicted via HOMFLY-PT homology).

There is still much to do...

Thanks for your attention!