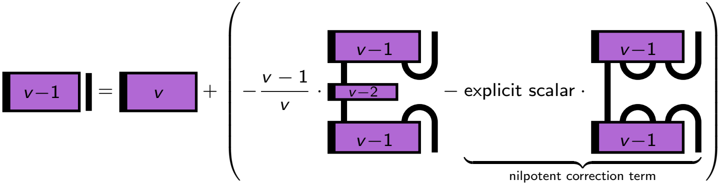
Fusion rules for SL_2

Or: A toy example of modular representation theory

Daniel Tubbenhauer

$$\mathtt{T}(v-1) \otimes \mathtt{T}(1) \cong \mathtt{T}(v) \oplus \mathtt{T}(v-2)$$

$$\Longleftrightarrow$$



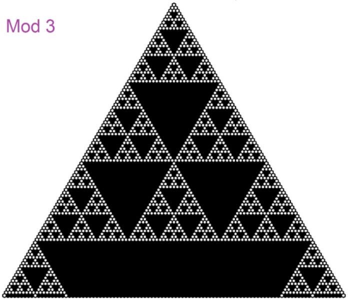
Joint with Lousie Sutton, Paul Wedrich, Jieru Zhu

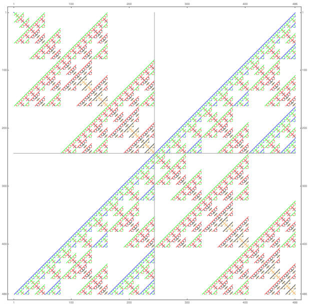
July 2021

Question. What can we say about finite-dimensional modules of SL_2...

- ...in the context of representations of classical groups? $\rightsquigarrow$ The modules and their structure.
- ...in the context of representations of Hopf algebras? $\rightsquigarrow$ Object fusion rules *i.e.* tensor products rules.
- ...in the context of categories? $\rightsquigarrow$ Morphisms of representations and their structure.

If the characteristic of the underlying field $\mathbb{K} = \overline{\mathbb{K}}$ of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{K})$ is finite we will see inverse fractals, *e.g.*





Question. What can we say about finite-dimensional modules of SL_2...

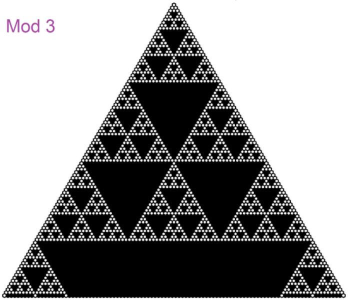
- ... in the context of representations of classical groups? $\rightsquigarrow$ The modules and
- es *i.e.*
- r

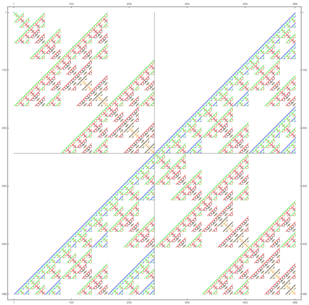
Spoiler. What will be the take away?

In some sense modular (char $p < \infty$) representation theory
is much harder than the classical one (char ∞ a.k.a. char 0 a.k.a. generically)
because secretly we are doing fractal geometry.

In my toy example SL_2 everything is explicit.

If the characteristic of the underlying field $\mathbb{K} = \mathbb{K}$ of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{K})$ is finite we will see inverse fractals, *e.g.*





Weyl $\sim$1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$ X^0Y^0

$\Delta(2-1)$ X^1Y^0 X^0Y^1

$\Delta(3-1)$ X^2Y^0 X^1Y^1 X^0Y^2

$\Delta(4-1)$ X^3Y^0 X^2Y^1 X^1Y^2 X^0Y^3

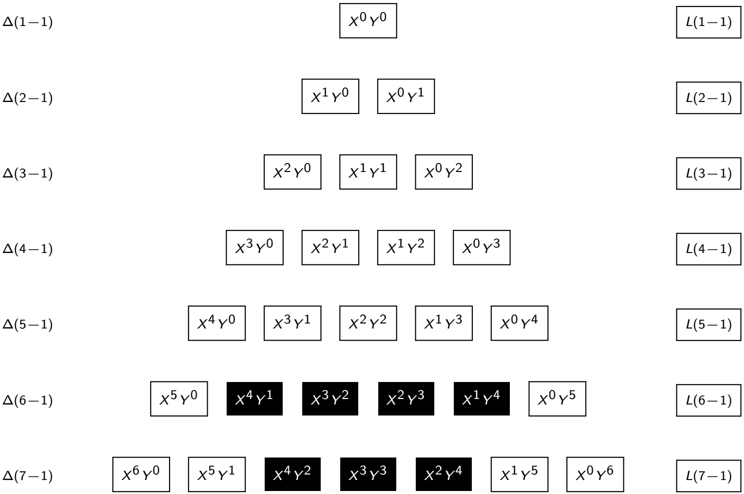
$\Delta(5-1)$ X^4Y^0 X^3Y^1 X^2Y^2 X^1Y^3 X^0Y^4

$\Delta(6-1)$ X^5Y^0 X^4Y^1 X^3Y^2 X^2Y^3 X^1Y^4 X^0Y^5

$\Delta(7-1)$ X^6Y^0 X^5Y^1 X^4Y^2 X^3Y^3 X^2Y^4 X^1Y^5 X^0Y^6

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's rows are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

Weyl $\sim$1923. The SL_2 simples $\mathtt{L}(v-1)$ in $\Delta(v-1)$ for $p=5$.



$\Delta(1-1)$ X^0Y^0 $L(1-1)$

$\Delta(2-1)$ X^1Y^0 X^0Y^1 $L(2-1)$

$\Delta(3-1)$ X^2Y^0 X^1Y^1 X^0Y^2 $L(3-1)$

$\Delta(4-1)$ X^3Y^0 X^2Y^1 X^1Y^2 X^0Y^3 $L(4-1)$

$\Delta(5-1)$ X^4Y^0 X^3Y^1 X^2Y^2 X^1Y^3 X^0Y^4 $L(5-1)$

$\Delta(6-1)$ X^5Y^0 X^4Y^1 X^3Y^2 X^2Y^3 X^1Y^4 X^0Y^5 $L(6-1)$

$\Delta(7-1)$ X^6Y^0 X^5Y^1 X^4Y^2 X^3Y^3 X^2Y^4 X^1Y^5 X^0Y^6 $L(7-1)$

$\Delta(7-1)$ has (its head) $\mathtt{L}(7-1)$ and $\mathtt{L}(3-1)$ as factors.

Weyl ∼1923. The SL_2 simples $\mathrm{L}(v{-}1)$ in $\Delta(v{-}1)$ for $p=5$.

$\Delta($1 -1)

$\Delta($2 -1)

$\Delta($3 -1)

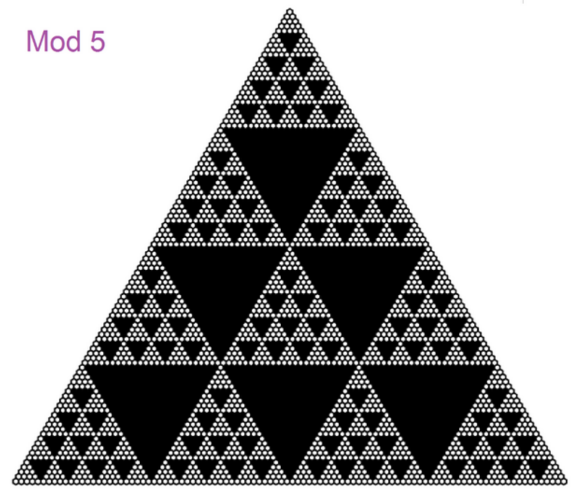
$\Delta($4 -1)

$\Delta($5 -1)

$\Delta($6 -1)

$\Delta($7 -1)

$\Delta(7{-}1)$ h



Pascals triangle modulo $p=5$ picks out the simples,
e.g. an unbroken east-west line is a Weyl module which is simple.

Weyl $\sim$1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

$\Delta(1-1)$ X^0Y^0

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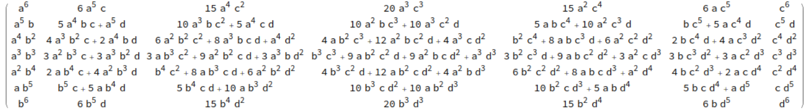
$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's rows are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

Weyl ∼1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

Example $\Delta(7-1) = \mathbb{K}X^6Y^0 \oplus \dots \oplus \mathbb{K}X^0Y^6$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts as



$$\begin{pmatrix}
a^6 & 6a^5c & 15a^4c^2 & 20a^3c^3 & 15a^2c^4 & 6ac^5 & c^6 \\
a^5b & 5a^4bc+a^5d & 10a^3bc^2+5a^4cd & 10a^2bc^3+10a^3c^2d & 5abc^4+10a^2c^3d & bc^5+5ac^4d & c^5d \\
a^4b^2 & 4a^3b^2c+2a^4bd & 6a^2b^2c^2+8a^3bcd+a^4d^2 & 4ab^2c^3+12a^2bc^2d+4a^3cd^2 & b^2c^4+8abc^3d+6a^2c^2d^2 & 2bc^4d+4ac^3d^2 & c^4d^2 \\
a^3b^3 & 3a^2b^3c+3a^3b^2d & 3ab^3c^2+9a^2b^2cd+3a^3bd^2 & b^3c^3+9ab^2c^2d+9a^2bcd^2+a^3d^3 & 3b^2c^3d+9abc^2d^2+3a^2cd^3 & 3bc^3d^2+3ac^2d^3 & c^3d^3 \\
a^2b^4 & 2ab^4c+4a^2b^3d & b^4c^2+8ab^3cd+6a^2b^2d^2 & 4b^3c^2d+12ab^2cd^2+4a^2bd^3 & 6b^2c^2d^2+8abcd^3+a^2d^4 & 4bc^2d^3+2acd^4 & c^2d^4 \\
ab^5 & b^5c+5ab^4d & 5b^4cd+10ab^3d^2 & 10b^3cd^2+10ab^2d^3 & 10b^2cd^3+5abd^4 & 5bcd^4+ad^5 & cd^5 \\
b^6 & 6b^5d & 15b^4d^2 & 20b^3d^3 & 15b^2d^4 & 6bd^5 & d^6
\end{pmatrix}$$

The rows are expansions of $(aX+cY)^{7-i}(bX+dY)^{i-1}$. Binomials!

$\Delta(4-1)$ X^3Y^0 X^2Y^1 X^1Y^2 X^0Y^3

$\Delta(5-1)$ X^4Y^0 X^3Y^1 X^2Y^2 X^1Y^3 X^0Y^4

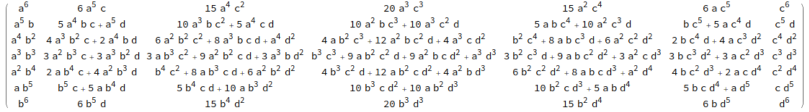
$\Delta(6-1)$ X^5Y^0 X^4Y^1 X^3Y^2 X^2Y^3 X^1Y^4 X^0Y^5

$\Delta(7-1)$ X^6Y^0 X^5Y^1 X^4Y^2 X^3Y^3 X^2Y^4 X^1Y^5 X^0Y^6

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matrix who's rows are expansions of $(aX+cY)^{v-i}(bX+dY)^{i-1}$.

Weyl ~**1923.** The SL_2 (dual) Weyl modules $\Delta(v{-}1)$.

Example $\Delta(7{-}1)=\mathbb{K}X^6Y^0\oplus\cdots\oplus\mathbb{K}X^0Y^6$**.**



$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts as

$$\begin{pmatrix}
a^6 & 6a^5c & 15a^4c^2 & 20a^3c^3 & 15a^2c^4 & 6ac^5 & c^6 \\
a^5b & 5a^4bc+a^5d & 10a^3bc^2+5a^4cd & 10a^2bc^3+10a^3c^2d & 5abc^4+10a^2c^3d & bc^5+5ac^4d & c^5d \\
a^4b^2 & 4a^3b^2c+2a^4bd & 6a^2b^2c^2+8a^3bcd+a^4d^2 & 4ab^2c^3+12a^2bc^2d+4a^3cd^2 & b^2c^4+8abc^3d+6a^2c^2d^2 & 2bc^4d+4ac^3d^2 & c^4d^2 \\
a^3b^3 & 3a^2b^3c+3a^3b^2d & 3ab^3c^2+9a^2b^2cd+3a^3bd^2 & b^3c^3+9ab^2c^2d+9a^2bcd^2+a^3d^3 & 3b^2c^3d+9abc^2d^2+3a^2cd^3 & 3bc^3d^2+3ac^2d^3 & c^3d^3 \\
a^2b^4 & 2ab^4c+4a^2b^3d & b^4c^2+8ab^3cd+6a^2b^2d^2 & 4b^3c^2d+12ab^2cd^2+4a^2bd^3 & 6b^2c^2d^2+8abcd^3+a^2d^4 & 4bc^2d^3+2acd^4 & c^2d^4 \\
ab^5 & b^5c+5ab^4d & 5b^4cd+10ab^3d^2 & 10b^3cd^2+10ab^2d^3 & 10b^2cd^3+5abd^4 & 5bcd^4+ad^5 & cd^5 \\
b^6 & 6b^5d & 15b^4d^2 & 20b^3d^3 & 15b^2d^4 & 6bd^5 & d^6
\end{pmatrix}$$

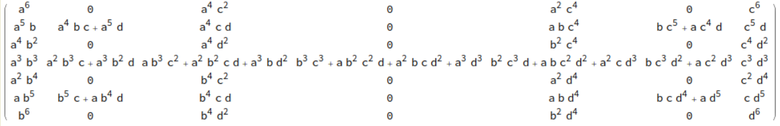
The rows are expansions of $(aX+cY)^{7-i}(bX+dY)^{i-1}$. Binomials!

$\Delta(4{-}1)$ X^3Y^0 X^2Y^1 X^1Y^2 X^0Y^3

Example $\Delta(7{-}1)$**, characteristic** 0**.**

No common eigensystem $\Rightarrow$ $\Delta(7{-}1)$ simple.

Example $\Delta(7{-}1)$**, characteristic** 2**.**



$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts as

$$\begin{pmatrix}
a^6 & 0 & a^4c^2 & 0 & a^2c^4 & 0 & c^6 \\
a^5b & a^4bc+a^5d & a^4cd & 0 & abc^4 & bc^5+ac^4d & c^5d \\
a^4b^2 & 0 & a^4d^2 & 0 & b^2c^4 & 0 & c^4d^2 \\
a^3b^3 & a^2b^3c+a^3b^2d & ab^3c^2+a^2b^2cd+a^3bd^2 & b^3c^3+ab^2c^2d+a^2bcd^2+a^3d^3 & b^2c^3d+abc^2d^2+a^2cd^3 & bc^3d^2+ac^2d^3 & c^3d^3 \\
a^2b^4 & 0 & b^4c^2 & 0 & a^2d^4 & 0 & c^2d^4 \\
ab^5 & b^5c+ab^4d & b^4cd & 0 & abd^4 & bcd^4+ad^5 & cd^5 \\
b^6 & 0 & b^4d^2 & 0 & b^2d^4 & 0 & d^6
\end{pmatrix}$$

$(0,0,0,1,0,0,0)$ is a common eigenvector, so we found a submodule.

Weyl ∼1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

X^0Y^0

X^1Y^0 X^0Y^1

X^2Y

When is $\Delta(v-1)$ simple?

$\Delta(v-1)$ is simple

$\Leftrightarrow$

$\binom{v-1}{w-1} \neq 0$ for all $w \leq v$

$\Leftrightarrow$ (Lucas' theorem)

$v = [a_r, 0, \dots, 0]_p$.

Lucas ∼1878.

"Binomials mod p are the product of binomials of the p-adic digits":

$\binom{a}{b} = \prod_{i=0}^{r} \binom{a_i}{b_i}$ mod p,

where $a = [a_r, \dots, a_0]_p = \sum_{i=0}^{r} a_i p^i$ *etc.*

Y^0

X^3Y^1 X^2Y^2 X^1Y^3 X^0Y^4

$\Delta($

$\Delta($

Y^6

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ matri

$dY)^{i-1}$.

General.

Weyl $\Delta(\lambda)$ and dual Weyl $\nabla(\lambda)$

are easy a.k.a. standard;

are parameterized by dominant integral weights;

are highest weight modules;

are defined over $\mathbb{Z}$;

have the classical Weyl characters;

form a basis of the Grothendieck group unitriangular w.r.t. simples;

satisfy (a version of) Schur's lemma $\dim_{\mathbb{K}} \mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu)) = \Delta_{i,0}\Delta_{\lambda,\mu}$;

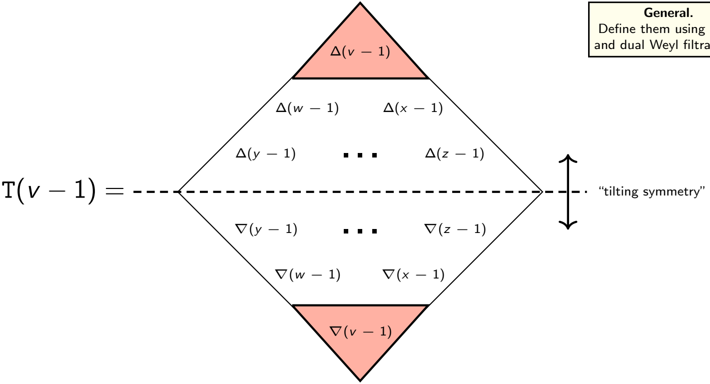
are simple generically;

have a root-binomial-criterion to determine whether they are simple (Jantzen's thesis ∼1973).

Ringel, Donkin ∼1991. There is a class of indecomposables $\mathtt{T}(v-1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

- They have Δ- and ∇ filtrations, which look the same if you tilt your head:

General.
Define them using Weyl and dual Weyl filtrations.



$\mathtt{T}(v-1) =$

$\Delta(v-1)$

$\Delta(w-1)$ $\Delta(x-1)$

$\Delta(y-1)$ ... $\Delta(z-1)$

"tilting symmetry"

$\nabla(y-1)$ ... $\nabla(z-1)$

$\nabla(w-1)$ $\nabla(x-1)$

$\nabla(v-1)$

- Play the role of projective modules.
- $\mathtt{T}(v-1) \cong \mathtt{L}(v-1) \cong \Delta(v-1) \cong \nabla(v-1)$ generically.
- They are a bit better behaved than simples.

Ringel, Donkin ∼1991. There is a class of indecomposables $\mathtt{T}(v{-}1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

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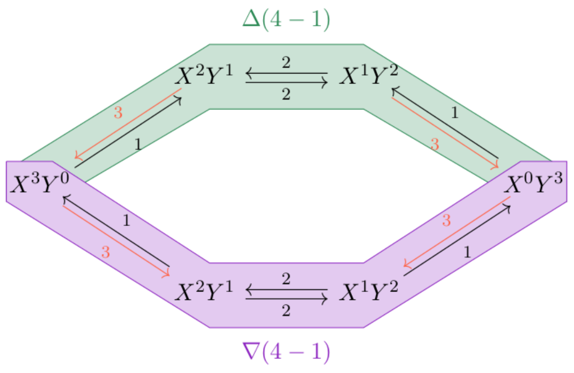
$\mathtt{T}(v$

Example. $\mathtt{T}(4{-}1)$ **in characteristic** 3.



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- They have Δ- and ∇ filtrations, which look the same if you tilt your head:

How many Weyl factors does $\mathtt{T}(v{-}1)$ have?

Weyl factors of $\mathtt{T}(v{-}1)$ is 2^k where

$k=\max\{\nu_p(\binom{v-1}{w-1}), w\leq v\}$. (Order of vanishing of $\binom{v-1}{w-1}$.)

determined by (Lucas's theorem)

non-zero non-leading digits of $v=[a_r,a_{r-1},...,a_0]_p$.

T

symmetry”

Example. $\mathtt{T}(220540{-}1)$ **for** $p=11$**?**

$v=220540=[1,4,0,7,7,1]_{11}$;

Maximal vanishing for $w=75594=[0,5,1,8,8,2]_{11}$;

$\binom{v-1}{w-1}=(\mathit{HUGE})=[...,\neq 0,0,0,0,0]_{11}$.

$\Rightarrow \mathtt{T}(220540{-}1)$ has 2^4 Weyl factors.

- Play the ro
- $\mathtt{T}(v{-}1)\cong$
- They are a

Ringel, Donkin ∼1991. There is a class of indecomposables $\mathtt{T}(v-1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

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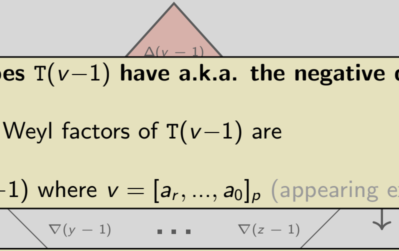

Which Weyl factors does $\mathtt{T}(v-1)$ have a.k.a. the negative digits game?

Weyl factors of $\mathtt{T}(v-1)$ are

$\Delta([a_r, \pm a_{r-1}, ..., \pm a_0]_p - 1)$ where $v = [a_r, ..., a_0]_p$ (appearing exactly once).

Example. $\mathtt{T}(220540-1)$ for $p = 11$?

$v = 220540 = [1, 4, 0, 7, 7, 1]_{11}$;

has Weyl factors $[1, \pm 4, 0, \pm 7, \pm 7, \pm 1]_{11}$;

e.g. $\Delta(218690 = [1, 4, 0, -7, -7, -1]_{11} - 1)$ appears.

- Play the rol
- $\mathtt{T}(v-1) \cong \mathtt{L}(v-1) = \Delta(v-1) = \nabla(v-1)$ generically.
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$\mathtt{T}(v$

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- $\mathtt{T}(v-1) \cong$
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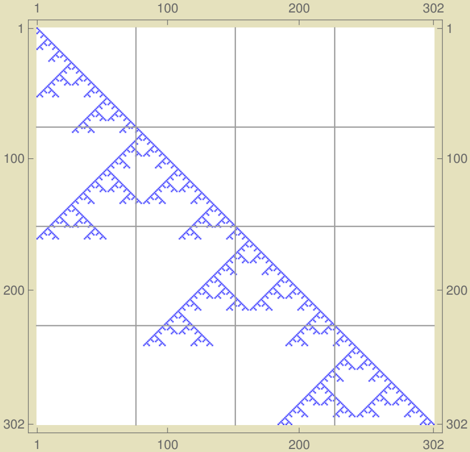
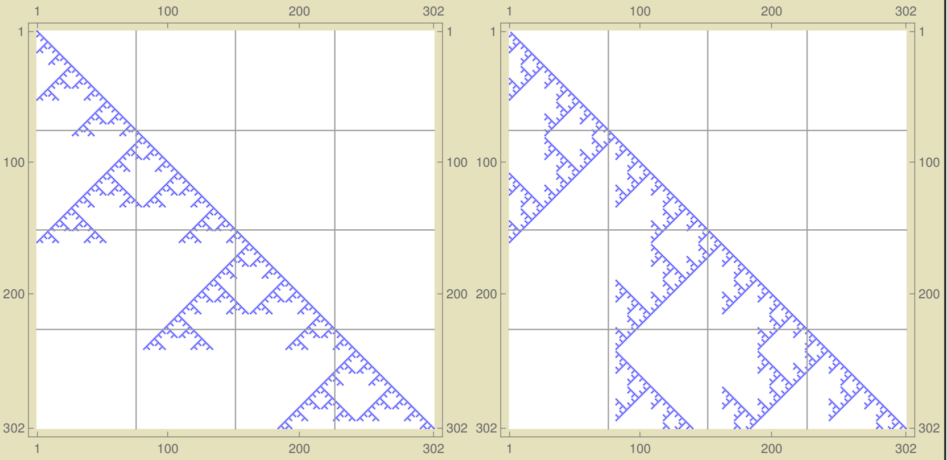
The tilting-Cartan matrix a.k.a. $\big(\mathtt{T}(v-1) : \Delta(w-1)\big)$.



This is characteristic 3.

Ringel, Donkin ∼1991. There is a class of indecomposables $\mathtt{T}(v-1)$ indexed by $\mathbb{N}$. They are a bit tricky to define, but:

$\big(\mathtt{T}(v-1):\Delta(w-1)\big)$ vs. $\big[\Delta(v-1):\mathtt{L}(w-1)\big]$ – flawed reciprocity.



This is characteristic 3.

- They are a bit better behaved than simples.

Tilting modules form a braided monoidal category $\mathcal{T}\mathrm{ilt}$.

Simple$\otimes$simple$\neq$simple, Weyl$\otimes$Weyl$\neq$Weyl, but tilting$\otimes$tilting$=$tilting.

The Grothendieck algebra $[\mathcal{T}\mathrm{ilt}]$ of $\mathcal{T}\mathrm{ilt}$ is a commutative algebra with basis $[\mathtt{T}(v-1)]$. So what I would like to answer on the object level, *i.e.* for $[\mathcal{T}\mathrm{ilt}]$:

- What are the fusion rules? I start here – fusion for $\mathtt{T}(1)$
- Find the $N_{v,w}^{x} \in \mathbb{N}_0$ in $\mathtt{T}(v-1) \otimes \mathtt{T}(w-1) \cong \bigoplus_x N_{v,w}^{x}\mathtt{T}(x-1)$.
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the structure constants.

  This appears to be tricky and I do not have an answer
- What are the thick $\otimes$-ideals?
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the ideals. This is discussed second

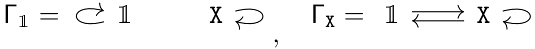
General.
These facts hold in general, and
tilting modules form the "nicest possible" monoidal subcategory.

Fusion graphs.

The fusion graph $\Gamma_v = \Gamma_{\mathtt{T}(v-1)}$ of $\mathtt{T}(v-1)$ is:

- Vertices of Γ_v are $w \in \mathbb{N}$, and identified with $\mathtt{T}(w-1)$.
- k edges $w \xrightarrow{k} x$ if $\mathtt{T}(x-1)$ appears k times in $\mathtt{T}(v-1) \otimes \mathtt{T}(w-1)$.
- $\mathtt{T}(v-1)$ is a $\otimes$-generator if Γ_v is strongly connected.
- This works for any reasonable monoidal category, with vertices being indecomposable objects and edges count multiplicities in $\otimes$-products.

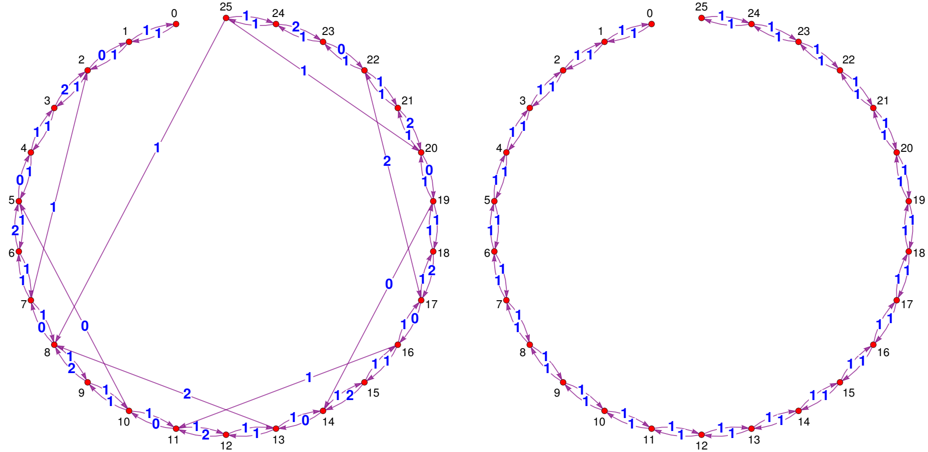
Baby example. Assume that we have two indecomposable objects $\mathbb{1}$ and $\mathtt{X}$, with $\mathtt{X}^{\otimes 2} = \mathbb{1} \oplus \mathtt{X}$. Then:



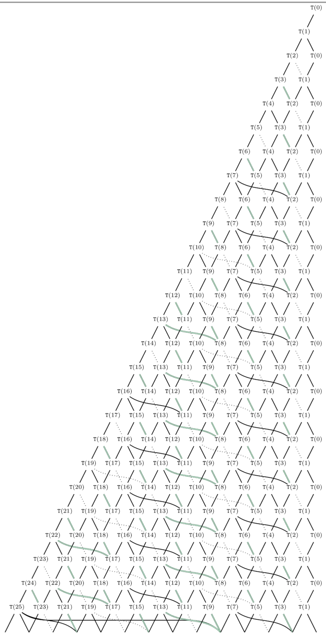
not a $\otimes$-generator

a $\otimes$-generator

Fusion graphs for $\mathtt{T}(1)$: char 3 vs. generic.



T(1)'s fusion graph via a Bratteli-type diagram



Formulas, for friends of formulas

Let $v = [a_j, ..., a_0]_p$. We have

$$\mathtt{T}(v-1) \otimes \mathtt{T}(1) \cong \mathtt{T}(v) \oplus \bigoplus_{i=0}^{tl} \mathtt{T}(v-2p^i)^{\oplus x_i}, x_i = \begin{cases} 0 & \text{if } a_i = 0 \text{ or } i = j \text{ and } a_j = 1, \\ 2 & \text{if } a_i = 1, \\ 1 & \text{if } a_i > 1. \end{cases}$$

tl=tail length=length of $[..., \neq p-1, p-1, p-1, ..., p-1]_p$

Proof strategy.

- Feed the problem into a machine;
- let it do a lot of calculations;
- guess the formula;
- prove the formula using character computations. Easy

Tilting modules form a braided monoidal category $\mathcal{T}\mathrm{ilt}$.

Simple$\otimes$simple$\neq$simple, Weyl$\otimes$Weyl$\neq$Weyl, but tilting$\otimes$tilting$=$tilting.

The Grothendieck algebra $[\mathcal{T}\mathrm{ilt}]$ of $\mathcal{T}\mathrm{ilt}$ is a commutative algebra with basis $[\mathtt{T}(v-1)]$. So what I would like to answer on the object level, *i.e.* for $[\mathcal{T}\mathrm{ilt}]$:

- What are the fusion rules? I start here – fusion for $\mathtt{T}(1)$
- Find the $N_{v,w}^{x} \in \mathbb{N}_0$ in $\mathtt{T}(v-1) \otimes \mathtt{T}(w-1) \cong \bigoplus_x N_{v,w}^{x}\mathtt{T}(x-1)$.
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the structure constants.

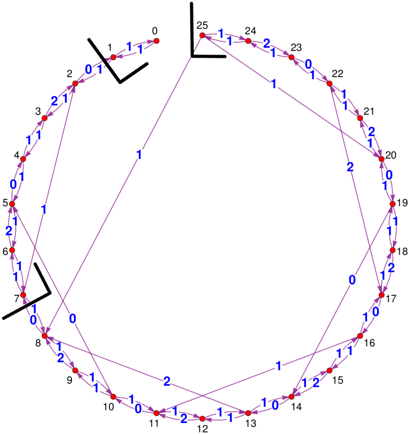
  This appears to be tricky and I do not have an answer
- What are the thick $\otimes$-ideals?
  - ▷ For $[\mathcal{T}\mathrm{ilt}]$ this means finding the ideals. This is discussed second

$\otimes$-ideals of $\mathcal{T}\mathrm{ilt}$ are indexed by prime powers.

Thick $\otimes$-ideal $=$ generated by identities on objects.
$\otimes$-ideal $=$ generated by any sets of morphism.

- Every $\otimes$-ideal is thick, and any non-zero thick $\otimes$-ideal is of the form $\mathcal{J}_{p^k} = \{\mathtt{T}(v-1) \mid v \geq p^k\}$.
- There is a chain of $\otimes$-ideals $\mathcal{T}\mathrm{ilt} = \mathcal{J}_1 \supset \mathcal{J}_p \supset \mathcal{J}_{p^2} \supset \dots$. The cells, *i.e.* $\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are the strongly connected components of Γ_1.

Example ($p = 3$).



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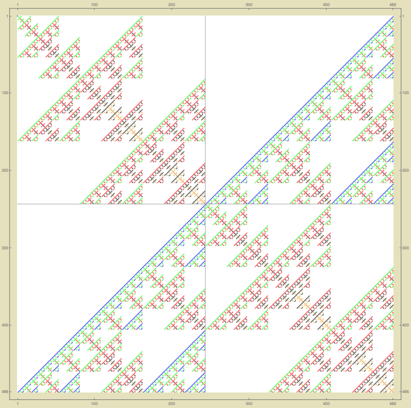
  $\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are th

Example ($p=3$).

Prime power Verlinde categories.

The ideal $\mathcal{J}_{p^k} \subset \mathcal{T}\mathrm{ilt}/\mathcal{J}_{p^{k+1}}$ is the cell of projectives.
The abelianizations $\mathcal{V}\mathrm{er}_{p^k}$ of $\mathcal{T}\mathrm{ilt}/\mathcal{J}_{p^{k+1}}$ are called Verlinde categories.
The Cartan matrix of $\mathcal{V}\mathrm{er}_{p^k}$ is a $p^k - p^{k-1}$-square matrix
with entries given by the common Weyl factors of $\mathtt{T}(v-1)$ and $\mathtt{T}(w-1)$.

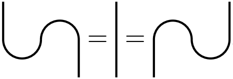
Example (Cartan matrix of $\mathcal{V}\mathrm{er}_{3^4}$).

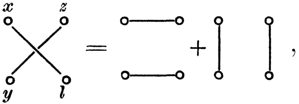


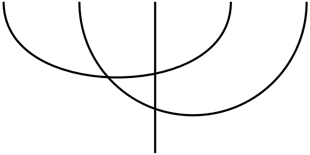
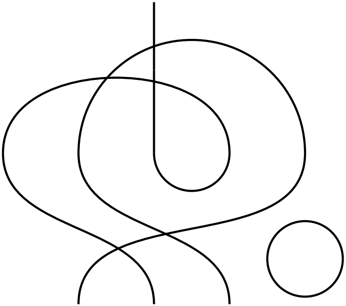
Rumer–Teller–Weyl $\sim$1932, Temperley–Lieb $\sim$1971, Kauffman $\sim$1987.

The category $\mathcal{TL}$ is the monoidal $\mathbb{Z}$-linear category monoidally generated by

object generators : $\bullet$, morphism generators : : $\mathbb{1} \to \bullet^{\otimes 2}$, : $\bullet^{\otimes 2} \to \mathbb{1}$,

relations :  $= -2$, .



$\begin{array}{c}\bullet^{\otimes 5}\\ \uparrow\\ \bullet^{\otimes 1}\end{array}$  , $\begin{array}{c}\bullet^{\otimes 1}\\ \uparrow\\ \bullet^{\otimes 3}\end{array}$  .

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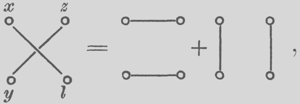
Theorem (folklore).

$\mathcal{TL}$ is an integral model of $\mathcal{T}\mathrm{ilt}$, *i.e.* fixing $\mathbb{K}$,

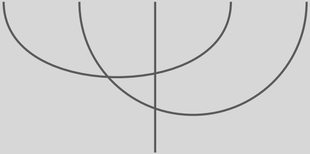
$$\mathcal{TL}\to\mathcal{T}\mathrm{ilt},\quad \bullet\mapsto \mathtt{T}(1)$$

induces an equivalence upon additive, idempotent completion.

: $\bullet^{\otimes 2}\to\mathbb{1}$,

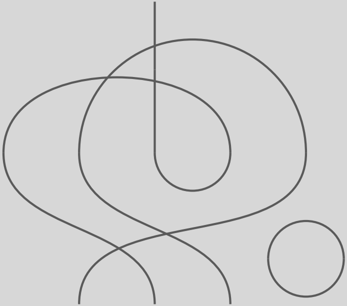


$\begin{matrix}\bullet^{\otimes 5}\\ \uparrow\\ \bullet^{\otimes 1}\end{matrix}$



,

$\begin{matrix}\bullet^{\otimes 1}\\ \uparrow\\ \bullet^{\otimes 3}\end{matrix}$



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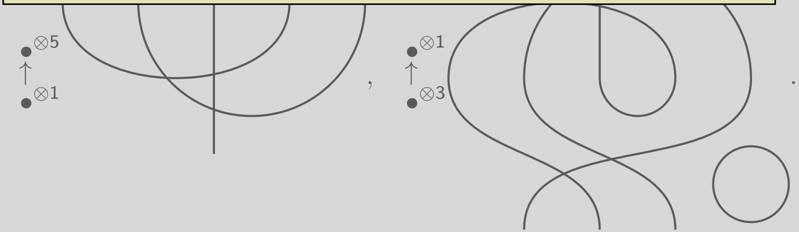
Burrull–Libedinsky–Sentinelli (∼2019).

Under this equivalence
$$\boxed{v-1} \mapsto \mathtt{T}(v-1)$$
where the purple box is an explicitly given idempotent in $\mathrm{End}_{\mathcal{TL}}(\bullet^{\otimes v-1})$ called *p*-Jones–Wenzl projector.



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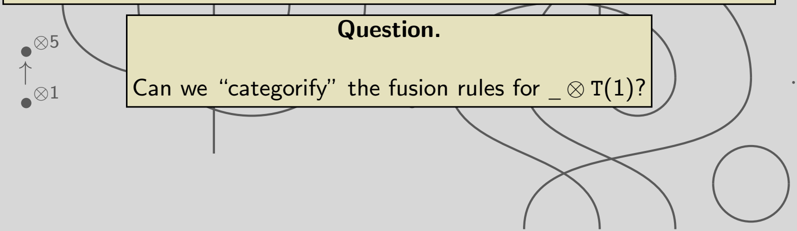


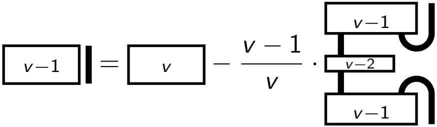
Question.

Can we “categorify” the fusion rules for $\_ \otimes \mathtt{T}(1)$?

Generically, using classical Jones–Wenzl projector (white boxes):

$$\Delta(v-1)\otimes\Delta(1)\cong\Delta(v)\oplus\Delta(v-2)$$

$$\Longleftrightarrow$$



In characteristic p using purple boxes, *e.g.*:

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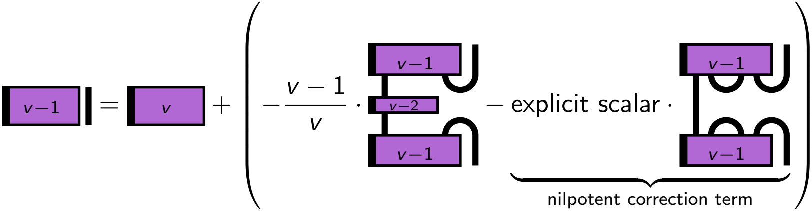

Yes we can!

Let $v = [a_j, \dots, a_0]_p$. We have

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$$\Longleftrightarrow$$

$$\boxed{v-1}\,| = \boxed{v} + \textstyle\sum_{i=0}^{tl} P_v^i \quad \text{where } P_v^i = \begin{cases} 0 & \text{if } a_i = 0, \\ \text{explicit diagrams} & \text{if } a_i = 1, \\ \text{other explicit diagrams} & \text{if } a_i > 1. \end{cases}$$

Proof strategy.

- Feed the problem into a machine;
- let it do a lot of calculations;
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- prove the formula using a huge inductive argument. Not so easy

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$= 0,$

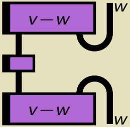
$= 1,$

$> 1.$

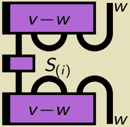
This is also fractal.

P_v^i is usually of the form

$$P_v^i = (\text{explicit scalar})' \cdot$$



$$- \text{explicit scalar} \cdot$$



$$, \quad w = p^i.$$

Roughly each strand “is blown up by p”

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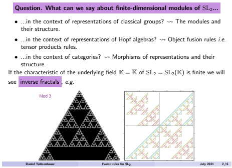
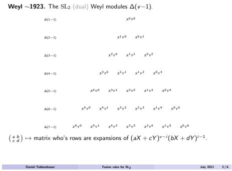


Question. What can we say about finite-dimensional modules of SL_2...

- ...in the context of representations of classical groups? ⇝ The modules and their structure.
- ...in the context of representations of Hopf algebras? ⇝ Object fusion rules *i.e.* tensor products rules.
- ...in the context of categories? ⇝ Morphisms of representations and their structure.

If the characteristic of the underlying field $\mathbb{K} = \overline{\mathbb{K}}$ of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{K})$ is finite we will see inverse fractals, *e.g.*

Daniel Tubbenhauer Fusion rules for SL_2 July 2021 2 / 6



Weyl ∼1923. The SL_2 (dual) Weyl modules $\Delta(v-1)$.

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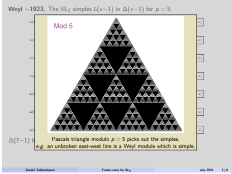
$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto$ matrix who's rows are expansions of $(aX + cY)^{v-i}(bX + dY)^{i-1}$.

Daniel Tubbenhauer Fusion rules for SL_2 July 2021 3 / 6



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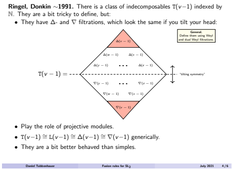
Daniel Tubbenhauer Fusion rules for SL_2 July 2021 3 / 6



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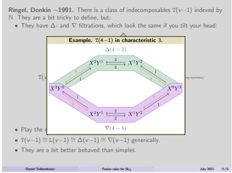
- They have Δ- and ∇ filtrations, which look the same if you tilt your head:

General. Define them using Weyl and dual Weyl filtrations.

- Play the role of projective modules.
- $\mathrm{T}(v-1) \cong \mathrm{L}(v-1) \cong \Delta(v-1) \cong \nabla(v-1)$ generically.
- They are a bit better behaved than simples.

Daniel Tubbenhauer Fusion rules for SL_2 July 2021 4 / 6



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Example. $\mathrm{T}(4-1)$ **in characteristic** 3.

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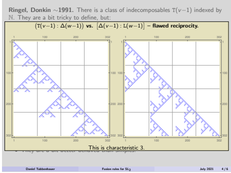
Daniel Tubbenhauer Fusion rules for SL_2 July 2021 4 / 6



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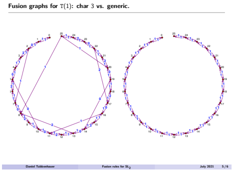
This is characteristic 3.

Daniel Tubbenhauer Fusion rules for SL_2 July 2021 4 / 6



Fusion graphs for $\mathrm{T}(1)$: char 3 vs. generic.

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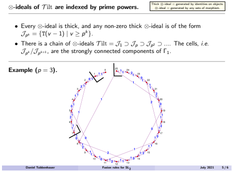

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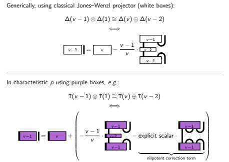

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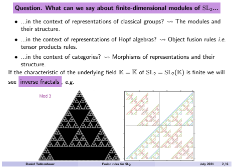
There is still much to do...



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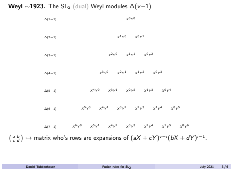

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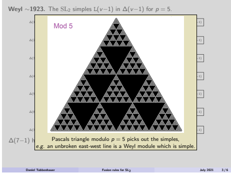

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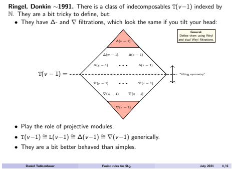

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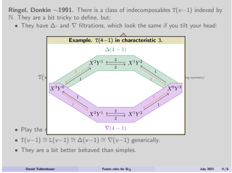

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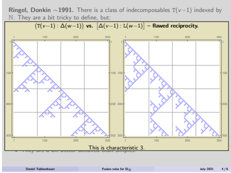

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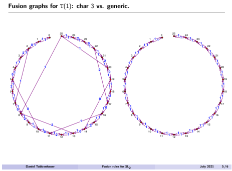
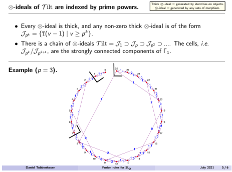
$(\mathrm{T}(v-1):\Delta(w-1))$ **vs.** $[\Delta(v-1):\mathrm{L}(w-1)]$ **– flawed reciprocity.**

This is characteristic 3.



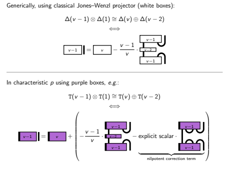
Fusion graphs for $\mathrm{T}(1)$: char 3 vs. generic.



$\otimes$-ideals of $\mathcal{T}\mathrm{ilt}$ are indexed by prime powers.

- Every $\otimes$-ideal is thick, and any non-zero thick $\otimes$-ideal is of the form $\mathcal{J}_{p^k}=\{\mathrm{T}(v-1)\mid v\geq p^k\}$.
- There is a chain of $\otimes$-ideals $\mathcal{T}\mathrm{ilt}=\mathcal{J}_1\supset\mathcal{J}_p\supset\mathcal{J}_{p^2}\supset\dots$. The cells, *i.e.* $\mathcal{J}_{p^k}/\mathcal{J}_{p^{k+1}}$, are the strongly connected components of Γ_1.

Example ($p=3$).



Generically, using classical Jones–Wenzl projector (white boxes):

$$\Delta(v-1)\otimes\Delta(1)\cong\Delta(v)\oplus\Delta(v-2)$$

$$\Longleftrightarrow$$

In characteristic p using purple boxes, *e.g.*:

$$\mathrm{T}(v-1)\otimes\mathrm{T}(1)\cong\mathrm{T}(v)\oplus\mathrm{T}(v-2)$$

$$\Longleftrightarrow$$

Thanks for your attention!