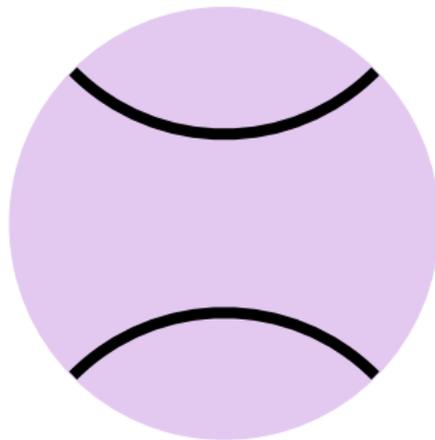
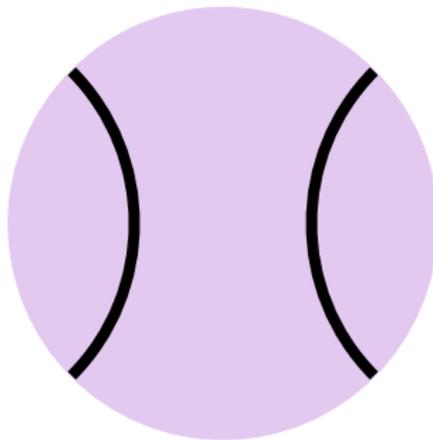
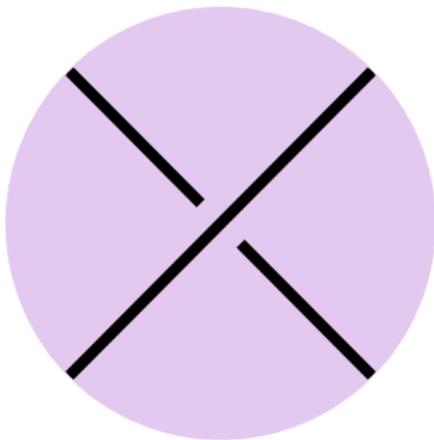


Temperley–Lieb times four

Or: Invariant theory, magnetism, subfactors and skein

Daniel Tubbenhauer



March 2022

Throughout

Please convince yourself that I haven't messed up while picking my quotations from my stolen material

The Temperley–Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, e.g.:

- ▶ Via valence bond theory Rumer–Teller–Weyl (RTW) ~1932
 - ▶ Via the Potts model Temperley–Lieb ~1971
 - ▶ Via subfactors Jones ~1983
 - ▶ Via skein theory Kauffman ~1987
-

Eine für die Valenztheorie geeignete Basis
der binären Vektorinvarianten.

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

Vorgelegt von H. WEYL in der Sitzung am 28 Oktober 1932.

Proc. Roy. Soc. Lond. A. 322, 251–280 (1971)
Printed in Great Britain

Relations between the ‘percolation’ and ‘colouring’
problem and other graph-theoretical problems associated with
regular planar lattices: some exact results for
the ‘percolation’ problem

BY H. N. V. TEMPERLEY

Department of Applied Mathematics, University College, Swansea, Wales, U.K.

AND E. H. LIEB†

*Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, Mass., U.S.A.*

(Communicated by S. F. Edwards, F.R.S.—Received 11 November 1970)

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Warning

I consider the two 1932 papers below as one

RUMER, G., Zur Theorie der Spinvalenz	337
— TELLER, E., und WEYL, H., Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten	499

They are quite similar and appeared in the same issue of
Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen
Mathematisch-Physikalische Klasse
(not continue from 1933 onward)

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

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the ‘percolation’ problem

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The Temperley

The TL calcul

► Via valenc

► Via the P

► Via subfac

► Via skein

Eine für die
der bi

G. Rumer (Moska

Vorgelegt von H. WEYL in der Sitzung am 28 Oktober 1932.

What we will see today

(2) Quantum chemistry

(3) Statistical mechanics

(4) Operator theory

(1) Quantum topology

The collage contains several diagrams:

- Top Left:** Molecular orbitals for a diatomic molecule. Two carbon atoms are shown with p-orbitals. A **Sigma bond (1 pair of electron)** is formed by the head-to-head overlap of orbitals, and a **pi bond (1 pair of electron)** is formed by the side-to-side overlap of orbitals.
- Top Right:** A **Box of Gas** containing particles, with a cartoon character holding a clipboard and a speech bubble containing U, V, N .
- Bottom Left:** A circular diagram with a central white circle and four grey sectors, each containing a star.
- Bottom Right:** A lattice of diagrams representing states. Each node contains a diagram of two particles and a mathematical expression of the form $g^{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle$.

percolation' and 'colouring'
oretical problems associated with
es: some exact results for
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V. TEMPERLEY
University College, Swansea, Wales, U.K.
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Massachusetts Institute of Technology,
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What we will see today

(2) Quantum chemistry

(3) Statistical mechanics

(4) Operator theory

(1) Quantum topology

The collage contains several diagrams:

- Top Left:** Molecular orbitals showing a sigma bond (1 pair of electron) and a pi bond (1 pair of electron) between two atoms.
- Top Right:** A 'Box of Gas' containing particles, with a scientist character and a speech bubble containing 'U, V, N'.
- Bottom Left:** A circular diagram with shaded sectors and stars, possibly representing a phase diagram or a specific mathematical model.
- Bottom Right:** A lattice of nodes connected by lines, with mathematical labels such as $g^{(i_1, \dots, i_n)}$ associated with the nodes.

percolation' and 'colouring'
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V. TEMPERLEY
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H. LIEB†
Massachusetts Institute of Technology,
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(1) is the newest incarnation of the TL calculus but easiest to explain, so I start with (1)

The Temperley–Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, e.g.:

▶ Via valence bond theory: Rumer, Teller, Wall (RTW) — 1932

▶ Via the Potts

▶ Via subfactor

▶ Via skein theory

Not discussed today, but honorable mentions

The TL calculus also appears in...

- ...the theory of quantum groups
- ...integrable models
- ...representation theory of reductive groups
- ...categorical quantum mechanics
- ...logic and computation
- ...probably more that I am not aware of

Eine für die Valenz
der binären Vektorinvarianten.

Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem

V. TEMPERLEY
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† F.R.S.—Received 11 November 1970)

Example (of a folk theorem in quantum group theory)

The TL calculus is equivalent to $\mathit{Tilt}_{\mathbb{K}}(U_q(\mathfrak{sl}_2))$
(after additive + idempotent completion)

G. Rumer (Moskva)

Vorgelegt von H.

The Temperley–Lieb (TL) calculus is everywhere

Today's talk is based on:

my memory (horrible reference...)

On the Number of Rumer Diagrams

Valentin Vankov Iliev *

The Increasingly Popular Potts Model

or

A Graph Theorist Does Physics (!)

Jo Ellis-Monaghan

Subfactors

in Memory of Vaughan Jones

Zhengwei Liu

Tsinghua University

Math-Science Literature Lecture Series

November 23, 2020, Harvard CMSA and Tsinghua YMSC

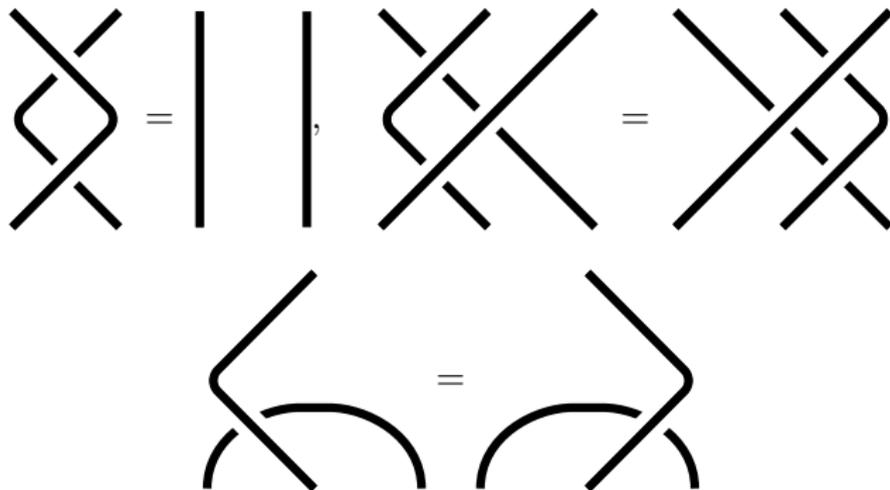
The above are easy to google (its worth it!)

Kauffman's construction ~1987

Step 1 Take the framed tangle calculus **Tan** with generators



and relations being the usual tangle relations, e.g.



Warning

I do not want to be precise what “calculus” means since it doesn't matter and is a bit messy in the literature, e.g.:

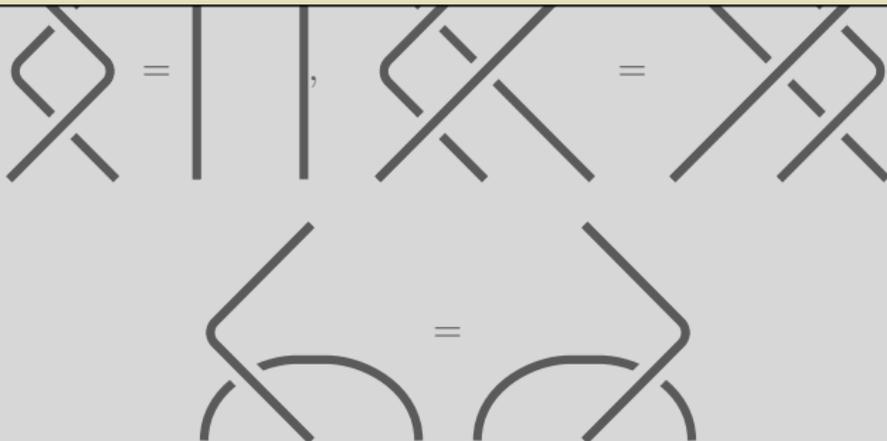
RTW never used any precise formulation

TL used algebras, but not using that terminology

Jones used algebras and operads, but not using the latter terminology

Kauffman used operads, but not using that terminology

Other researchers might prefer monoidal categories (e.g. following Turaev ~1990)



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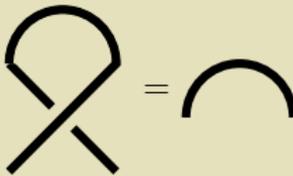
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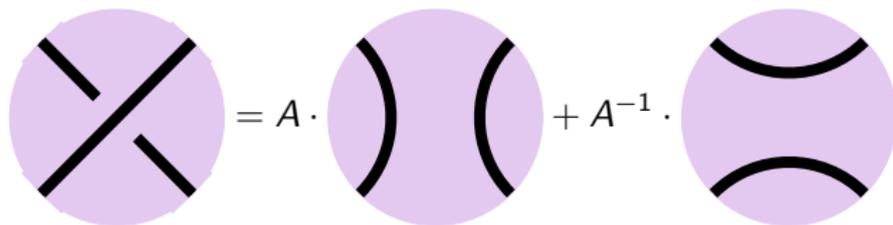
Other researchers might prefer monoidal categories (e.g. following Turaev ~1990)

Note that *Tan* is framed
so no relations of the form



Kauffman's construction ~ 1987

Step 2 Make **Tan** $\mathbb{Z}[A, A^{-1}]$ -linear and impose



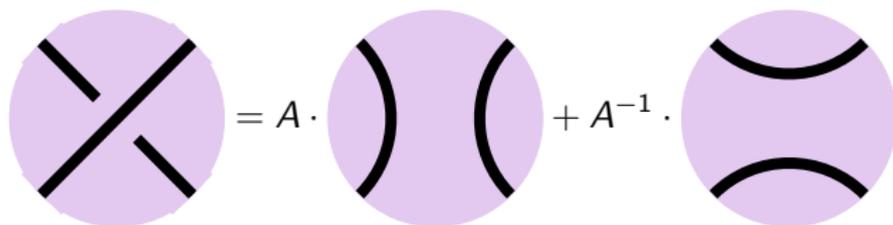
The diagram illustrates the Kauffman skein relation. It shows three purple circular regions. The first region on the left contains two thick black lines crossing each other. This is followed by an equals sign, then the letter 'A' multiplied by a region containing two thick black arcs on the left side of the circle. This is followed by a plus sign, then 'A inverse' multiplied by a region containing two thick black arcs on the right side of the circle.

$$\text{Crossing} = A \cdot \text{Left Arcs} + A^{-1} \cdot \text{Right Arcs}$$

Kauffman skein relation

Kauffman's construction ~ 1987

Step 2 Make **Tan** $\mathbb{Z}[A, A^{-1}]$ -linear and impose



Kauffman skein relation

Kauffman skein relation
 \longleftrightarrow
averaging over ways to get rid of the crossings

Here I am faithfully reproducing a constant disagreement in the literature over the meaning of the “quantum parameter”
In quantum group theory $q = A^2$

Why the A?

The A in Kauffman's formula only became clear in the light of Jones' paper

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS¹

BY VAUGHAN F. R. JONES²

that appeared a bit earlier than Kauffman's paper

STATE MODELS AND THE JONES POLYNOMIAL

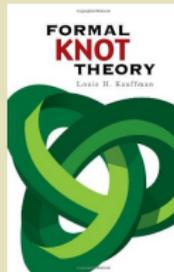
LOUIS H. KAUFFMAN

(Received in revised form 1 September 1986)

§1. INTRODUCTION

IN THIS PAPER I construct a state model for the (original) Jones polynomial [5]. In [6] a state model was constructed for the Conway polynomial.

Before 1985 Kauffman tried but didn't quite get there; [6] ~1983 is:



Kauffman's construction ~ 1987

Step 3 Realize that one also need impose to impose

$$\bigcirc = \delta = -A^2 - A^{-2}$$

Then you are done and we have the TL calculus $\mathbf{TL}_{\mathbb{Z}[A, A^{-1}]}(\delta)$

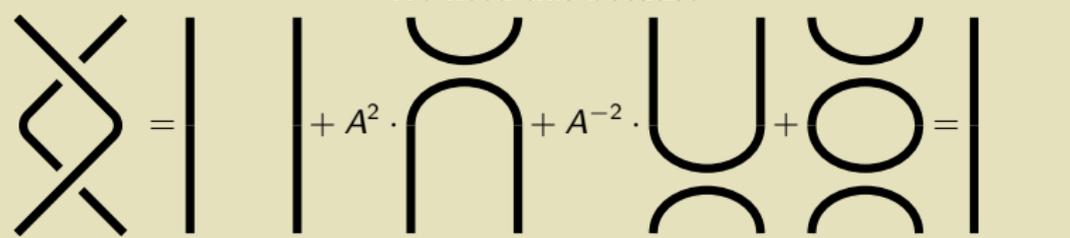
Kauffman's construction ~ 1987

Step 3 Realize that one also need impose to impose

$$\bigcirc = \delta = -A^2 - A^{-2}$$

Then you are done and we have the TL calculus $TL_{\mathbb{Z}[A, A^{-1}]}(\delta)$

We need this because



implies $\delta = -A^2 - A^{-2}$

This is Kauffman's famous calculation

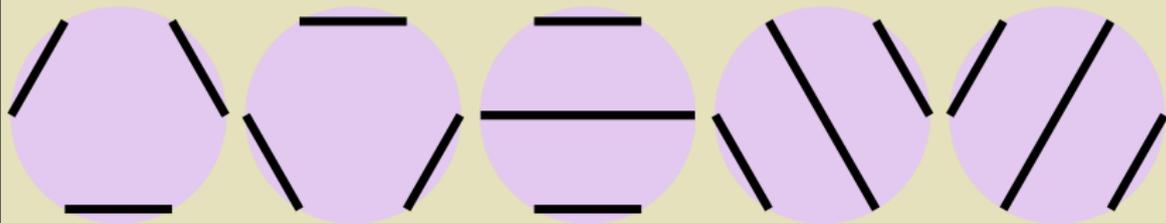
Kauffman's construction ~ 1987

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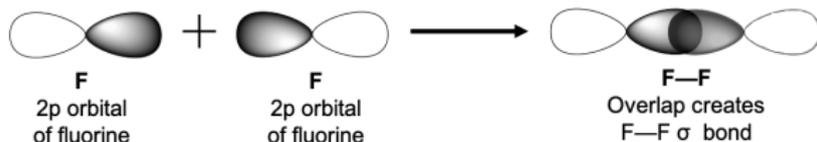
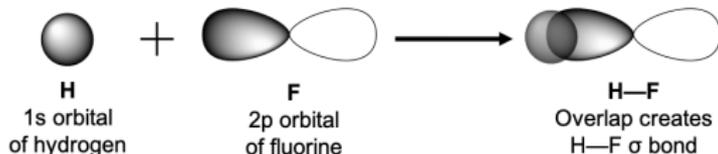
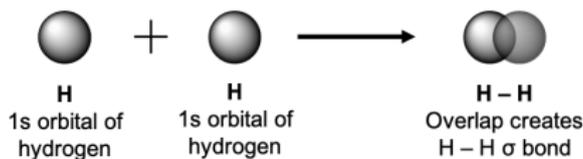
The

Example (6 points)



This is a basis of the six strand case
In general, the Catalan numbers give the dimension

The RTW construction ~ 1932



- ▶ **Problem** Find a model for chemical bonding
- ▶ Valence bond theory uses methods of quantum mechanics to explain bonding
- ▶ The RTW paper models valence bonds using $SL_2(\mathbb{C})$

The RTW construction ~ 1932

- ▶ Each atom is a vector $x = (x_1, x_2) \in \mathbb{C}^2$
- ▶ Each bond $[x, y]$ is a matrix determinant

$$[x, y] = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

viewed as a polynomial with four variables

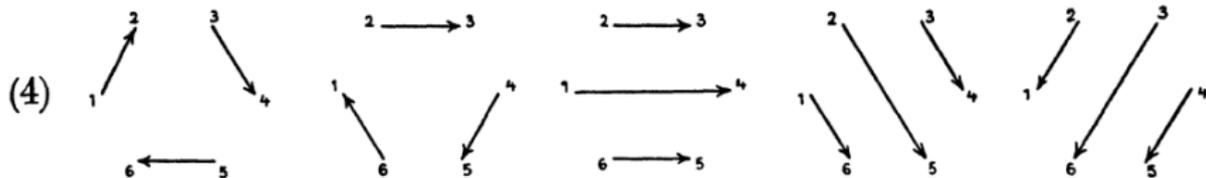
- ▶ These determinants are the building blocks of all $f: \mathbb{C}^{2n} \rightarrow \mathbb{C}$ that are invariant under transformation with determinant 1
- ▶ First fundamental theorem of invariant theory

Any $SL_2(\mathbb{C})$ -invariant function is a linear combination of products

$$[x^{(1)}, y^{(1)}] \cdot \dots \cdot [x^{(k)}, y^{(k)}]$$

The RTW construction ~1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind



- ▶ RTW now put the atoms on a circle
- ▶ Then RTW draw bonds as lines
- ▶ The result is TL diagrams coming from valence theory:

atoms=points and bonds=strands

The Kauffman bracket via valence bonds

aus N Strichen zwischen den n Punkten x, y, \dots, z . Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

$$(3) \quad \begin{array}{c} x & z \\ \diagdown & / \\ & \diagup & \diagdown \\ y & l \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} | \\ | \end{array},$$

Kreuzungen auflösen kann¹⁾. Natürlich ist mit dieser Bemerkung

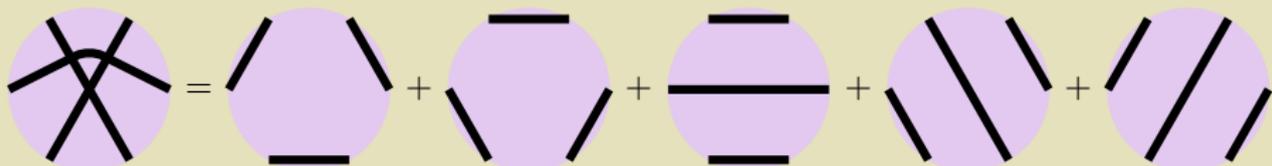
The Kauffman bracket follows easily from the RTW setting:

$$[x, l][y, z] = (x_1 l_2 - x_2 l_1)(y_1 z_2 - y_2 z_1) = [x, z][y, l] + [x, y][l, z]$$

► RTW now put the atoms on a circle

► Then RTW draw bonds as lines

Example



The Kauffman bracket via valence bonds

aus N Strichen zwischen den n Punkten x, y, \dots, z . Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

$$(3) \quad \begin{array}{c} x & z \\ \diagdown & / \\ & \\ / & \diagdown \\ y & t \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} + \begin{array}{c} \circ \\ | \\ \circ \end{array} \begin{array}{c} \circ \\ | \\ \circ \end{array},$$

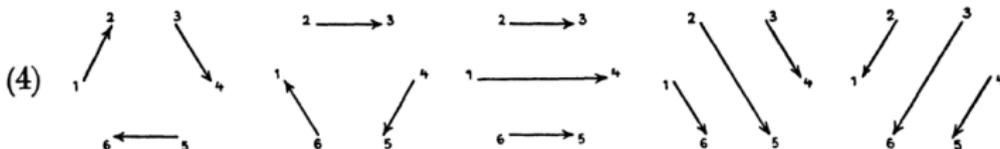
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Second fundamental theorem of invariant theory via valence bonds

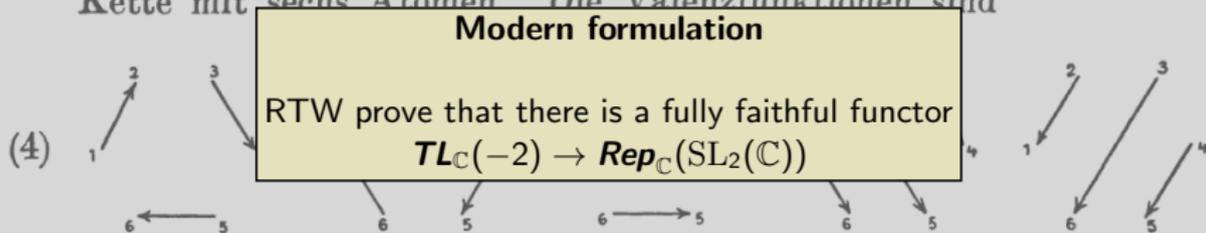
3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind



RTW also prove that crossingless matching form a basis

The RTW construction ~1932

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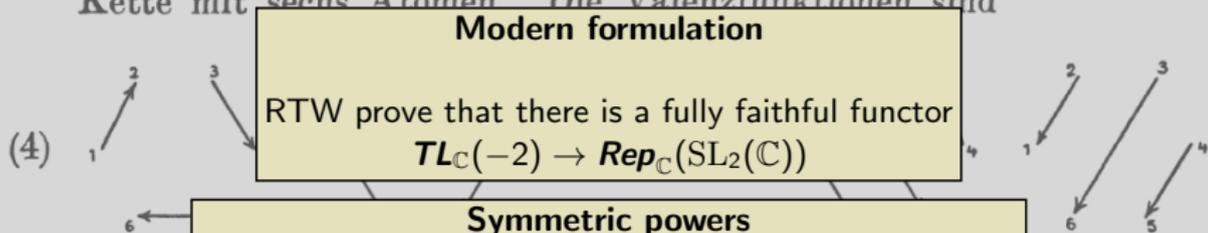


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The RTW construction ~1932

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind



- ▶ RTW nov
- ▶ Then RTW
- ▶ The result

Symmetric powers

Als Beispiel betrachten wir das Hydrazin NH_2-NH_2 . Wir bezeichnen mit a, b die beiden N-Atome, mit $1, 2, 3, 4$ die vier H-Atome. Ordnen wir die Atome auf einem Kreis an, so erhalten wir nach der Anweisung folgende sechs Valenzzustände als Basis³⁾:

(3)

Actually it is more general:
 RTW also address these questions for symmetric powers with $Sym^k \mathbb{C}^2$ corresponding to a k -valence bond

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

Let's ask SAGEMath whether the RTW basis has the correct number of elements:

Type some Sage code below and press Evaluate.

```
1 A1 = WeylCharacterRing(['A', 1])
2 A1(1,0)*A1(1,0)*A1(1,0)
```

Evaluate

$2 \cdot A1(2,1) + A1(3,0)$

Type some Sage code below and press Evaluate.

```
1 A1 = WeylCharacterRing(['A', 1])
2 A1(3,0)*A1(1,0)*A1(1,0)
```

Evaluate

$A1(3,2) + 2 \cdot A1(4,1) + A1(5,0)$

Indeed, $1^2 + 2^2 = 5$ and $1^2 + 2^2 + 1^2 = 6$

3. Als zweites Beispiel behandeln wir die zyklische einvalentige Kette mit sechs Atomen. Die Valenzfunktionen sind

Edward Teller is the big name here

Edward Teller - Wikipedia <https://en.wikipedia.org/v>

WIKIPEDIA

Edward Teller

Edward Teller (Hungarian: *Teller Ede*; January 15, 1908 – September 9, 2003) was a Hungarian-American theoretical physicist who is known colloquially as "the father of the hydrogen bomb" (see the Teller-Ulam design), although he did not care for the title, considering it to be in poor taste.^[1] Throughout his life, Teller was known both for his scientific ability and for his difficult interpersonal relations and volatile personality.

Edward Teller



UNCLASSIFIED

LOS ALAMOS SCIENTIFIC LABORATORY
OF
THE UNIVERSITY OF CALIFORNIA

March 9, 1951 LAM-1285

This document consists of 20 pages.
No. 20 of 20 copies. Series A

OR ENRHOLOGICALS: DEVIATIONS I. (U)
Hydrodynamic Lenses and Radiation Mirrors

Work done by:
E. Teller
S. Glas

Report written by:
E. Teller
S. Glas

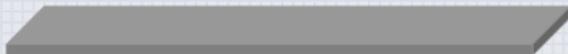
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WRAPUP DATA

Teller's Wikipedia page has 25 printed pages (01.Mar.2022); it is very readable

The Ising Model

Consider a sheet of metal:



It has the property that at low temperatures it is magnetized, but as the temperature increases, the magnetism “melts away”.*

We would like to model this behavior. We make some simplifying assumptions to do so.

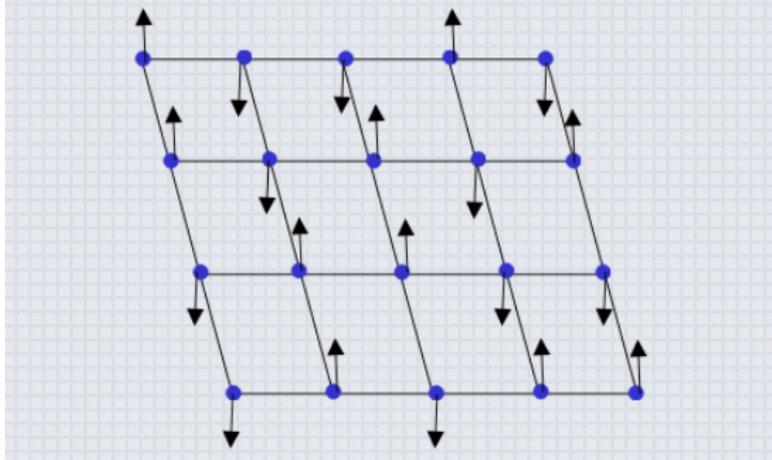
- The individual atoms have a “spin”, i.e., they act like little bar magnets, and can either point up (a spin of +1), or down (a spin of -1).
- Neighboring atoms with the same spins have an interaction energy, which we will assume is constant.
- The atoms are arranged in a regular lattice.

▶ The Ising model’s interpretation is explained above **Magnetism**

▶ The Potts model is a generalization of the Ising model

▶ TL studied these **Solid-state physics**

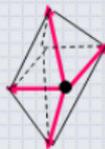
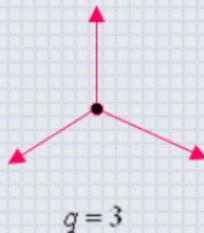
A choice of 'spin' at each lattice point.



- ▶ The Ising model is a lattice model
- ▶ The states are spins (up and down)
- ▶ Recall that what we want to know is Z_S Partition function

$q=Q!$ The Potts Model

Now let there be q possible states....



$q = 4$

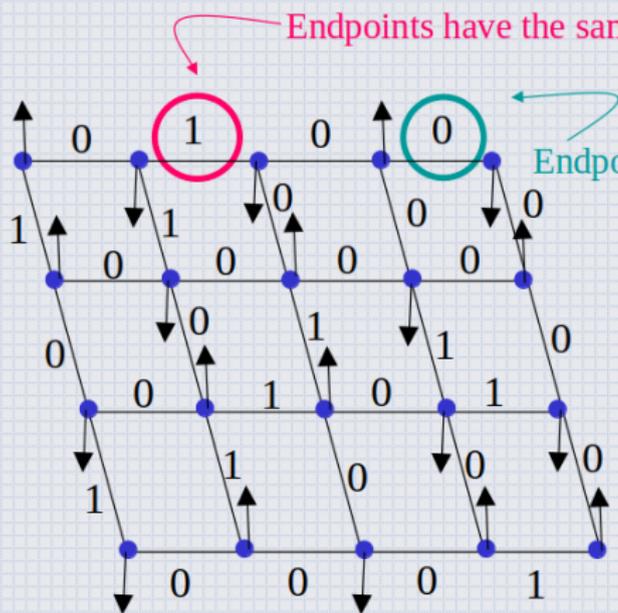


Orthogonal vectors,
with δ replaced by dot
product

Colorings of the points
with q colors

- ▶ The Potts model is a lattice model
- ▶ The states are “spins” from 1 to Q (Ising $Q = 2$)
- ▶ We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

The TL construction ~1971



Endpoints have the same spins, so δ is 1.

Endpoints have different spins, so δ is 0.

$$H(w) = \sum_{\text{edges}} -J \delta_{S_i, S_j}$$

$H(w)$ of this system is $-10J$

$$Z_X = \sum_{\text{states}} \exp(-\beta H)$$

$$H = \sum_{\text{edges}} \delta_{\sigma_i, \sigma_j} J, \quad J = \text{energy (a number)}$$

► We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

The Potts model is very applicable:

Applications of the Potts Model

- **Liquid-gas transitions**
- **Foam behaviors**
- **Magnetism**
- **Biological Membranes**
- **Social Behavior**
- **Separation in binary alloys**
- **Spin glasses**
- **Neural Networks**
- **Flocking birds**
- **Beating heart cells**

These are all complex systems with nearest neighbor interactions.

These microscale interactions determine the macroscale behaviors of the system, in particular phase transitions.

▶ The

▶ The

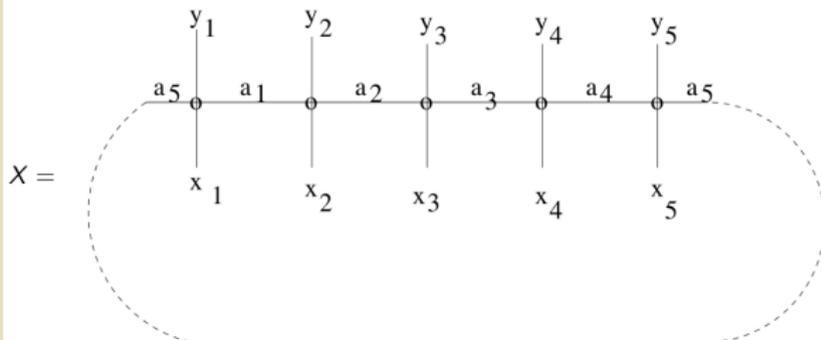
For all of these there is some form of the TL calculus

▶ We want to know $Z_S = Z_S(\beta = 1/kT)$ Partition function

The TL construction ~1971

Recall from last time that solving the model “is equivalent to having good expressions for transfer matrices”

Transfer matrices



$$T^{\text{row}} = \sum_{a_i} R(a_n, a_1 | x_1, y_1) \dots R(a_{n-1}, a_n | x_n, y_n)$$

- ▶ For \mathbb{Z}^2 with periodic boundary use transfer matrix above
- ▶ **Problem** Computing the largest eigenvalue becomes infeasible

▶ The states are “spins” from 1 to Q (Ising $Q = 2$)

▶ We want to know $Z_S = Z_S(\beta = 1/kT)$ **Partition function**

The TL construction ~1971

$$\left. \begin{aligned} (1 - T_{12})[1\ 2] &= 2[1\ 2], \\ (1 - T_{23})[1\ 2][3\ 4] &= [4\ 1][2\ 3], \\ (1 - T_{23})[4\ 1][2\ 3] &= 2[4\ 1][2\ 3]. \end{aligned} \right\}$$

- ▶ $V = \mathbb{C}^Q$; the operators below are on tensor powers of V
- ▶ $p =$ multiplication by $1/\sqrt{Q}$, $d_{i,i+1}(v_i \otimes v_j) = \delta_{ij} v_i \otimes v_j$
 $Q = (A + A^{-1})^2$, e.g. $Q = 2$ implies $A = (-1)^{1/4}$
- ▶ $E_{2i-1} = 1 \otimes \dots \otimes 1 \otimes p \otimes 1 \otimes \dots \otimes 1$ (p in the i th entry)
- ▶ $E_{2i} = 1 \otimes \dots \otimes 1 \otimes d_{i,i+1} \otimes 1 \otimes \dots \otimes 1$ ($d_{i,i+1}$ in the i th entry)
- ▶ Up to scaling, $E_k = 1 - T_{k(k+1)}$ **Kauffman bracket**
- ▶ The transfer matrix with free horizontal boundary conditions is a multiple of $(\prod_{i=1}^{n-1} aE_{2i} + 1)(\prod_{i=1}^n bE_{2i-1} + 1)$ where a and b determined by the boundary condition

The operators satisfy the TL relations

$$E_k \leftrightarrow \text{cup with label } k$$

$$E_k E_k = \sqrt{Q} E_k \leftrightarrow \text{cup over cup} = \sqrt{Q} \text{cup}$$

$$E_k E_{k\pm 1} E_k = E_k \leftrightarrow \text{cup over cup over cup} = \text{cup}$$

$$E_k E_l = E_l E_k \text{ for } |k - l| > 1 \leftrightarrow \text{far commutativity}$$

▶ $V = \mathbb{C}^Q$; th

▶ $p = \text{multiplic}$

$$Q = (A + A)$$

▶ $E_{2i-1} = 1 \otimes$

▶ $E_{2i} = 1 \otimes \dots$

▶ Up to scalin

▶ The transfer

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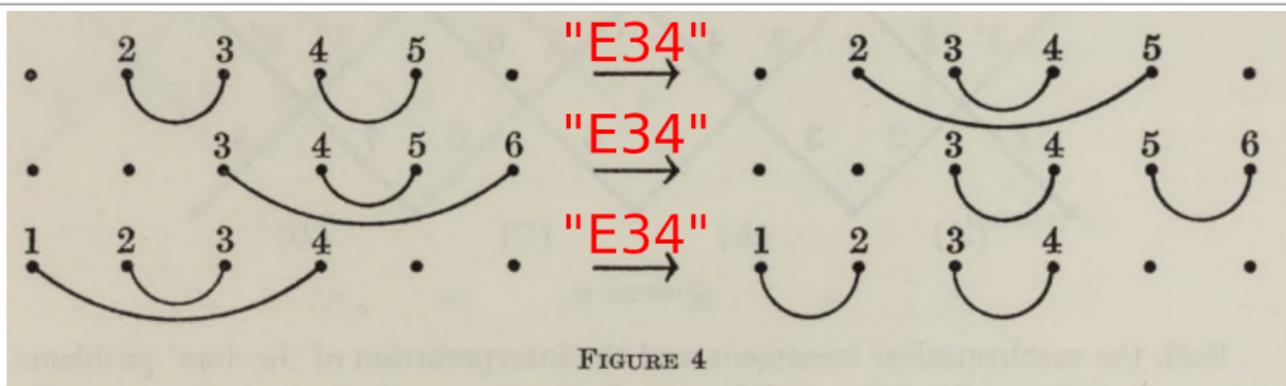
$$E_k E_l = E_l E_k \text{ for } |k - l| > 1 \leftrightarrow \text{far commutativity}$$

TL then show that Z_S is determined by the TL relations

- ▶ $V = \mathbb{C}^Q$; th
- ▶ $p = \text{multiplic}$
- ▶ $Q = (A + A$
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- ▶ The transfer

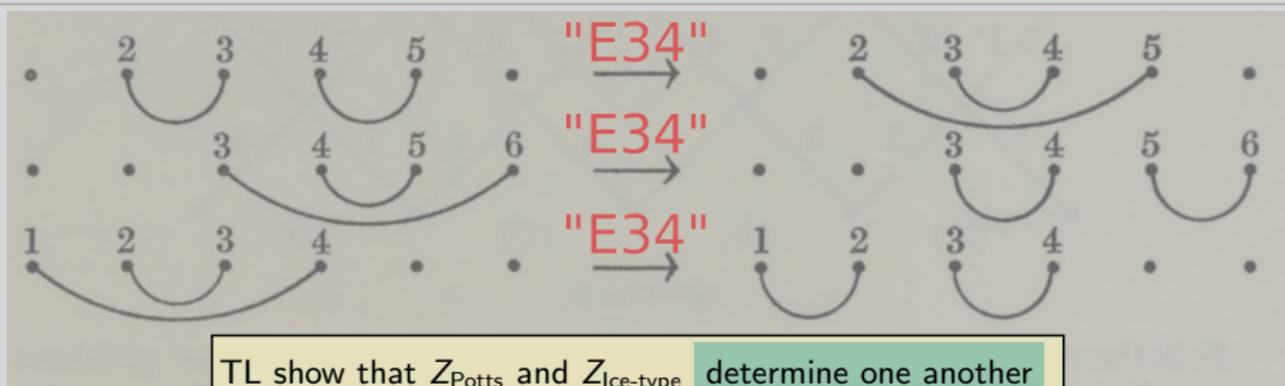
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The TL construction ~1971



- ▶ TL also write down and study cell modules
- ▶ They use the usual diagrammatics to describe these
- ▶ They did not use diagrammatics to describe $TL_C(\sqrt{Q})$ itself
- ▶ They do not compute the dimension of $TL_C(\sqrt{Q})$

The TL construction ~1971



General solution of the Potts model

$$Z_G \approx T(a, b)$$

where $T(x, y)$ is the Tutte polynomial

for $a = (|Q| + \exp(\beta J) - 1) / (\exp(\beta J) - 1)$ and $b = \exp(\beta J)$

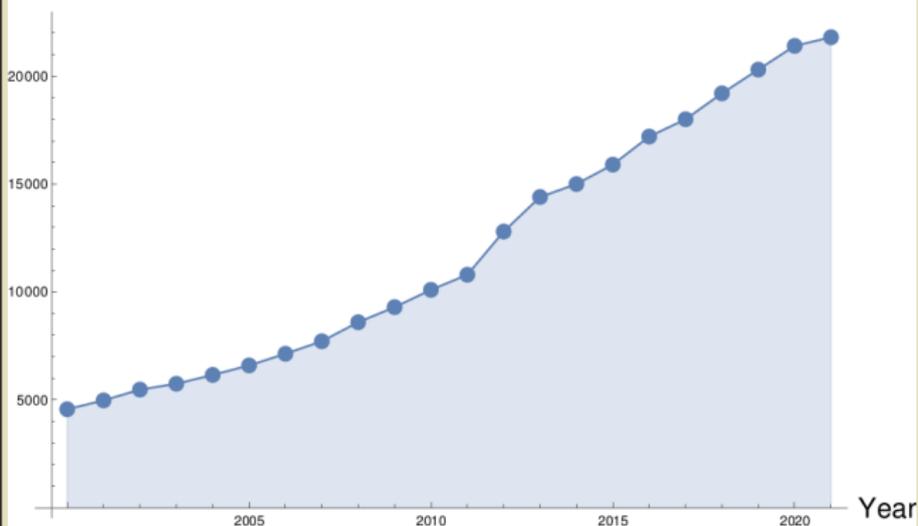
- ▶ TL also
- ▶ They us
- ▶ They did not use diagrammatics to describe $\text{TL}_{\mathbb{C}}(\sqrt{Q})$ itself
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The TL construction ~1971

Impressive!

Articles mentioning Potts models found by google scholar 01.Mar.2022

Count



More than 250000 hits in total

This is a widely spread incarnation of the TL calculus

- ▶ TL also
- ▶ They u
- ▶ They d
- ▶ They d

Index for Subfactors

V.F.R. Jones

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA

- ▶ Factor = von Neumann algebra with trivial center
- ▶ A subfactor is an inclusion of factors $N \subset M$
- ▶ Murray–von Neumann ~1930+ classified factors by types: I , II_1 , II_∞ and III
- ▶ II_1 are the “most exciting” ones They have a unique trace!
We will stick with these (I drop the “of type II_1 ” – it should appear everywhere)

Invent. math. 72, 1–25 (1983)

The word “algebra” is highlighted because there will be representations!

*Inventiones
mathematicae*

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Example

M is a factor, G a nice group, then $M^G \subset M$ is a subfactor

Vague slogan Subfactors \leftrightarrow fixed points of a “quantum group” G action

The is also a version of Galois correspondence and one can recover G from $M^G \subset M$

In this sense subfactor theory generalizes finite group theory

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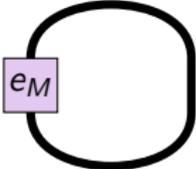
▶ A subfactor is an inclusion of factors $N \subset M$

▶ Jones' paper was one of the starting point of transferring subfactors from functional analysis to algebra/combinatorics and III

▶ II_1 are the “most exciting” ones They have a unique trace!

We will stick with these (I drop the “of type II_1 ” – it should appear everywhere)

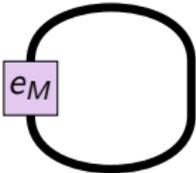
Jones' construction ~ 1983

$$[M : N] \longleftrightarrow \text{tr}(e_M) \in \mathbb{R}_{\geq 0}$$


- ▶ Subfactor $N \subset M$, M is a N -module by left multiplication
- ▶ Assume that M is finitely generated projective N -module
The index $[M : N] \in \mathbb{R}_{\geq 0}$ is the trace of the idempotent e_M for M
- ▶ Jones' index theorem ~ 1983 The index is an invariant of $N \subset M$ and

$$[M : N] \in \left\{ 4 \cos^2\left(\frac{\pi}{k+2}\right) \mid k \in \mathbb{N} \right\} \cup [4, \infty]$$

Jones' construction ~ 1983

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Note the “quantization” below 4:

This was a weird/exciting result!

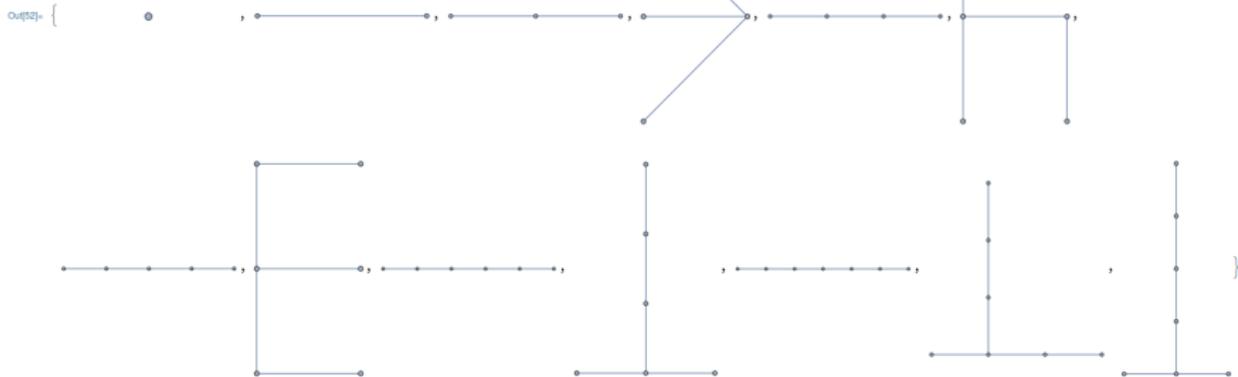
Jones: The most challenging part is constructing these subfactors

Sketch of the quantization argument

Associated a graph G to $N \subset M$ and " $[M : N] = pf(G)$ "

Then use Kronecker's theorem

```
In[51]:= findgraphs4[n_] := Select[GraphData /@ Flatten[Table[GraphData["Connected", m], {m, 1, n}], 1], Last[Sort[Eigenvalues[AdjacencyMatrix[#]], Less]] < 2 &];
findgraphs4[7]
```



§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

Jones' projectors satisfy the scaled TL relations for $\delta = [M : N] = 4 \cos^2\left(\frac{\pi}{k+2}\right)$

$$e_k \iff \frac{1}{[M:N]} \text{ (cup with } k \text{ strands)}$$

$$\blacktriangleright e_k e_k = e_k \iff \text{(two cups)} = \text{(one cup)}$$

$$\blacktriangleright e_k e_{k\pm 1} e_k = \frac{1}{[M:N]} e_k \iff \text{(cup and cap)} = \frac{1}{[M:N]} \text{ (cup)}$$

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$\{e_i | i \geq n\}$ generate a factor R_n
 $R_2 \subset R_1$ is a subfactor of index $4 \cos^2\left(\frac{\pi}{k+2}\right)$

$$\blacktriangleright e_k e_{k \pm 1} e_k = \frac{1}{[M:N]} e_k \iff \text{ (cup and cap) } = \frac{1}{[M:N]} \text{ (cup)}$$

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The Markov property!

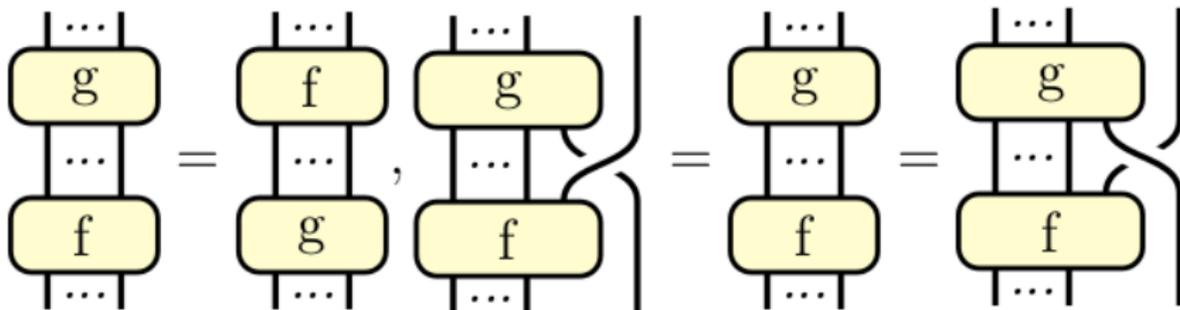
They satisfy the relations

- (I) $e_i^2 = e_i, e_i^* = e_i,$
- (II) $e_i e_{i\pm 1} e_i = t/(1+t)^2 e_i,$ **TL relations**
- (III) $e_i e_j = e_j e_i$ if $|i-j| \geq 2.$

Here t is a complex number. It has been shown by H. Wenzl [24] that an arbitrarily large family of such projections can only exist if t is either real and positive or $e^{\pm 2\pi i/k}$ for some $k = 3, 4, 5, \dots$. When t is one of these numbers, there exists such an algebra for all n possessing a trace $\text{tr}: A_n \rightarrow \mathbb{C}$ completely determined by the normalization $\text{tr}(1) = 1$ and

- (IV) $\text{tr}(ab) = \text{tr}(ba),$
 - (V) $\text{tr}(we_{n+1}) = t/(1+t)^2 \text{tr}(w)$ if w is in $A_n,$
 - (VI) $\text{tr}(a^*a) > 0$ if $a \neq 0$ **Markov trace**
- (note $A_0 = \mathbb{C}$).

In braid pictures (crossing is given by Kauffman skein formula)



In hindsight the crucial result Jones' proved about $TL_C(\delta)$ is the existence of a Markov trace

§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

Jones' projectors satisfy the scaled TL relations for $\delta = [M : N] = 4 \cos^2(\frac{\pi}{k+2})$

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In hindsight the crucial result Jones' proved about $TL_C(\delta)$ is the existence of a Markov trace

8.4 Possible Values of the Index

Why? Well, because of the (Birman-)Jones polynomial:

Jones Polynomial

Jones met Birman in 1984:

Markov trace on the braid group \Rightarrow knot invariant

It was surprising that the Markov trace naturally comes from the trace of the II_1 factor,

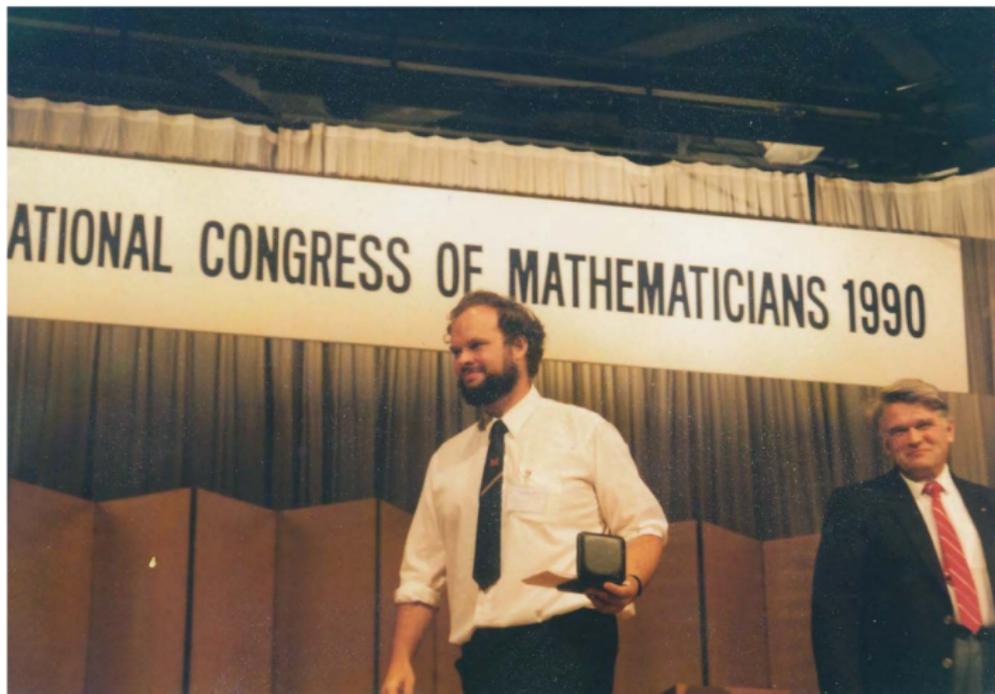
$$\tau(x\sigma_n) = \tau(x), \forall x \in TL_n, \longrightarrow \text{Reidemeister move I} = \left| \begin{array}{c} \text{crossing} \\ \text{vertical line} \end{array} \right|$$

therefore leading to a knot invariant, well-known as the Jones polynomial, by which Jones answered a series of old questions in knot theory in 1985.

$$\text{Diagram 1} = t + t^3 - t^4 \quad \text{Diagram 2} = t^{-1} + t^{-3} - t^{-4}$$

Reflection: $t \rightarrow t^{-1}$.

Jones was awarded the Fields Medal at Kyoto in 1990 for these breakthroughs.



► $e_k e_l = e_l e_k$ for $|k - l| > 1 \iff$ far commutativity

§4. Possible Values of the Index

§4.1. Certain Algebras Generated by Projections

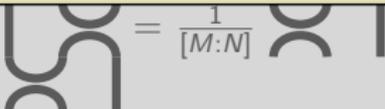
Jones also implicitly coined the name "TL algebra" ~ 1985 :

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS¹

BY VAUGHAN F. R. JONES²

For real t , D. Evans pointed out that an explicit representation of A_n on \mathbb{C}^{2n+2} was discovered by H. Temperley and E. Lieb [23], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A

▶ $e_k e_{k\pm 1} e_k = \frac{1}{[M:N]} e_k \iff$  $= \frac{1}{[M:N]} \text{ (crossing) }$

▶ $e_k e_l = e_l e_k$ for $|k - l| > 1 \iff$ far commutativity

The Temperley-Lieb (TL) calculus is everywhere

The TL calculus was discovered several times, e.g.:

- Via valence bond theory **Rumer-Teller-Maxwell (RTM) –1932**
- Via the Potts model **Temperley-Lieb –1971**
- Via subfactor **Jones –1983**
- Via skein theory **Kauffman –1987**

Eine für die Valenztheorie geeignete Basis der dualen Vektorraumvarianten.

Von
O. Rumer (Moskau), E. Teller und H. West (Ottawa)
 Topologie im 1. Welt: in der Sitzung am 03. Oktober 1932.



Basel Universität | Universität Bonn | Max-Planck-Gesellschaft

The Temperley-Lieb calculus

What we will see today

- (2) Quantum chemistry
- (3) Statistical mechanics
- (4) Operator theory
- (1) Quantum topology

Referred to as *face of a wheel*

Einige für die Valenztheorie geeignete Basis der dualen Vektorraumvarianten.

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(1) is the most incarnation of the TL calculus but easiest to explain, so I start with (1)

The RTM

The Kauffman bracket via valence bonds

Wir haben nun einen Satz, der ein mit TL in Relation ist.

Erinnern wir uns an n Teilchen mit dieser Darstellung

The Kauffman bracket follows easily from the RTM setting: $\langle \mathbb{R}^n, \mathbb{R}^n \rangle = \langle \mathbb{R}^n, \mathbb{R}^n \rangle = \langle \mathbb{R}^n, \mathbb{R}^n \rangle$

RTM now put the strings on a **circle**

Example

Jones' construction –1983

14 Possible Values of the Index

J.A.S. Conjecture: Algebra Generated by Projectors

Jones' projectors satisfy the scaled **TL relation** for $\delta = |M - N| = 4 \cos^2(\pi/5)$

$e_k \circ e_k = \delta e_k$

$e_k \circ e_{k+1} = \delta e_{k+1}$

$e_k \circ e_{k+2} = \delta e_{k+2}$

$e_k \circ e_{k+3} = \delta e_{k+3}$

$e_k \circ e_{k+4} = \delta e_{k+4}$

$e_k \circ e_{k+5} = \delta e_{k+5}$

Kauffman's construction –1987

Step 2: Make Yao $\sum(A, A^{-1})$ -linear and impose

$\text{Crossing} = A \cdot \text{Loop} + A^{-1} \cdot \text{Loop}$

Kauffman skein relation

Kauffman skein relation

averaging over ways to get rid of the crossing

Here I am faithfully reproducing a contour diagram in the literature over the meaning of the "quantum parameter" in quantum group theory: $q = -A^2$

The TL construction –1971

q=Q! The Potts Model

Now let there be q possible states...

Orthogonal vectors, with $\langle \mathbb{R}^q, \mathbb{R}^q \rangle = \delta_{ij}$

Colorings of the points with q -colors

- The Potts model is a lattice model
- The states are "spins" from 1 to Q (hinc $Q = 2$)
- We want to know $Z_q = Z_q(\beta = 1/AT)$ **Partition function**

Jones' construction –

In **highlight**: the crucial result: Jones' proved about $TL_n(\delta)$ is the existence of a Markov trace

Why? Well, because of the **diminishing**-Jones polynomial

Jones Polynomial

Jones met Birman in 1984

Markov trace on the braid group \rightarrow knot invariant

It was surprising that the Markov trace naturally comes from the trace of the H_q factor.

$\tau(\sigma_{ij}) = \tau(\sigma_{ji}), \forall i, j \in \{1, \dots, n\}$ (Hecke algebra more!) $\tau(\sigma_{ij}^2) = \tau(\sigma_{ij})^2, \forall i, j \in \{1, \dots, n\}$

therefore leading to a knot invariant, well known as the Jones polynomial, which Jones announced a series of old questions in last theory in 1985.

Reflection: $\tau = \tau^{-1}$

Thanks for your attention!