

Diagrammatics and cryptography

Or: Not too small, please!

Daniel Tubbenhauer

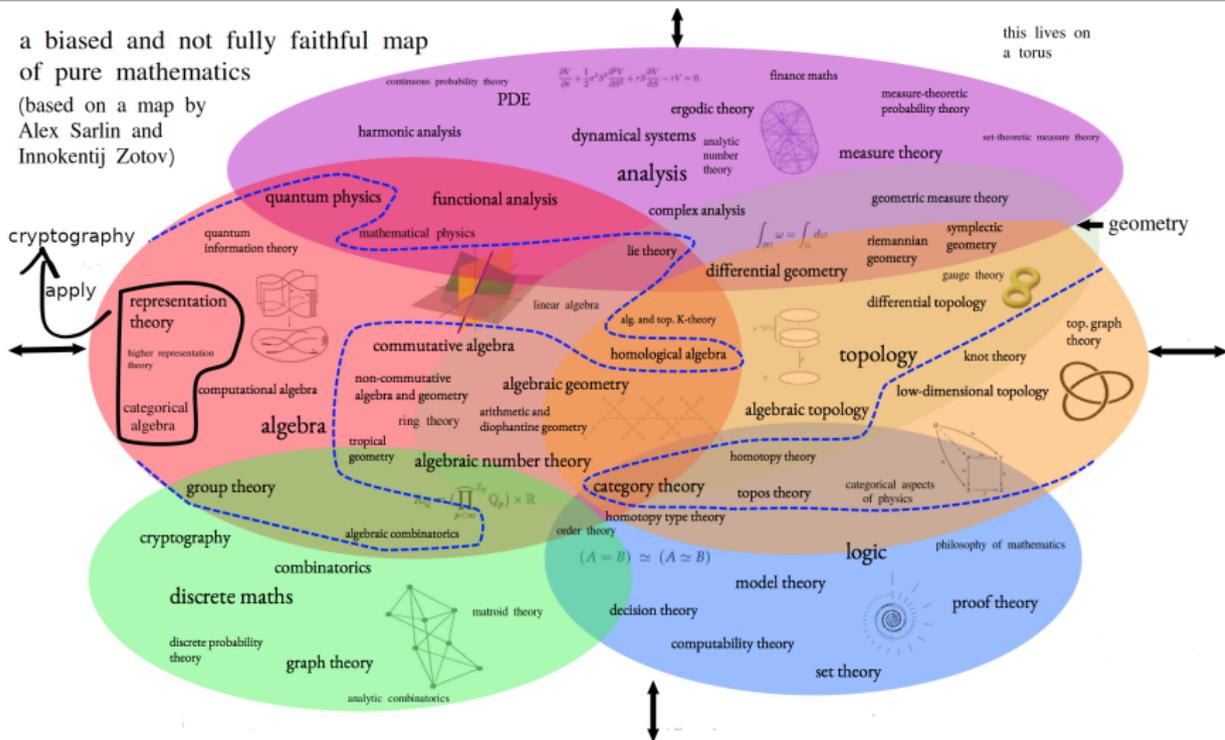
Symbol	Diagrams	Useful?	Symbol	Diagrams	Useful?
pPa_n		YES*	Pa_n		YES*
Mo_n		YES	$RoBr_n$		YES*
TL_n		YES	Br_n		YES*
pRo_n		YES*	Ro_n		YES*
pS_n		EX	S_n		NO

Joint with Mikhail Khovanov and Maithreya Sitaraman

December 2021

Where are we?

a biased and not fully faithful map
of pure mathematics
(based on a map by
Alex Sarlin and
Innokentij Zotov)



The six main fields of pure mathematics **algebra**, **analysis**, **geometry**, **topology**, **logic**, **discrete mathematics**

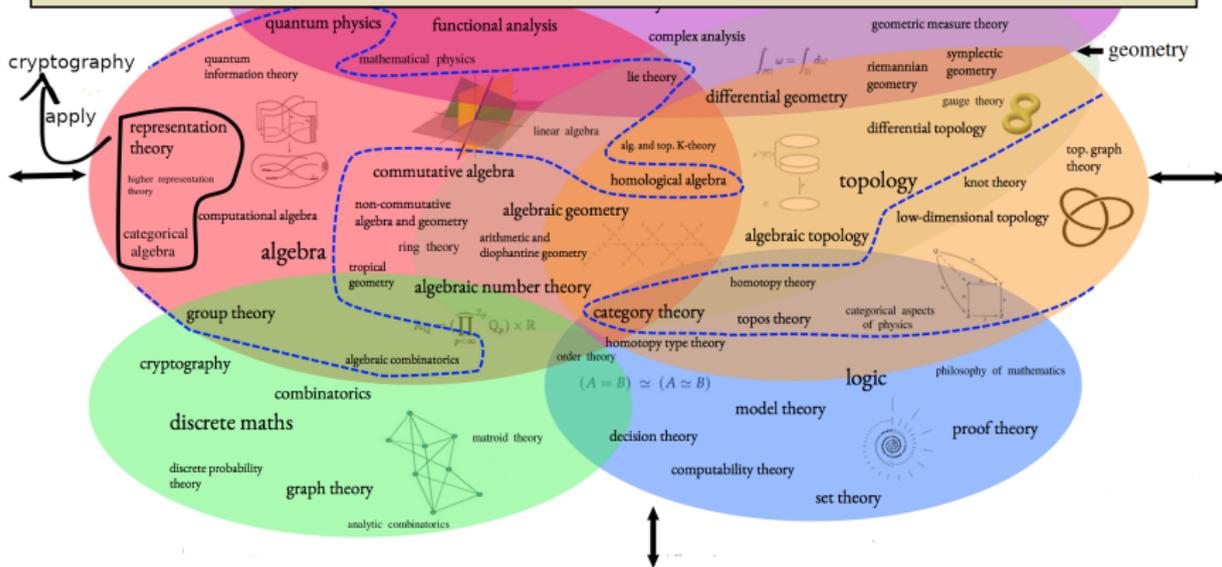
Wh

The map of (pure) mathematics

Black box. Representation theory and its categorical analog My research area

Dashed box. Where I usually apply them My research outreach

Applications beyond my current research? The future (within NU?)



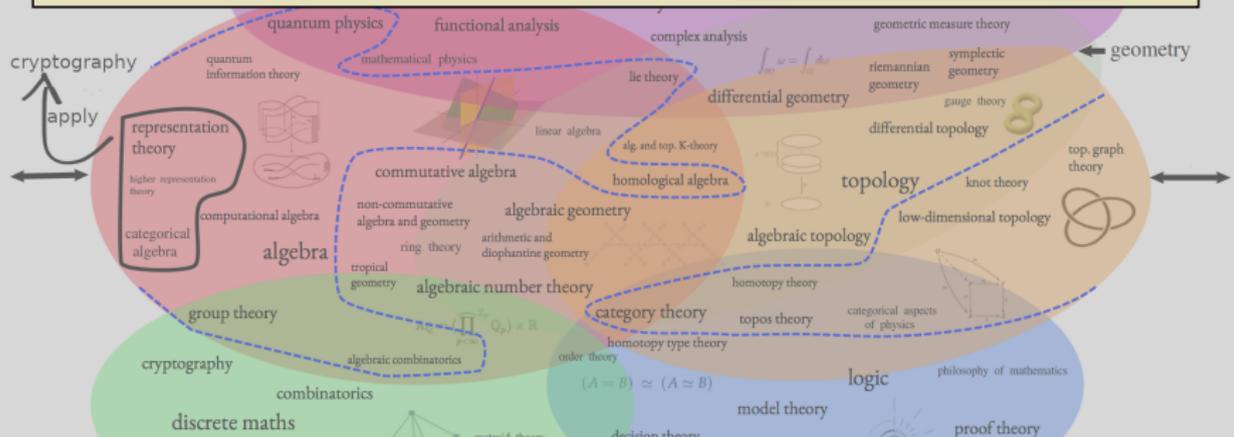
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Today

An applications of my work to cryptography The future (within NU?)

Why? Because it is neat (judge yourself!)
summarizes my research areas
and fits well to NU

End-to-end encryption



- ▶ **E2EE** Only the two communicating parties should decrypt the message
- ▶ **Problem** How to transfer the encryption key?
- ▶ **Diffie–Hellman (DH)** Addresses this problem

End-to-end encryption

Symmetric encryption

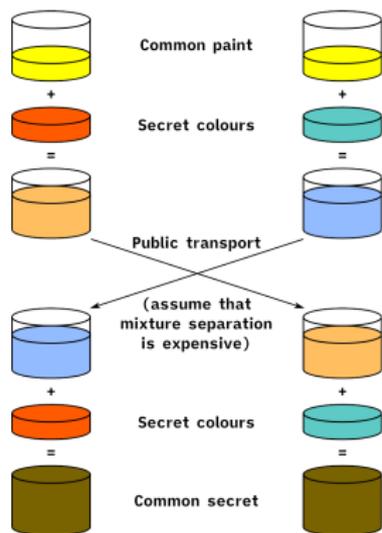


Asymmetric encryption



- ▶ **Symmetric** Both parties use the same secret key
- ▶ **Problem (still)** How to transfer the encryption key?
- ▶ **Asymmetric** Both parties have a public and a private key, no sharing needed

End-to-end encryption



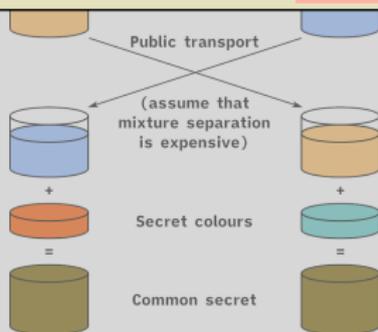
- ▶ **DH** Two secrets a, b , public g , send g^a or g^b and get $(g^b)^a = g^{ab} = (g^a)^b$
- ▶ **Catch** Relies on the mixtures to be hard to decompose (discrete log problem)
- ▶ **BTW** Using colors is not very practical ;-), so usually take $a, b, g \in (\mathbb{Z}/p\mathbb{Z})^\times$

Colors!

The color picture makes it clear that this can easily be generalized

For example, one could take a different group

Varying the protocol and one can even allow arbitrary monoids



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Public transport

(assume that)

Example (Shpilrain–Ushakov (SU) key exchange protocol)

The public data is a monoid S , and two sets $A, B \subset S$ of commuting elements and $g \in S$

Party A chooses privately $a, a' \in A$ and party B chooses privately $b, b' \in A$

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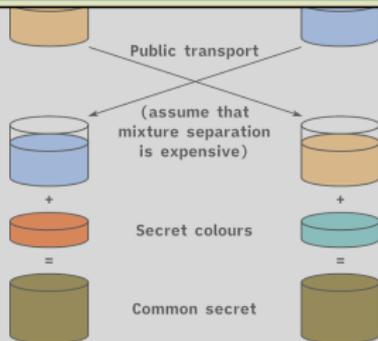
Note that S can be an arbitrary monoid in this protocol

The complexity of S determines how difficult it is to find the common secret from the public data.

Linear attack (Myasnikov–Roman'kov ~2015)

“All” protocol involving monoids can be attacked if the monoid admits a small non-trivial representation

Enter representation theory



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Our idea

Systematically study and construct monoids with no small non-trivial representations

The abstract theory is governed by Green's theory of cells (Green's relations)

The good finite examples come from quantum topology and monoidal categories

The representation theory group at NU knows these very well!

The good infinite examples are Artin, Thompson, Grigorchuk groups and alike

The geometric group theory group at NU knows these very well!

Other examples we know come from 2-representation theory and fusion categories

Vaguely related to the operator algebras group at NU

Examples and non-examples

Dynkin Diagrams of Simple Lie Algebras

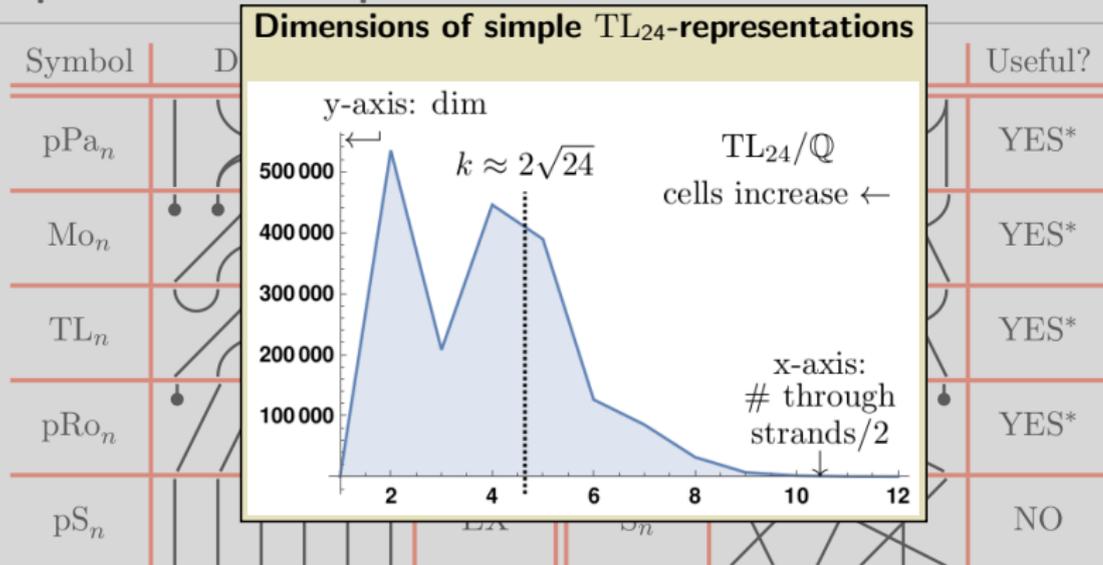
A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	A_{18}	A_{19}	A_{20}	A_{21}	A_{22}	A_{23}	A_{24}	A_{25}	A_{26}	A_{27}	A_{28}	A_{29}	A_{30}	A_{31}	A_{32}	A_{33}	A_{34}	A_{35}	A_{36}	A_{37}	A_{38}	A_{39}	A_{40}	A_{41}	A_{42}	A_{43}	A_{44}	A_{45}	A_{46}	A_{47}	A_{48}	A_{49}	A_{50}	A_{51}	A_{52}	A_{53}	A_{54}	A_{55}	A_{56}	A_{57}	A_{58}	A_{59}	A_{60}	A_{61}	A_{62}	A_{63}	A_{64}	A_{65}	A_{66}	A_{67}	A_{68}	A_{69}	A_{70}	A_{71}	A_{72}	A_{73}	A_{74}	A_{75}	A_{76}	A_{77}	A_{78}	A_{79}	A_{80}	A_{81}	A_{82}	A_{83}	A_{84}	A_{85}	A_{86}	A_{87}	A_{88}	A_{89}	A_{90}	A_{91}	A_{92}	A_{93}	A_{94}	A_{95}	A_{96}	A_{97}	A_{98}	A_{99}	A_{100}	A_{101}	A_{102}	A_{103}	A_{104}	A_{105}	A_{106}	A_{107}	A_{108}	A_{109}	A_{110}	A_{111}	A_{112}	A_{113}	A_{114}	A_{115}	A_{116}	A_{117}	A_{118}	A_{119}	A_{120}	A_{121}	A_{122}	A_{123}	A_{124}	A_{125}	A_{126}	A_{127}	A_{128}	A_{129}	A_{130}	A_{131}	A_{132}	A_{133}	A_{134}	A_{135}	A_{136}	A_{137}	A_{138}	A_{139}	A_{140}	A_{141}	A_{142}	A_{143}	A_{144}	A_{145}	A_{146}	A_{147}	A_{148}	A_{149}	A_{150}	A_{151}	A_{152}	A_{153}	A_{154}	A_{155}	A_{156}	A_{157}	A_{158}	A_{159}	A_{160}	A_{161}	A_{162}	A_{163}	A_{164}	A_{165}	A_{166}	A_{167}	A_{168}	A_{169}	A_{170}	A_{171}	A_{172}	A_{173}	A_{174}	A_{175}	A_{176}	A_{177}	A_{178}	A_{179}	A_{180}	A_{181}	A_{182}	A_{183}	A_{184}	A_{185}	A_{186}	A_{187}	A_{188}	A_{189}	A_{190}	A_{191}	A_{192}	A_{193}	A_{194}	A_{195}	A_{196}	A_{197}	A_{198}	A_{199}	A_{200}	A_{201}	A_{202}	A_{203}	A_{204}	A_{205}	A_{206}	A_{207}	A_{208}	A_{209}	A_{210}	A_{211}	A_{212}	A_{213}	A_{214}	A_{215}	A_{216}	A_{217}	A_{218}	A_{219}	A_{220}	A_{221}	A_{222}	A_{223}	A_{224}	A_{225}	A_{226}	A_{227}	A_{228}	A_{229}	A_{230}	A_{231}	A_{232}	A_{233}	A_{234}	A_{235}	A_{236}	A_{237}	A_{238}	A_{239}	A_{240}	A_{241}	A_{242}	A_{243}	A_{244}	A_{245}	A_{246}	A_{247}	A_{248}	A_{249}	A_{250}	A_{251}	A_{252}	A_{253}	A_{254}	A_{255}	A_{256}	A_{257}	A_{258}	A_{259}	A_{260}	A_{261}	A_{262}	A_{263}	A_{264}	A_{265}	A_{266}	A_{267}	A_{268}	A_{269}	A_{270}	A_{271}	A_{272}	A_{273}	A_{274}	A_{275}	A_{276}	A_{277}	A_{278}	A_{279}	A_{280}	A_{281}	A_{282}	A_{283}	A_{284}	A_{285}	A_{286}	A_{287}	A_{288}	A_{289}	A_{290}	A_{291}	A_{292}	A_{293}	A_{294}	A_{295}	A_{296}	A_{297}	A_{298}	A_{299}	A_{300}	A_{301}	A_{302}	A_{303}	A_{304}	A_{305}	A_{306}	A_{307}	A_{308}	A_{309}	A_{310}	A_{311}	A_{312}	A_{313}	A_{314}	A_{315}	A_{316}	A_{317}	A_{318}	A_{319}	A_{320}	A_{321}	A_{322}	A_{323}	A_{324}	A_{325}	A_{326}	A_{327}	A_{328}	A_{329}	A_{330}	A_{331}	A_{332}	A_{333}	A_{334}	A_{335}	A_{336}	A_{337}	A_{338}	A_{339}	A_{340}	A_{341}	A_{342}	A_{343}	A_{344}	A_{345}	A_{346}	A_{347}	A_{348}	A_{349}	A_{350}	A_{351}	A_{352}	A_{353}	A_{354}	A_{355}	A_{356}	A_{357}	A_{358}	A_{359}	A_{360}	A_{361}	A_{362}	A_{363}	A_{364}	A_{365}	A_{366}	A_{367}	A_{368}	A_{369}	A_{370}	A_{371}	A_{372}	A_{373}	A_{374}	A_{375}	A_{376}	A_{377}	A_{378}	A_{379}	A_{380}	A_{381}	A_{382}	A_{383}	A_{384}	A_{385}	A_{386}	A_{387}	A_{388}	A_{389}	A_{390}	A_{391}	A_{392}	A_{393}	A_{394}	A_{395}	A_{396}	A_{397}	A_{398}	A_{399}	A_{400}	A_{401}	A_{402}	A_{403}	A_{404}	A_{405}	A_{406}	A_{407}	A_{408}	A_{409}	A_{410}	A_{411}	A_{412}	A_{413}	A_{414}	A_{415}	A_{416}	A_{417}	A_{418}	A_{419}	A_{420}	A_{421}	A_{422}	A_{423}	A_{424}	A_{425}	A_{426}	A_{427}	A_{428}	A_{429}	A_{430}	A_{431}	A_{432}	A_{433}	A_{434}	A_{435}	A_{436}	A_{437}	A_{438}	A_{439}	A_{440}	A_{441}	A_{442}	A_{443}	A_{444}	A_{445}	A_{446}	A_{447}	A_{448}	A_{449}	A_{450}	A_{451}	A_{452}	A_{453}	A_{454}	A_{455}	A_{456}	A_{457}	A_{458}	A_{459}	A_{460}	A_{461}	A_{462}	A_{463}	A_{464}	A_{465}	A_{466}	A_{467}	A_{468}	A_{469}	A_{470}	A_{471}	A_{472}	A_{473}
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Examples and non-examples

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- ▶ **New examples** Finite monoids coming from quantum topology
- ▶ **More specific** Submonoids of the partition monoid above
- ▶ **Completely open** I claim your favorite example from quantum topology and geometric group theory will also work - lets work on this at NU!

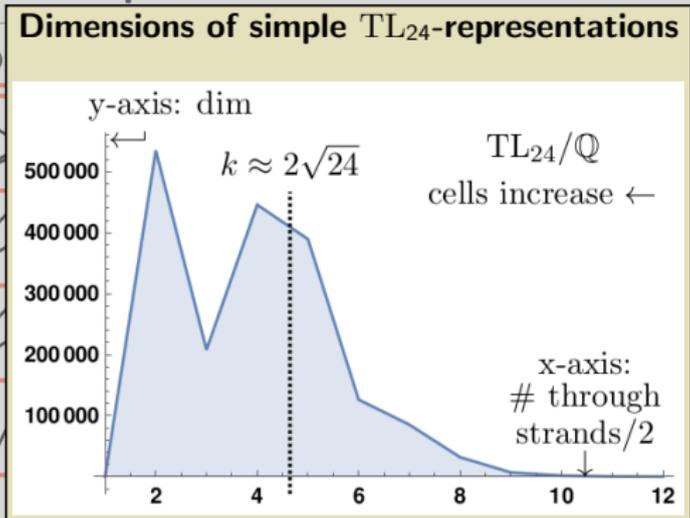
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Example (following Spencer ~2021)

After appropriate truncation

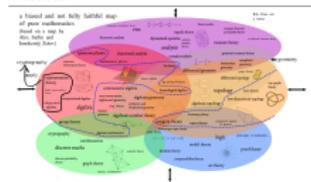
the representation gap of TL_n is bounded from below by

$$\frac{4}{(n + \lfloor 2\sqrt{n} \rfloor + 2)(n + \lfloor 2\sqrt{n} \rfloor + 4)} \binom{n}{(\lfloor 2\sqrt{n} \rfloor)/2}$$

topology and

geometric group theory will also work - lets work on this at NU!

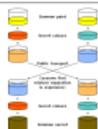
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The six main fields of pure mathematics: **algebra**, **analysis**, **geometry**, **topology**, **logic**, **discrete mathematics**.

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End-to-end encryption



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M_n		YES	Rel_n		YES
$\mathbb{T}L_n$		YES	Br_n		YES
$\mu\mathbb{B}_n$		YES	\mathbb{B}_n		YES
$\mu\mathbb{S}_n$		YES	\mathbb{S}_n		NO

- ▶ **Classical example** Cyclic groups have only big representations over \mathbb{F}_p
- ▶ **Non-examples** Groups of Lie type have all very small representations
- ▶ **Non-examples** Sporadic groups are too small to be useful

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Example (Sylva-Gabaiak (SU) key exchange protocol)

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Note that S can be an arbitrary monoid in this protocol

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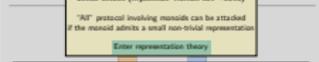
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- ▶ **Symmetric** Both parties use the same secret key
- ▶ **Problem (still)** How to transfer the encryption key?
- ▶ **Asymmetric** Both parties have a public and a private key, no sharing needed

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Enter representation theory

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The abstract theory is governed by Green's theory of cells (Green's relations)

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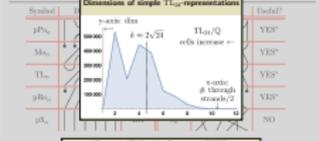
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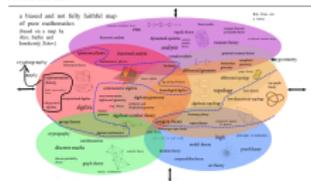


- ▶ **New example** After appropriate truncation the representation gap of $\mathbb{T}L_n/\mathbb{Q}$ is bounded from below by $\frac{1}{2}n - \sqrt{2n}$
- ▶ **More specific** β is a root of $x^2 - 2x + 1$
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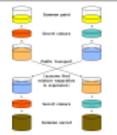
There is still much to do...

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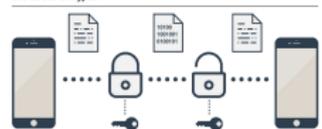
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μS_n		EX	S_n		NO

- ▶ **New examples** Finite monoids coming from quantum topology
- ▶ **More specific** Submonoids of the partition monoid above
- ▶ **Complexity open** I claim your favorite example from quantum topology and geometric group theory will also work - lets work on this at NU!

End-to-end encryption



- ▶ **Symmetric** Both parties use the same secret key
- ▶ **Problem (still)** How to transfer the encryption key?
- ▶ **Asymmetric** Both parties have a public and a private key, no sharing needed

End-to-end encryption

Linear attack (Myasnikov-Roman'kov - 2015)

"AI" protocol involving monoids can be attacked if the monoid admits a small non-trivial representation

Enter representation theory

Our idea

Systematically study and construct monoids with no small non-trivial representations

The abstract theory is governed by Green's theory of cells (Green's relations)

The good finite examples come from quantum topology and monoidal categories

The representation theory group at NU knows these very well!

The good infinite examples are Artin, Thompson, Garguchuk groups and allies

The geometric group theory group at NU knows these very well!

Other examples we know come from 2-representation theory and fusion categories

Vaguely related to the operator algebra group at NU!

Examples and non-examples

Dimension of simple TL_n -representations

y -axis: d_n

x -axis: n

$\beta = 2\sqrt{2}$

cells (increase \leftarrow)

TL_n/\mathbb{Q}

n -cells β through attack/2

Symbol	Useful?
μP_n	YES
M_n	YES
TL_n	YES
μB_n	YES
μS_n	NO

- ▶ **New class** After appropriate truncation the representation gap of TL_n is bounded from below by $\frac{1}{2} \log_2 \frac{1}{\beta}$
- ▶ **More specific** Submonoids of the partition monoid above
- ▶ **Complexity open** I claim your favorite example from quantum topology and geometric group theory will also work - lets work on this at NU!

Thanks for your attention!