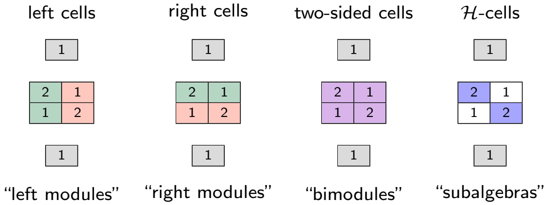
2-representation theory of Soergel bimodules

Or: Mind your groups

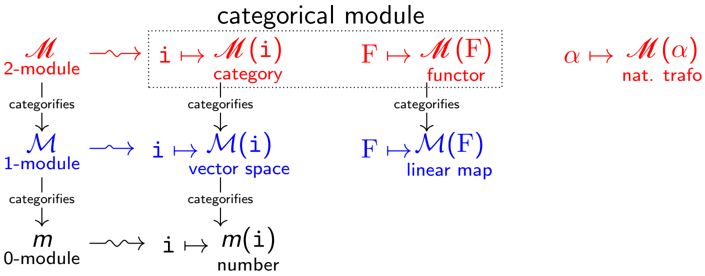
Daniel Tubbenhauer



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

June 2019

2-representation theory in a nutshell

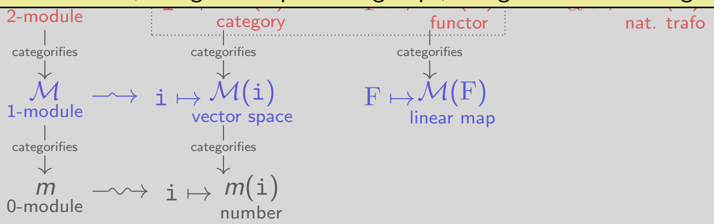


Examples of 2-categories.

Monoidal categories, module categories $\mathscr{R}\mathrm{ep}(G)$ of finite groups G,

module categories of Hopf algebras, fusion or modular tensor categories,

Soergel bimodules $\mathscr{S}$, categorified quantum groups, categorified Heisenberg algebras.



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2-module   category   functor   nat. trafo

Examples of 2-representation of these.

Categorical modules, functorial actions,

(co)algebra objects, conformal embeddings of affine Lie algebras,

the LLT algorithm, cyclotomic Hecke/KLR algebras, categorified (anti-)spherical module.

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Applications of 2-representations.

Representation theory (classical and modular), link homology, combinatorics

TQFTs, quantum physics, geometry.

2-representation theory in a nutshell

categorical module

$\mathscr{M}$ (2-module) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$ (category) $\quad \mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$ (functor) $\quad \alpha \mapsto \mathscr{M}(\alpha)$ (nat. trafo)

Plan for today.

1) Give an overview of the main ideas of 2-representation theory.

2) Discuss the group-like example $\mathscr{R}\mathrm{ep}(G)$.

3) Discuss the semigroup-like example $\mathscr{S}$.

Representation theory is group theory in vector spaces

Let C be a finite-dimensional algebra.

Frobenius ∼1895++, Burnside ∼1900++, Noether ∼1928++.
Representation theory is the useful? study of algebra actions

$$\mathcal{M}\colon \mathrm{C} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The "atoms" of such an action are called simple.

Maschke ∼1899, Noether, Schreier ∼1928. All modules are built out of simples ("Jordan–Hölder" filtration).

Basic question: Find the periodic table of simples.

Let $\mathscr{C}$ be a finitary 2-category.

Etingof–Ostrik, Chuang–Rouquier, many others ∼2000++. 2-representation theory is the useful? study of actions of 2-categories:

$$\mathscr{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with $\mathcal{V}$ being some finitary category. (Called 2-modules or 2-representations.)

The “atoms” of such an action are called 2-simple (“simple transitive”).

Mazorchuk–Miemietz ∼2014. All 2-modules are built out of 2-simples (“weak 2-Jordan–Hölder filtration”).

Basic question: Find the periodic table of 2-simples.

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Empirical fact.

Most of the fun happens already for monoidal categories (one-object 2-categories);

I will stick to this case for the rest of the talk,

but what I am going to explain works for 2-categories.

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Th

Mazorchuk–Miemietz ∼2014. All 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Basic question: Find the periodic table of 2-simples.

A category $\mathcal{V}$ is called finitary if its equivalent to $\mathrm{C}\text{-}p\mathcal{M}\mathrm{od}$. In particular:

- It has finitely many indecomposable objects M_j (up to $\cong$).
- It has finite-dimensional hom-spaces.
- Its Grothendieck group $[\mathcal{V}] = [\mathcal{V}]_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ is finite-dimensional.

A finitary, monoidal category $\mathscr{C}$ can thus be seen as a categorification of a finite-dimensional algebra.
Its indecomposable objects C_i give a distinguished basis of $[\mathscr{C}]$.

A finitary 2-representation of $\mathscr{C}$:

- A choice of a finitary category $\mathcal{V}$.
- (Nice) endofunctors $\mathscr{M}(\mathrm{C}_i)$ acting on $\mathcal{V}$.
- $[\mathscr{M}(\mathrm{C}_i)]$ give $\mathbb{N}$-matrices acting on $[\mathcal{V}]$.

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- It has finite-dimensiona
- Its Grothendieck group … dimensional.

The atoms (decat).

A C module is called simple

if it has no C-stable ideals.

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Dictionary.

cat	finitary	finitary+monoidal	fiat	functors
decat	vector space	algebra	self-injective	matrices

A finita

finite-dimensional algebra.

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Instead of studying C and its action via matrices,

study $\mathrm{C}\text{-}p\mathcal{M}\mathrm{od}$ and its action via functors.

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- Its Grothendie… …nal.

Example (decat).

$\mathrm{C} = \mathbb{C} = 1$ acts on any vector space via $\lambda \cdot \_$.

It has only one simple $\mathtt{V} = \mathbb{C}$.

A finitary, monoida… …ification of a finite-dimensional algebra.

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A finitary 2-…

- A choic…
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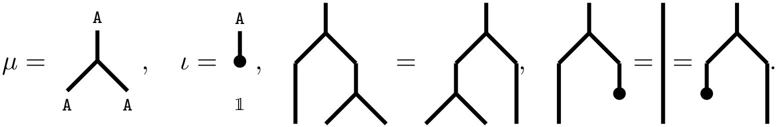
- $[\mathscr{M}(\mathrm{C}_i)]$ give $\mathbb{N}$-matrices acting on $[\mathcal{V}]$.

Example (cat).

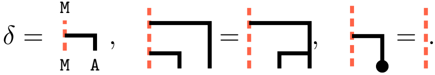
$\mathscr{C} = \mathscr{V}\mathrm{ec} = \mathscr{R}\mathrm{ep}(1)$ acts on any finitary category via $\mathbb{C} \otimes_{\mathbb{C}} \_$

It has only one 2-simple $\mathcal{V} = \mathcal{V}\mathrm{ec}$.

An algebra $\mathtt{A} = (\mathtt{A}, \mu, \iota)$ in $\mathscr{C}$:



Its (right) modules $(\mathtt{M}, \delta)$:



Example. Algebras in $\mathscr{V}\mathrm{ec}$ are algebras; modules are modules.

Example. Algebras in $\mathscr{R}\mathrm{ep}(G)$ and their modules ▸ Click.

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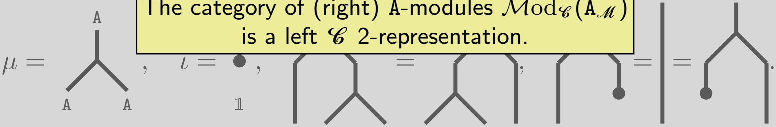

The category of (right) $\mathtt{A}$-modules $\mathcal{M}\mathrm{od}_{\mathscr{C}}(\mathtt{A}_{\mathscr{M}})$ is a left $\mathscr{C}$ 2-representation.

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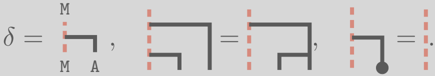

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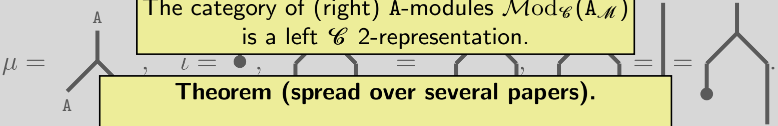

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Theorem (spread over several papers).

Completeness. For every 2-simple $\mathscr{M}$ there exists a simple algebra object $\mathtt{A}_{\mathscr{M}}$ in (a quotient of) $\mathscr{C}$ (fiat) such that $\mathscr{M} \cong \mathcal{M}\mathrm{od}_{\mathscr{C}}(\mathtt{A}_{\mathscr{M}})$.

Non-redundancy. $\mathscr{M} \cong \mathscr{N}$ if and only if $\mathtt{A}_{\mathscr{M}}$ and $\mathtt{A}_{\mathscr{N}}$ are Morita–Takeuchi equivalent.

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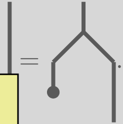
$\mu =$  , $\iota = \bullet$, $=$, $=$ $=$  .

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Exa

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Example.

Simple algebra objects in $\mathscr{V}\mathrm{ec}$ are simple algebras.
Up to Morita–Takeuchi equivalence these are just $\mathbb{C}$; and $\mathcal{M}\mathrm{od}_{\mathscr{V}\mathrm{ec}}(\mathbb{C}) \cong \mathcal{V}\mathrm{ec}$.

The above theorem is a vast generalization of this.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ (G a finite group).
- $\mathscr{C}$ is monoidal and finitary (and fiat). For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation $\mathbb{1}$.
- The regular 2-representation $\mathscr{M} \colon \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array} .$$

- The decategorification is a $\mathbb{N}$-representation, the regular representation.
- The associated algebra object is $\mathtt{A}_{\mathscr{M}} = \mathbb{1} \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $K \subset G$ be a subgroup.
- $\mathcal{R}\mathrm{ep}(K)$ is a 2-representation of $\mathscr{R}\mathrm{ep}(G)$, with action

$$\mathcal{R}es_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}(\mathcal{R}\mathrm{ep}(K))$$

  which is indeed a 2-action because $\mathcal{R}es_K^G$ is a $\otimes$-functor.
- The decategorifications are $\mathbb{N}$-representations.
- The associated algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd_K^G(\mathbb{1}_K) \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K, \psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g, h)\rho(gh), \text{ for all } g, h \in K.$$

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes\colon \mathcal{V}(K, \phi) \boxtimes \mathcal{V}(K, \psi) \to \mathcal{V}(K, \phi\psi).$$

- $\mathcal{V}(K, \psi)$ is also a 2-representation of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\otimes} \mathcal{V}(K, \psi).$$

- The decategorifications are $\mathbb{N}$-representations. Example
- The associated algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd_K^G(\mathbb{1}_K) \in \mathscr{C}$, but with ψ-twisted multiplication.

Example ($\mathscr{R}\mathrm{ep}(G)$).

Theorem (folklore?).

Completeness. All 2-simples of $\mathscr{R}\mathrm{ep}(G)$ are of the form $\mathcal{V}(K,\psi)$.

Non-redundancy. We have $\mathcal{V}(K,\psi)\cong\mathcal{V}(K',\psi')$

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the subgroups are conjugate or $\psi'=\psi^{g}$, where $\psi^{g}(k,l)=\psi(gkg^{-1},glg^{-1})$.

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Note that $\mathscr{R}\mathrm{ep}(G)$ has only finitely many 2-simples.

This is no coincidence.

- $\mathcal{V}(K,\psi)$ is ...

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\otimes} \mathcal{V}(K,\psi).$$

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Theorem (Etingof–Nikshych–Ostrik ∼2004); the group-like case.

If $\mathscr{C}$ is fusion (fiat and semisimple),
then it has only finitely many 2-simples.

This is false if one drops the semisimplicity. ▸ Example

Clifford, Munn, Ponizovskiĭ, Green ∼1942++. ▸ Semigroups

Write $\mathtt{X} \leq_L \mathtt{Y}$ if $\mathtt{Y}$ is a direct summand of $\mathtt{ZX}$ for $\mathtt{Z} \in \mathscr{C}$, *i.e.* $\mathtt{Y} \subset_{\oplus} \mathtt{ZX}$. $\mathtt{X} \sim_L \mathtt{Y}$ if $\mathtt{X} \leq_L \mathtt{Y}$ and $\mathtt{Y} \leq_L \mathtt{X}$. $\sim_L$ partitions $\mathscr{C}$ into left cells $\mathcal{L}$. Similarly for right $\mathcal{R}$, two-sided cells $\mathcal{J}$ or 2-modules.

An apex is a maximal two-sided cell not annihilating a 2-module.

Fact (Chan–Mazorchuk ∼2016). Any 2-simple has a unique apex.

Mackaay–Mazorchuk–Miemietz–Zhang ∼2018. For any fiat 2-category $\mathscr{C}$ (semigroup-like) there exists a fiat 2-subcategory $\mathscr{A}_{\mathcal{H}}$ (almost group-like) such that

$$\begin{Bmatrix} \text{2-simples of } \mathscr{C} \\ \text{with apex } \mathcal{J} \end{Bmatrix} \xleftrightarrow{\text{one-to-one}} \begin{Bmatrix} \text{2-simples of } \mathscr{A}_{\mathcal{H}} \\ \text{with apex } \mathcal{H} \subset \mathcal{J} \end{Bmatrix}$$

Catch. In general $\mathscr{A}_{\mathcal{H}}$ is not fusion.

Clifford

Write X ... $_L$ Y if

X $\leq_L$ Y ... ,

two-sided cells $\mathcal{J}$ or 2-modules.

Example (group-like).

Fusion categories, *e.g.* $\mathscr{R}\mathrm{ep}(G)$, have only one cell. $\mathscr{A}_{\mathcal{H}}$ is everything.

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Soergel bimodules $\mathscr{S}(S_n)$ for the symmetric group
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Example (Taft algebra T_2).

T_2-$\mathcal{M}\mathrm{od}$ has two cells – the lowest cell containing the
trivial representation; the biggest containing the projectives.

Let Γ be a Coxeter graph.

Artin ∼1925, Tits ∼1961++. The Artin–Tits group and its Coxeter group quotient are given by generators-relations:

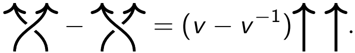
$$AT = \langle b_i \mid \underbrace{\cdots b_i b_j b_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots b_j b_i b_j}_{m_{ij} \text{ factors}} \rangle .$$

$$\downarrow$$

$$W = \langle s_i \mid s^2 = 1, \underbrace{\cdots s_i s_j s_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots s_j s_i s_j}_{m_{ij} \text{ factors}} \rangle$$

▸ Generalize classical braid groups, or ▸ generalize polyhedron groups, respectively.

H is the quotient of $\mathbb{Z}[v, v^{-1}]AT$ by the quadratic relations, *e.g.*



Fact (Kazhdan–Lusztig ∼1979, Soergel–Elias–Williamson ∼1990,2012). H has a distinguished basis, called the ▸ KL basis, which is a decategorification of indecomposable objects of $\mathscr{S}$.

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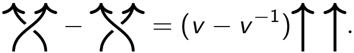
Question. What can one say about simples of H using KL cells?

m_{ij} factors m_{ij} factors .

$$W = \langle s_i \mid s^2 = 1, \underbrace{\cdots s_i s_j s_i}_{m_{ij}\text{ factors}} = \underbrace{\cdots s_j s_i s_j}_{m_{ij}\text{ factors}} \rangle$$

Generalize classical braid groups, or generalize polyhedron groups, respectively.

H is the quotient of $\mathbb{Z}[v, v^{-1}]AT$ by the quadratic relations, *e.g.*



Fact (Kazhdan–Lusztig ∼1979, Soergel–Elias–Williamson ∼1990,2012). H has a distinguished basis, called the KL basis, which is a decategorification of indecomposable objects of $\mathscr{S}$.

Example (type B_2).

$W = \langle s, t \mid s^2 = t^2 = 1, tsts = stst \rangle$. Number of elements: 8. Number of cells: 3, named 0 (lowest) to 2 (biggest).

Cell order:

$$0 — 1 — 2$$

Size of the cells:

cell	0	1	2
size	1	6	1

Cell structure:

1

s, sts	st
ts	t, tst

w_0

$\xrightarrow{\text{number of elements}}$

1

2	1
1	2

1

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$$c_s \cdot c_s = (1+\text{bigger powers})c_s.$$
$$c_{sts} \cdot c_s = (1+\text{bigger powers})c_{sts}.$$
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Cell structure:

s, sts	st	number of elements	2	1
				2

Example (SAGE).

$$c_{w_0} \cdot c_{w_0} = (1+\text{bigger powers})c_{w_0}.$$

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Fact (Lusztig ∼1984++).

For any Coxeter group W there is a well-defined function

$$a \colon W \to \mathbb{N}$$

which is constant on two-sided cells.

Big example

Cell order:

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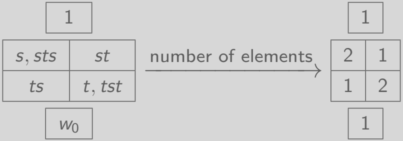
Cell structure:



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▸ Big example

Idea (Lusztig ∼1984).

Ignore everything except the leading coefficient of the classical KL basis shifted by a(two-sided cell). Those shifted versions are what I denote by c_w.

The asymptotic limit $\mathrm{A}_0(W)$ of $\mathrm{H}_v(W)$ is defined as follows.

As a free $\mathbb{Z}$-module:

$$\mathrm{A}_0(W) = \bigoplus_{\mathcal{J}} \mathbb{Z}\{a_w \mid w \in \mathcal{J}\}. \quad \text{vs.} \quad \mathrm{H}_v(W) = \mathbb{Z}[v, v^{-1}]\{c_w \mid w \in W\}.$$

Multiplication.

$$a_x a_y = \sum_{z \in \mathcal{J}} \gamma_{x,y}^{z} a_z. \quad \text{vs.} \quad c_x c_y = \sum_{z \in \mathcal{J}} v^{a(z)} h_{x,y}^{z} c_z + \text{bigger friends.}$$

where $\gamma_{x,y}^{z} \in \mathbb{N}$ is the leading coefficient of $h_{x,y}^{z} \in \mathbb{N}[v, v^{-1}]$.

Example (type B_2).

The multiplication tables (empty entries are 0 and $[2] = 1 + v^2$) in 1:

	a_s	a_{sts}	a_{st}	a_t	a_{tst}	a_{ts}
a_s	a_s	a_{sts}	a_{st}			
a_{sts}	a_{sts}	a_s	a_{st}			
a_{ts}	a_{ts}	a_{ts}	$a_t + a_{tst}$			
a_t				a_t	a_{tst}	a_{ts}
a_{tst}				a_{tst}	a_t	a_{ts}
a_{st}				a_{st}	a_{st}	$a_s + a_{sts}$

	c_s	c_{sts}	c_{st}	c_t	c_{tst}	c_{ts}
c_s	$[2]c_s$	$[2]c_{sts}$	$[2]c_{st}$	c_{st}	$c_{st} + c_{w_0}$	$c_s + c_{sts}$
c_{sts}	$[2]c_{sts}$	$[2]c_s + [2]^2 c_{w_0}$	$[2]c_{st} + [2]c_{w_0}$	$c_s + c_{sts}$	$c_s + [2]^2 c_{w_0}$	$c_s + c_{sts} + [2]c_{w_0}$
c_{ts}	$[2]c_{ts}$	$[2]c_{ts} + [2]c_{w_0}$	$[2]c_t + [2]c_{tst}$	$c_t + c_{tst}$	$c_t + c_{tst} + [2]c_{w_0}$	$2c_{ts} + c_{w_0}$
c_t	c_{ts}	$c_{ts} + c_{w_0}$	$c_t + c_{tst}$	$[2]c_t$	$[2]c_{tst}$	$[2]c_{ts}$
c_{tst}	$c_t + c_{tst}$	$c_t + [2]^2 c_{w_0}$	$c_t + c_{tst} + [2]c_{w_0}$	$[2]c_{tst}$	$[2]c_t + [2]^2 c_{w_0}$	$[2]c_{ts} + [2]c_{w_0}$
c_{st}	$c_s + c_{sts}$	$c_s + c_{sts} + [2]c_{w_0}$	$2c_{st} + c_{w_0}$	$[2]c_{st}$	$[2]c_{st} + [2]c_{w_0}$	$[2]c_s + [2]c_{sts}$

(Note the "subalgebras".)

The asymptotic algebra is much simpler!

Big example

Fact (Lusztig ∼1984++).

$\mathrm{A}_0(W) = \bigoplus_{\mathcal{J}} \mathrm{A}_0^{\mathcal{J}}(W)$ with the a_w basis
and all its summands $\mathrm{A}_0^{\mathcal{J}}(W) = \mathbb{Z}\{a_w \mid w \in \mathcal{J}\}$
are multifusion algebras. (Group-like.)

Multifusion algebras = decategorifications of multifusion categories.

The asym

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where $\gamma_{x,y}^z \in \mathbb{N}$ is the leading coefficient of $h_{x,y}^z \in \mathbb{N}[v, v^{-1}]$.

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Multiplication.

$a_x a_y = \sum$ ... r friends.

Surprising fact 1 (Lusztig ∼1984++).
It seems one throws almost away everything, but:

There is an explicit embedding

$$\mathrm{H}_v(W) \hookrightarrow \mathrm{A}_0(W) \otimes_{\mathbb{Z}} \mathbb{Z}[v, v^{-1}]$$

which is an isomorphism after scalar extension to $\mathbb{C}(v)$.

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Surprising fact 2 – $\mathcal{H}$-cell-theorem (Lusztig ∼1984++).

There is an explicit one-to-one correspondence

$$\{\text{simples of } \mathrm{H}_v(W) \text{ with apex } \mathcal{J}\} \xleftrightarrow{\text{one-to-one}} \{\text{simples of } \mathrm{A}_0^{\mathcal{H}}(W)\}.$$

▸ Example

Categorified picture – Part 1.

Theorem (Soergel–Elias–Williamson ∼1990,2012).
There exists a monoidal category $\mathscr{S}$ such that:

- (1) For every $w \in W$, there exists an indecomposable object $\mathtt{C}_w$.
- (2) The $\mathtt{C}_w$, for $w \in W$, form a complete set of pairwise non-isomorphic indecomposable objects up to shifts.
- (3) The identity object is $\mathtt{C}_1$, where 1 is the unit in W.
- (4) $\mathscr{C}$ categorifies H with $[\mathtt{C}_w] = c_w$.

Examples in type A_1; polynomial ring.

Let $\mathrm{R} = \mathbb{C}[x]$ with $W = S_2$ action given by $s.x = -x$; $\mathrm{R}^s = \mathbb{C}[x^2]$.

The indecomposable Soergel bimodules over R are
$\mathtt{C}_1 = \mathbb{C}[x]$ and $\mathtt{C}_s = \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]$.

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Examples in type A_1; coinvariant algebra.

$$\mathtt{C}_s \otimes_{\mathrm{R}_W} \mathtt{C}_s = \big(\mathbb{C}[x]/x^2 \otimes \mathbb{C}[x]/x^2\big) \otimes_{\mathbb{C}[x]/x^2} \big(\mathbb{C}[x]/x^2 \otimes \mathbb{C}[x]/x^2\big).$$

Which gives $\mathtt{C}_s\mathtt{C}_s \cong \mathtt{C}_s \oplus \mathtt{C}_s\langle 2\rangle = (1+v^2)\mathtt{C}_s$.

Categorified picture – Part 2.

Theorem (Lusztig, Elias–Williamson ∼2012).
Let $\mathcal{H}$ be an $\mathcal{H}$-cell of W. There exists a fusion category $\mathscr{A}_{\mathcal{H}}$ such that:

- (1) For every $w \in \mathcal{H}$, there exists a simple object $\mathtt{A}_w$.
- (2) The $\mathtt{A}_w$, for $w \in \mathcal{H}$, form a complete set of pairwise non-isomorphic simple objects.
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$$\mathtt{A}_x\mathtt{A}_y = \bigoplus_{z\in\mathcal{J}} \gamma_{x,y}^{z}\mathtt{A}_z. \quad \text{vs.} \quad \mathtt{C}_x\mathtt{C}_y = \bigoplus_{z\in\mathcal{J}} v^{a(z)}h_{x,y}^{z}\mathtt{C}_z + \text{bigger friends.}$$

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Examples in type A_1; coinvariant algebra.

$\mathtt{C}_1 = \mathbb{C}[x]/x^2$ and $\mathtt{C}_s = \mathbb{C}[x]/x^2 \otimes \mathbb{C}[x]/x^2$. (Positively graded, but non-semisimple.)

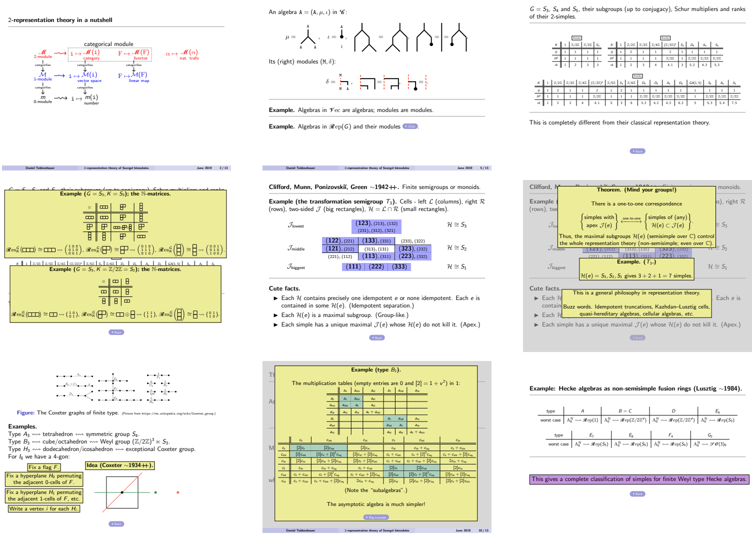
$\mathtt{A}_1 = \mathbb{C}$ and $\mathtt{A}_s = \mathbb{C} \otimes \mathbb{C}$. (Degree zero part.)

Categorified picture – Part 2.

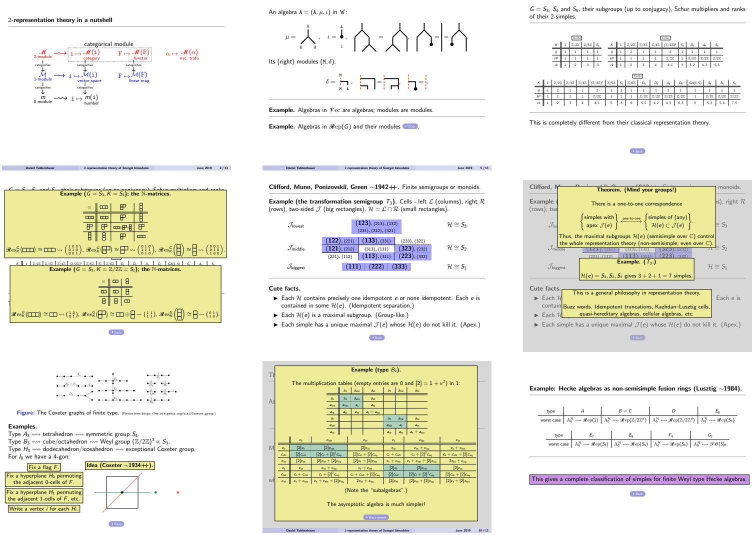
Theorem (June 2019 on arXiv).
For any finite Coxeter group W and any $\mathcal{H} \subset \mathcal{J}$ of W, there is an injection

$$\Theta \colon \big(\{\text{2-simples of } \mathscr{A}_{\mathcal{H}}\} \,/\, \cong \big) \hookrightarrow \big(\{\text{graded 2-simples of } \mathscr{S} \text{ with apex } \mathcal{J}\} \,/\, \cong \big)$$

- We conjecture Θ to be a bijection.
- We have proved the conjecture for all $\mathcal{H}$ which contain the longest element of a parabolic subgroup of W.
- If true, the conjecture implies that there are finitely many equivalence classes of 2-simples of $\mathscr{S}$.
- For almost all W, we would get a complete classification of the 2-simples.



There is still much to do...



Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across mathematics, and beyond.

VERY considerable advances in the theory of groups of

But this wasn't clear at all when Frobenius started it.

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Back

Simple objects in $\mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z})$ are $\mathbb{1}$ (trivial) and $-\mathbb{1}$ (sign).

Algebra object 1. $\mathtt{A}_1 = \mathbb{1}$:

μ	$\mathbb{1} \otimes \mathbb{1}$
$\mathbb{1}$	1

.

Two modules $\mathtt{M}_1 = \mathbb{1}$ and $\mathtt{M}_2 = -\mathbb{1}$, so $\mathcal{M}\mathrm{od}_{\mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z})}(\mathbb{1}) \cong \mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z})$.

Algebra object 2. $\mathtt{A}_2 = \mathbb{1} \oplus -\mathbb{1}$:

μ	$\mathbb{1} \otimes \mathbb{1}$	$\mathbb{1} \otimes -\mathbb{1}$	$-\mathbb{1} \otimes \mathbb{1}$	$-\mathbb{1} \otimes -\mathbb{1}$
$\mathbb{1}$	1			1
$-\mathbb{1}$		1	1	

.

One module $\mathtt{M}_3 = \mathbb{1} \oplus -\mathbb{1}$, so $\mathcal{M}\mathrm{od}_{\mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z})}(\mathbb{1} \oplus -\mathbb{1}) \cong \mathscr{R}\mathrm{ep}(1)$.

Both are 2-representation of $\mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z})$ since *e.g.*

$$-\mathbb{1} \otimes (\mathbb{1} \oplus -\mathbb{1}) \cong -\mathbb{1} \oplus \mathbb{1} \cong \mathbb{1} \oplus -\mathbb{1}.$$

Back

$G = S_3$, S_4 and S_5, their subgroups (up to conjugacy), Schur multipliers and ranks of their 2-simples.

$\mathscr{R}\mathrm{ep}(S_3)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	S_3
$\#$	1	1	1	1
H^2	1	1	1	1
rk	1	2	3	3

$\mathscr{R}\mathrm{ep}(S_4)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	S_3	D_4	A_4	S_4
$\#$	1	2	1	1	2	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	3	5, 2	4, 3	5, 3

$\mathscr{R}\mathrm{ep}(S_5)$

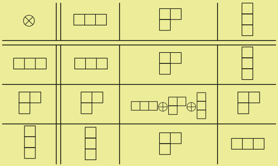
K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
$\#$	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

This is completely different from their classical representation theory.

Back

…their subgroups (up to conjugacy), Schur multipliers and ranks

Example ($G = S_3, K = S_3$); the $\mathbb{N}$-matrices.



$\mathscr{R}\mathrm{es}^G_K(\ydiagram{3}) \cong \ydiagram{3} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathscr{R}\mathrm{es}^G_K(\ydiagram{2,1}) \cong \ydiagram{2,1} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathscr{R}\mathrm{es}^G_K(\ydiagram{1,1,1}) \cong \ydiagram{1,1,1} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
#	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

This is completely different from their classical representation theory.

Back

$G = S_3$, S_4 and S_5, their subgroups (up to conjugacy), Schur multipliers and ranks

Example ($G = S_3, K = S_3$); the $\mathbb{N}$-matrices.

$\otimes$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{3}$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{2,1}$	$\ydiagram{2,1}$	$\ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}$	$\ydiagram{2,1}$
$\ydiagram{1,1,1}$	$\ydiagram{1,1,1}$	$\ydiagram{2,1}$	$\ydiagram{3}$

$$\mathscr{R}\mathrm{es}^G_K(\ydiagram{3}) \cong \ydiagram{3} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathscr{R}\mathrm{es}^G_K(\ydiagram{2,1}) \cong \ydiagram{2,1} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathscr{R}\mathrm{es}^G_K\left(\ydiagram{1,1,1}\right) \cong \ydiagram{1,1,1} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5

Example ($G = S_3, K = \mathbb{Z}/2\mathbb{Z} = S_2$); the $\mathbb{N}$-matrices.

$\otimes$	$\ydiagram{2}$	$\ydiagram{1,1}$
$\ydiagram{2}$	$\ydiagram{2}$	$\ydiagram{1,1}$
$\ydiagram{1,1}$	$\ydiagram{1,1}$	$\ydiagram{2}$

$$\mathscr{R}\mathrm{es}^G_K(\ydiagram{3}) \cong \ydiagram{2} \rightsquigarrow \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \mathscr{R}\mathrm{es}^G_K(\ydiagram{2,1}) \cong \ydiagram{2} \oplus \ydiagram{1,1} \rightsquigarrow \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right), \mathscr{R}\mathrm{es}^G_K\left(\ydiagram{1,1,1}\right) \cong \ydiagram{1,1} \rightsquigarrow \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

Back

The Taft Hopf algebra:

$$\mathrm{T}_2 = \mathbb{C}\langle g, x\rangle/(g^2 = 1,\ x^2 = 0,\ gx = -xg) = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]\hat{\otimes}\mathbb{C}[x]/(x^2).$$

T_2-$p\mathcal{M}\mathrm{od}$ is a non-semisimple fiat category.

$$\text{simples} : \{S_0, S_{-1}\} \begin{cases} g.m = \pm m, \\ x.m = 0, \end{cases} \qquad \text{indecomposables} : \{P_0, P_{-1}\}.$$

Tensoring with the projectives P_0 or P_{-1} gives a 2-representation of T_2-$p\mathcal{M}\mathrm{od}$ which however can be twisted by a scalar $\lambda \in \mathbb{C}$. The algebra objects are

$$\mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \otimes \mathbb{C}[x]/(x^2 - \lambda) \quad \text{and} \quad \mathbb{C}[1] \otimes \mathbb{C}[x]/(x^2 - \lambda).$$

This gives a one-parameter family of non-equivalent 2-simples of T_2-$p\mathcal{M}\mathrm{od}$.

Back

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Back

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$$\mathbb{C}[\mathbb{Z}/2 \ldots \quad \mathbb{C}[1]\otimes\mathbb{C}[x]/(x^2 - \lambda).$$

One crucial problem.

There can be infinitely many categorifications.
The decategorifications $[\mathscr{M}_i^{\lambda}]$ are all the same.

This gives a one-pa ... of T_2-$p\mathcal{M}\mathrm{od}$.

Back

Clifford, Munn, Ponizovskiĭ, Green ~1942++. Finite semigroups or monoids.

Example. $\mathbb{N}$, $\mathrm{Aut}(\{1,2,3\}) = S_3 \subset T_3 = \mathrm{End}(\{1,2,3\})$, groups, groupoids, categories, any $\cdot$ closed subsets of matrices, "anything you will ever meet", *etc.*

The cell orders and equivalences:

$$x \leq_L y \Leftrightarrow \exists z\colon zx = y, \quad x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x),$$
$$x \leq_R y \Leftrightarrow \exists z'\colon xz' = y, \quad x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x),$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z'\colon zxz' = y, \quad x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x).$$

Left, right and two-sided cells: Equivalence classes.

Example (group-like). The unit 1 is always in the lowest cell – *e.g.* $1 \leq_L y$ because we can take $z = y$. Invertible elements g are always in the lowest cell – *e.g.* $g \leq_L y$ because we can take $z = yg^{-1}$.

Back

Example (the transformation semigroup T_3). Cells - left $\mathcal{L}$ (columns), right $\mathcal{R}$ (rows), two-sided $\mathcal{J}$ (big rectangles), $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ (small rectangles).

$\mathcal{J}_{\text{lowest}}$	**(123)**, (213), (132), (231), (312), (321)			$\mathcal{H} \cong S_3$
$\mathcal{J}_{\text{middle}}$	**(122)**, (221)	**(133)**, (331)	(233), (322)	$\mathcal{H} \cong S_2$
	(121), (212)	(313), (131)	**(323)**, (232)	
	(221), (112)	**(113)**, (311)	**(223)**, (332)	
$\mathcal{J}_{\text{biggest}}$	**(111)**	**(222)**	**(333)**	$\mathcal{H} \cong S_1$

Cute facts.

- Each $\mathcal{H}$ contains precisely one idempotent e or none idempotent. Each e is contained in some $\mathcal{H}(e)$. (Idempotent separation.)
- Each $\mathcal{H}(e)$ is a maximal subgroup. (Group-like.)
- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ do not kill it. (Apex.)

Back

Clifford, M … monoids.

Theorem. (Mind your groups!)

There is a one-to-one correspondence

$$\begin{Bmatrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{Bmatrix} \xleftrightarrow{\text{one-to-one}} \begin{Bmatrix}\text{simples of (any)}\\ \mathcal{H}(e)\subset\mathcal{J}(e)\end{Bmatrix}.$$

Thus, the maximal subgroups $\mathcal{H}(e)$ (semisimple over $\mathbb{C}$) control the whole representation theory (non-semisimple; even over $\mathbb{C}$).

Example (… ns), right $\mathcal{R}$ (rows), tw…

$\mathcal{J}_{\text{low…}}$				… $\cong S_3$
$\mathcal{J}_{\text{middle}}$	**(121)**, (212)	(313), (131)	**(323)**, (232)	$\mathcal{H}\cong S_2$
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Example (... ns), right $\mathcal{R}$ (rows), two ...

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Example. (T_3.)

$\mathcal{H}(e) = S_3, S_2, S_1$ gives $3 + 2 + 1 = 7$ simples.

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- Each $\mathcal{H}$ contains precisely one idempotent e or none idempotent. Each e is contained in some $\mathcal{H}(e)$. (Idempotent separation.)
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Back

Clifford, M ... monoids.

Example (... ns), right $\mathcal{R}$ (rows), two ...

$\mathcal{J}_{low}$ ... $\cong S_3$

$\mathcal{J}_{middle}$ | $(121), (212)$ | $(313), (131)$ | $(323), (232)$ | $\mathcal{H} \cong S_2$

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$\mathcal{H}(e) = S_3, S_2, S_1$ gives $3 + 2 + 1 = 7$ simples.

Cute facts.

- Each $\mathcal{H}$ ... . Each e is contain...
- Each $\mathcal{H}$ ...
- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ do not kill it. (Apex.)

This is a general philosophy in representation theory.

Buzz words. Idempotent truncations, Kazhdan–Lusztig cells, quasi-hereditary algebras, cellular algebras, *etc.*

Back

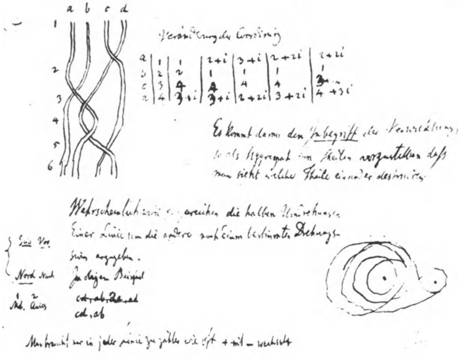


Figure: The first ever "published" braid diagram. (Page 283 from Gauß' handwritten notes, volume seven, $\leq$1830).

Tits $\sim$1961++. Gauß' braid group is the type A case of more general groups.

Back

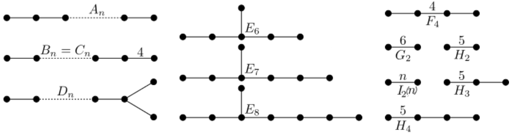
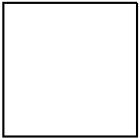


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.
Type A_3 $\leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S_4.
Type B_3 $\leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
Type H_3 $\leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.
For I_8 we have a 4-gon:

Idea (Coxeter ∼1934++).



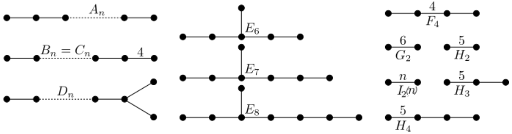


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type $A_3 \leftrightsquigarrow$ tetrahedron $\leftrightsquigarrow$ symmetric group S

Fact. The symmetries are given by exchanging flags.

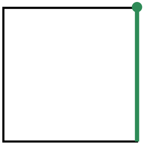
Type $B_3 \leftrightsquigarrow$ cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z}) \rtimes S_3$.

Type $H_3 \leftrightsquigarrow$ dodecahedron/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group.

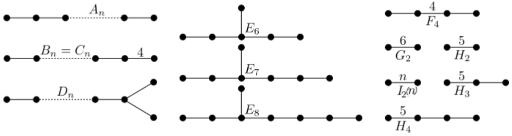
For I_8 we have a 4-gon:

Fix a flag F.

Idea (Coxeter ∼1934++).



Back



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Examples.

Type A_3 ⇝ tetrahedron ⇝ symmetric group S_4.
Type B_3 ⇝ cube/octahedron ⇝ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$.
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Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

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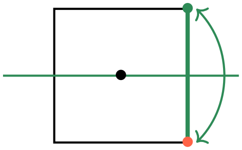
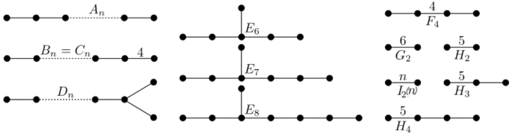


Back



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Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

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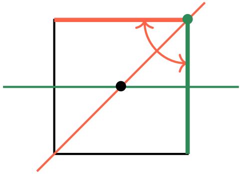

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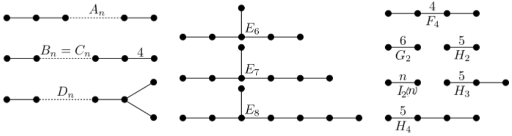

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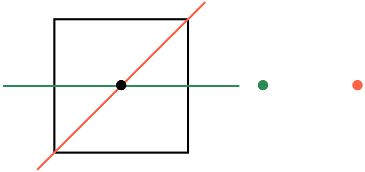
Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Idea (Coxeter ∼1934++).



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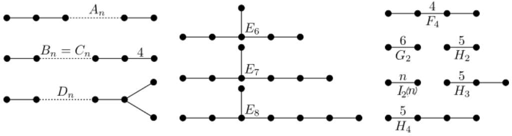


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

This gives a generator-relation presentation.

Type A_3 ⟿ tetrahedron ⟿ symmetric group S_4.

Type B_3 ⟿ And the braid relation measures the angle between hyperplanes.

Type H_3 ⟿ dodecahedron/icosahedron ⟿ exceptional Coxeter group.

For I_8 we have a 4-gon:

Fix a flag F.

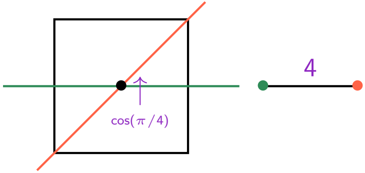
Fix a hyperplane H_0 permuting the adjacent 0-cells of F.

Fix a hyperplane H_1 permuting the adjacent 1-cells of F, etc.

Write a vertex i for each H_i.

Connect i, j by an n-edge for H_i, H_j having angle $\cos(\pi/n)$.

Idea (Coxeter ∼1934++).



Back

Example (type B_2).

$W = \langle s, t \mid s^2 = t^2 = 1, tsts = stst \rangle.$

$W = \{1, s, t, st, ts, sts, tst, w_0\}$

$\mathrm{H}(W) = \mathbb{C}(v)\langle H_s, H_t \mid H_s^2 = (v^{-1} - v)H_s + 1, H_t^2 = (v^{-1} - v)H_t + 1, H_tH_sH_tH_s = H_sH_tH_sH_t \rangle$

KL basis:

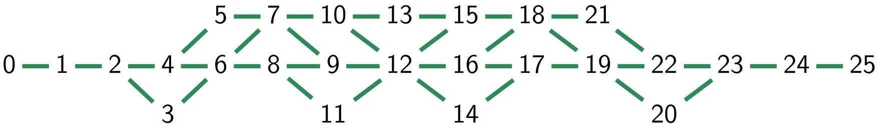
$$c_1 = 1, c_s = vH_s + v^2, c_t = vH_t + v^2, etc.$$

$c_s^2 = (1 + v^2)c_s$. (Quasi-idempotent, but "positively graded".)

Back

Example (SAGE). The Weyl group of type B_6. Number of elements: 46080. Number of cells: 26, named 0 (lowest) to 25 (biggest).

Cell order:



Size of the cells and *a*-value:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	15	11	16	17	12	15	25	25	36

Back

Example (cell 12).

Cell 12 is a bit scary:

$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{5,5}$	$\mathbf{4}_{5,5}$	$\mathbf{1}_{5,20}$	$\mathbf{2}_{5,25}$	$\mathbf{2}_{5,25}$
$\mathbf{1}_{20,5}$	$\mathbf{1}_{20,5}$	$\mathbf{4}_{20,20}$	$\mathbf{2}_{20,25}$	$\mathbf{2}_{20,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{4}_{25,25}$	$\mathbf{1}_{25,25}$
$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,5}$	$\mathbf{2}_{25,20}$	$\mathbf{1}_{25,25}$	$\mathbf{4}_{25,25}$

Example (SAGE). The W er of elements: 46080.
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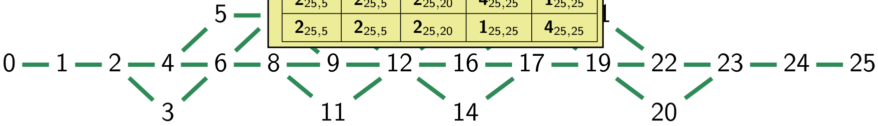
Cell order:



Size of the cells and *a*-value:

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
a	0	1	2	3	3	4	4	5	5	6	6	6	7	9	10	10	10	15	11	16	17	12	15	25	25	36

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$c_d c_d =$$
$$(1 + 5v^2 + 12v^4 + 18v^6 + 18v^8 + 12v^{10} + 5v^{12} + v^{14})c_d$$
$$+(v^2 + 4v^4 + 7v^6 + 7v^8 + 4v^{10} + v^{12})c_u$$
$$+(v^{-4} + 5v^{-2} + 11 + 14v^2 + 11v^4 + 5v^6 + v^8)c_{121232123565}$$

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565, \ u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$a_d a_d =$$
$$(1 + 5v^2 + 12v^4 + 18v^6 + 18v^8 + 12v^{10} + 5v^{12} + v^{14})c_d$$
$$+(v^2 + 4v^4 + 7v^6 + 7v^8 + 4v^{10} + v^{12})c_u$$
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Bigger friends.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$a_d a_d = \\ (1 + 5v^2 + 12v^4 + 18v^6 + 18v^8 + 12v^{10} + 5v^{12} + v^{14})c_d \\ +(v^2 + 4v^4 + 7v^6 + 7v^8 + 4v^{10} + v^{12})c_u$$

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565, \ u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$a_d a_d =$$
$$(1+5v^2+12v^4+18v^6+18v^8+12v^{10}+5v^{12}+v^{14})c_d$$
$$+(v^2+4v^4+7v^6+7v^8+4v^{10}+v^{12})c_u$$

Killed in the limit $v \to 0$.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example (SAGE). Here is a random calculation in the cell 12 for type B_6.

$$a_d a_d =$$
$$a_d$$

Looks much simpler.

Graph:

$$1 \overset{4}{—} 2 — 3 — 4 — 5 — 6$$

Elements (shorthand $s_i = i$):

$$d = d^{-1} = 132123565, \; u = u^{-1} = 12132123565.$$

Back

Example: Hecke algebras as non-semisimple fusion rings (Lusztig ∼1984).

type	A	$B = C$	D	E_6
worst case	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{R}\mathrm{ep}(1)$	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z}^d)$	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z}^d)$	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{R}\mathrm{ep}(S_3)$

type	E_7	E_8	F_4	G_2
worst case	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{R}\mathrm{ep}(S_3)$	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{R}\mathrm{ep}(S_5)$	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{R}\mathrm{ep}(S_4)$	$\mathrm{A}_0^{\mathcal{H}} \leftrightsquigarrow \mathscr{SO}(3)_6$

This gives a complete classification of simples for finite Weyl type Hecke algebras.

Back