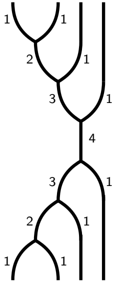
$\mathbf{U}_q(\mathfrak{sl}_n)$ diagram categories via q-Howe duality

Or: "Howe" to make diagrammatic categories work!

Daniel Tubbenhauer

$\mathcal{JW}_4 = \frac{1}{[4]!}$ 

Joint work with David Rose

February 2015

1. $\mathfrak{sl}_2$-spider and representation theory
   - Graphical calculus via Temperley-Lieb diagrams
   - The $\mathfrak{sl}_2$-spider is representation theory

2. Its cousins: The $\mathfrak{sl}_n$-spiders
   - The $\mathfrak{sl}_n$-spiders and representation theory
   - Proof? Quantum skew Howe duality!

3. More cousins: The symmetric $\mathfrak{sl}_2$-spider
   - The symmetric $\mathfrak{sl}_2$-spider and representation theory
   - Proof? Quantum symmetric Howe duality!

The $\mathfrak{sl}_2$-web space

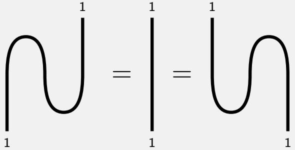
Definition(Rumer-Teller-Weyl 1932)

The $\mathfrak{sl}_2$-web space $W_2(b,t)$ is the free $\mathbb{C}(q) = \mathbb{C}_q$-vector space generated by non-intersecting arc diagrams with b bottom and t top boundary points modulo:

- The circle removal

$$= -q - q^{-1} = -[2]$$

- The isotopy relations



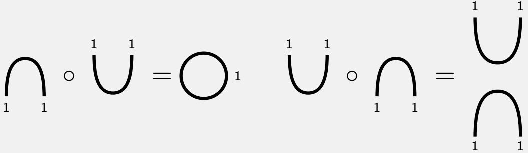
Note that $W_2(b,t)$ is a finite dimensional $\mathbb{C}_q$-vector space!

The $\mathfrak{sl}_2$-spider

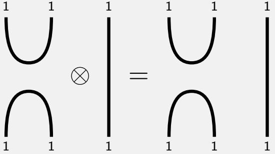
Definition(Kuperberg 1995)

The $\mathfrak{sl}_2$-spider $\mathbf{Sp}(\mathfrak{sl}_2)$ is the monoidal $\mathbb{C}_q$-linear category with:

- Objects are natural numbers and morphisms are $\mathrm{Hom}_{\mathbf{Sp}(\mathfrak{sl}_2)}(k,l) = W_2(k,l)$.
- Composition $\circ$:



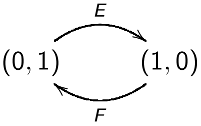
- Tensoring $\otimes$:



Connection to representation theory

Recall that $\mathbf{U}_q(\mathfrak{sl}_2)$ is generated by E, F, K.
Let $V = \mathbb{C}_q^2$ the vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$. Morally:

$$K = \begin{pmatrix} q^{+1} & 0 \\ 0 & q^{-1} \end{pmatrix}$$



$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Fact: All irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-modules are summands of $V^{\otimes k}$ for some $k \in \mathbb{N}$.

Let $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge$ be the monoidal, $\mathbb{C}_q$-linear category consisting of:

- Objects are tensor products $V^{\otimes k} = V \otimes \cdots \otimes V$ of finite length and morphisms are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners between these.
- Composition $\circ$ is composition of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Tensoring $\otimes$ is tensoring of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.

Diagrams for intertwiners

Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}\colon V \otimes V \to \mathbb{C}_q \quad \text{and} \quad \mathrm{cup}\colon \mathbb{C}_q \to V \otimes V,$$

projecting $V \otimes V$ onto $\mathbb{C}_q$ respectively embedding $\mathbb{C}_q$ into $V \otimes V$.
Define a functor $\Gamma^2_\wedge \colon \mathbf{Sp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_\wedge$:

- On objects: k is send to $V^{\otimes k} = V \otimes \cdots \otimes V$.
- On morphisms:

$\mapsto \mathrm{cap}$

$\mapsto \mathrm{cup}$

Theorem(Folklore)

The functor $\Gamma^2_\wedge \colon \mathbf{Sp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{Mod}_\wedge$ is an equivalence of monoidal categories.

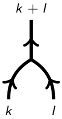
Kuperberg (1995): Let us try the same for other $\mathfrak{g}$'s!

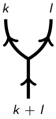
In 1995 Kuperberg rigorously defined "spiders" and introduced spiders for $\mathfrak{sl}_3$, B_2 and G_2. These spiders are diagrammatic categories for $\mathbf{U}_q(\mathfrak{g})$-module categories. His work was very influential: Spiders naturally appear in representation theory, combinatorics, low dimensional topology and algebraic geometry.

- Khovanov and Kuperberg gave a connection to dual canonical bases of $\mathbf{U}_q(\mathfrak{g})$.
- Fontaine, Kamnitzer and Kuperberg identified relations to the geometry of affine Grassmannians via the geometric Satake correspondence.
- Via this, there are relations to affine buildings over these Grassmannians.
- The Reshetikhin-Turaev's invariant of links "live" in spiders.
- Similarly from the Witten-Reshetikhin-Turaev invariants of 3-manifolds.
- $1+1$ or $2+1$-TQFT's and cobordism theories very often bound spiders.
- Via this connections to link homologies and related topics.
- More...

The main step beyond $\mathfrak{sl}_2$: Trivalent vertices

A $\mathfrak{sl}_n$-web is an oriented, labeled trivalent graph locally made of

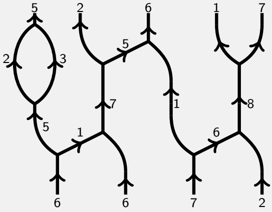
$\mathrm{m}^{k+l}_{k,l} =$ 

$\mathrm{s}^{k,l}_{k+l} =$ 

$k, l, k+l \in \{0, \dots, n\}$

Plus mirrors and sign issues that we skip today. Ask an expert, aka not me.

Example($n > 7$)

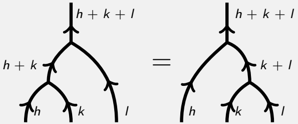


Let us try the same for $\mathfrak{sl}_n$: the $\mathfrak{sl}_n$-web space

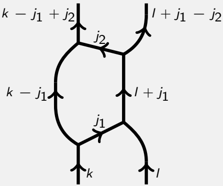
Definition(Cautis-Kamnitzer-Morrison 2012)

The $\mathfrak{sl}_n$-web space $W_n(\vec{k},\vec{l})$ is the free $\mathbb{C}_q$-vector space generated by $\mathfrak{sl}_n$-webs with $\vec{k}$ and $\vec{l}$ at the bottom and top modulo:

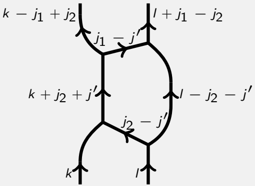
- Isotopy and associativity relations



- Others. Most notably the scary square switches:



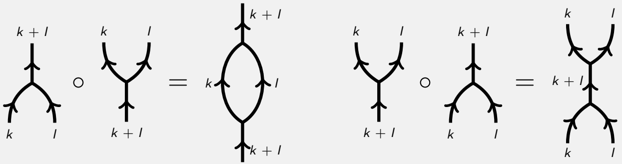
$$= \sum_{j'} \begin{bmatrix} k - j_1 - l + j_2 \\ j' \end{bmatrix}$$



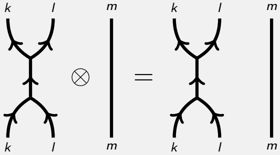
Definition(Cautis-Kamnitzer-Morrison 2012)

The $\mathfrak{sl}_n$-spider $\mathbf{Sp}(\mathfrak{sl}_n)$ is the monoidal $\mathbb{C}_q$-linear category with:

- Objects are $\vec{k} \in \mathbb{Z}^m_{\{0,\dots,n\}}$ and morphisms are $\mathrm{Hom}_{\mathbf{Sp}(\mathfrak{sl}_n)}(\vec{k},\vec{l}) = W_n(\vec{k},\vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



Connection to representation theory - yet again

Let $V = \mathbb{C}_q^n$ the vector representation of $\mathbf{U}_q(\mathfrak{sl}_n)$. For $k \in \{0, \dots, n\}$ let $\bigwedge_q^k \mathbb{C}_q^n$ denote the k-th fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representation.

Fact: All irreducible $\mathbf{U}_q(\mathfrak{sl}_n)$-modules are summands of

$$\textstyle\bigwedge_q^{\vec{k}} \mathbb{C}_q^n = \bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n$$

for some suitable vector $\vec{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m_{\{0,\dots,n\}}$.

Let $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ be the monoidal, $\mathbb{C}_q$-linear category consisting of:

- Objects are tensor products $\bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n$ of finite length and morphisms are $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners between these.
- Composition $\circ$ is composition of $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners.
- Tensoring $\otimes$ is tensoring of $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners.

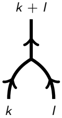
Diagrams for intertwiners - next try

Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners

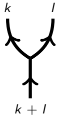
$$\mathrm{m}_{k,l}^{k+l}\colon \textstyle\bigwedge_q^k \mathbb{C}_q^n \otimes \bigwedge_q^l \mathbb{C}_q^n \to \bigwedge_q^{k+l} \mathbb{C}_q^n \quad \text{and} \quad \mathrm{s}_{k+l}^{k,l}\colon \bigwedge_q^{k+l} \mathbb{C}_q^n \to \bigwedge_q^k \mathbb{C}_q^n \otimes \bigwedge_q^l \mathbb{C}_q^n$$

given by projection and inclusion again.
Define a functor $\Gamma_\wedge^n\colon \mathbf{Sp}(\mathfrak{sl}_n) \to \mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$:

- On objects: $\vec{k}$ is send to $\bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n$.
- On morphisms:



$\mapsto \mathrm{m}_{k,l}^{k+l}$



$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem(Cautis-Kamnitzer-Morrison 2012)

The functor $\Gamma_\wedge^n\colon \mathbf{Sp}(\mathfrak{sl}_n) \to \mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$ is an equivalence of monoidal categories.

The quantum algebra $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$

For each $\mathfrak{gl}_m$-weight $\vec{k} \in \mathbb{Z}^{m-1}$ adjoin an idempotent $1_{\vec{k}}$ (Think: projection to the $\vec{k}$-weight space!) to $\mathbf{U}_q(\mathfrak{gl}_m)$.

Definition(Beilinson-Lusztig-MacPherson 1990)

The idempotented quantum general linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_m) = \bigoplus_{\vec{k},\vec{k}' \in \mathbb{Z}^{m-1}} 1_{\vec{k}'} \mathbf{U}_q(\mathfrak{gl}_m) 1_{\vec{k}}.$$

It is generated by F_i, E_i for $i = 1, \ldots, m-1$ suspect to some relations. These relations are just "cleaned-up" versions of the ones from $\mathfrak{gl}_m$.

We want to consider $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ as a category with objects $\vec{k}, \vec{k}' \in \mathbb{Z}^{m-1}$ and morphisms spaces $1_{\vec{k}'} \mathbf{U}_q(\mathfrak{gl}_m) 1_{\vec{k}}$.

“Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ on

$$\begin{aligned}\textstyle\bigwedge_q^N(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) &\cong \bigoplus_{k_1+\cdots+k_m=N} (\textstyle\bigwedge_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m}\mathbb{C}_q^n)\\ &\cong \bigoplus_{l_1+\cdots+l_n=N} (\textstyle\bigwedge_q^{l_1}\mathbb{C}_q^m \otimes \cdots \otimes \bigwedge_q^{l_n}\mathbb{C}_q^m)\end{aligned}$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_m)$-action f on the first term and an $\mathbf{U}_q(\mathfrak{sl}_n)$-action on the second. Howe: our $\bigwedge_q^{\vec{k}}\mathbb{C}_q^n$ is the $\vec{k}$-weight space of this.

In particular, there is a functorial action

$$\Phi_m^n \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$$

$$\vec{k} \mapsto \textstyle\bigwedge_q^{\vec{k}}\mathbb{C}_q^n, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge}(\textstyle\bigwedge_q^{\vec{k}}\mathbb{C}_q^n, \bigwedge_q^{\vec{l}}\mathbb{C}_q^n)$$

Howe: Φ_m^n is full. Or in words: all relations in $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ follow from the (natural) ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and the ones in the kernel of Φ_m^n.

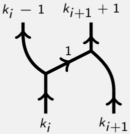
So how? "Howe"!

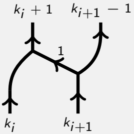
Theorem(Cautis-Kamnitzer-Morrison 2012)

There is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_m) & \xrightarrow{\quad \Phi_m^n \quad} & \mathfrak{sl}_n\text{-}\mathbf{Mod}_{\wedge} \\ & {\scriptstyle \Upsilon_m} \searrow \qquad \nearrow {\scriptstyle \Gamma_{\wedge}^n} & \\ & \mathbf{Sp}(\mathfrak{sl}_n) & \end{array}$$

with

$\Upsilon_m(F_i 1_{\vec{k}}) \mapsto$ 

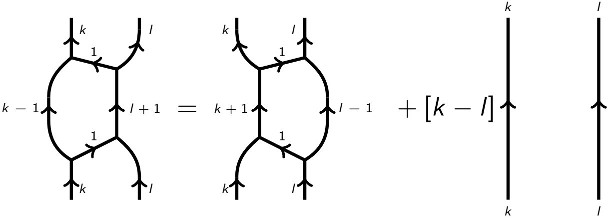
$\Upsilon_m(E_i 1_{\vec{k}}) \mapsto$ 

$\ker \Phi_m^n$ consists exactly of the $\mathfrak{gl}_m$-weights $\vec{k}$ with entries outside of $\{0, \dots, n\}$.

In words: all the relations in $\mathbf{Sp}(\mathfrak{sl}_n)$ follow from the ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$.

Exempli gratia

The mysterious square switch



is just

$$EF1_{(k,l)} - FE1_{(k,l)} = [k-l]1_{(k,l)}$$
$$\approx$$
$$EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$$

This needs to be on one slide...

Some additional remarks.

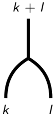
- One can do slightly better: the $\mathfrak{sl}_n$-webs form a $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$-module of a certain highest weight. Thus, playing with $\mathfrak{sl}_n$-webs is doing highest weight representation theory of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$.
- Cautis, Kamnitzer and Morrison show that the R-matrix braiding on $\mathfrak{sl}_n$-$\mathbf{Mod}_{\wedge}$ and Lusztig's Weyl group braiding on $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ coincide.
- As a consequence, the Reshetikhin-Turaev polynomials of links obtained from $\mathfrak{sl}_n$-$\mathbf{Mod}_{\wedge}$ come (for all n) from highest weight representation theory of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ (for a suitable fixed m depending on the link L).
- Another consequence of this: for a fixed link L the whole family of all Reshetikhin-Turaev polynomials (for all possible n and colors) contains only a finite amount of information about L.
- Up to here: we can categorify everything in sight!

Our story is easier in some sense...

A symmetric $\mathfrak{sl}_2$-web is a labeled trivalent graph locally made of

$\mathrm{cap}_k =$ 

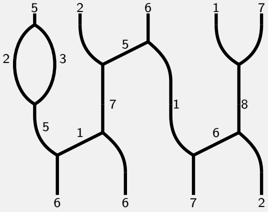
$\mathrm{cup}_k =$ 

$\mathrm{m}_{k,l}^{k+l} =$ 

$\mathrm{s}_{k+l}^{k,l} =$ 

No mirrors and sign issues, but $k, l, k+l \in \{0, 1, \dots\}$.

Example

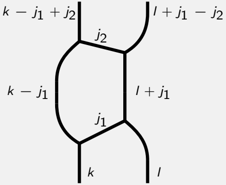


Never change a winning team: let us do the same again!

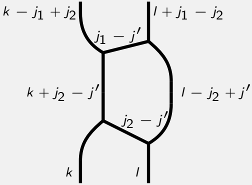
Definition

Given $\vec{k} \in \mathbb{Z}^k_{\geq 0}$ and $\vec{l} \in \mathbb{Z}^l_{\geq 0}$. The symmetric $\mathfrak{sl}_2$-web space $W_2^s(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by symmetric $\mathfrak{sl}_2$-webs between $\vec{k}$ and $\vec{l}$ modulo:

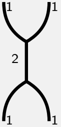
- Isotopy, associativity and "classical" relations, e.g. the scary square switches:



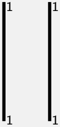
$$= \sum_{j'} \begin{bmatrix} k-j_1-l+j_2 \\ j' \end{bmatrix}$$



- New, symmetric relations. For example dumbbells:



$= [2]$



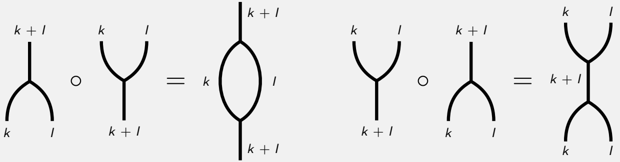
$+$

The symmetric $\mathfrak{sl}_2$-spider

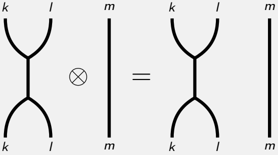
Definition

The symmetric $\mathfrak{sl}_2$-spider **SymSp($\mathfrak{sl}_2$)** is the monoidal $\mathbb{C}_q$-linear category with:

- Objects are $\vec{k} \in \mathbb{Z}^m_{\{0,1\dots,\}}$ and morphisms are $\mathrm{Hom}_{\mathbf{Sp}(\mathfrak{sl}_n)}(\vec{k},\vec{l}) = W_2^s(\vec{k},\vec{l})$.
- Composition $\circ$:



- Tensoring $\otimes$:



Connection to representation theory - yet again

Let $V = \mathbb{C}_q^2$ the vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$. For $k \in \{0, 1 \ldots\}$ let $\mathrm{Sym}_q^k \mathbb{C}_q^2$ denote the k-th symmetric $\mathbf{U}_q(\mathfrak{sl}_2)$-representation.

Let $\mathfrak{sl}_2$-*fd***Mod** be the monoidal, $\mathbb{C}_q$-linear category consisting of:

- Objects are tensor products $\mathrm{Sym}_q^{k_1} \mathbb{C}_q^2 \otimes \cdots \otimes \mathrm{Sym}_q^{k_m} \mathbb{C}_q^2$ of finite length and morphisms are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners between these.
- Composition $\circ$ is composition of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Tensoring $\otimes$ is tensoring of $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.

Note that $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge \subsetneq \mathfrak{sl}_2$-*fd***Mod**.

Fact: All irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-modules are of the form $\mathrm{Sym}_q^k \mathbb{C}_q^2$ for some k. Thus, $\mathfrak{sl}_2$-*fd***Mod** contains all finite dimensional representations, aka: no splitting of tensor products is necessary.

Diagrams for intertwiners - I am not bored yet

Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners

$$\mathrm{cap}_k\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \to \mathbb{C}_q \quad \mathrm{m}_{k,l}^{k+l}\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \to \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2$$

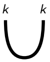
$$\mathrm{cup}_k\colon \mathbb{C}_q \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2 \quad \mathrm{s}_{k+l}^{k,l}\colon \mathrm{Sym}_q^{k+l}\mathbb{C}_q^2 \to \mathrm{Sym}_q^k\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^l\mathbb{C}_q^2$$

(guess where they come from...)

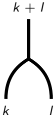
Define a functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}fd\mathbf{Mod}$:

- On objects: $\vec{k}$ is send to $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^2 \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^2$.
- On morphisms:

$\mapsto \mathrm{cap}_k$



$\mapsto \mathrm{cup}_k$



$\mapsto \mathrm{m}_{k,l}^{k+l}$

$\mapsto \mathrm{s}_{k+l}^{k,l}$

Theorem

Our $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}fd\mathbf{Mod}$ is an equivalence of monoidal categories.

“Howe” to prove this? You know “Howe”, right?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ on

$$\mathrm{Sym}_q^N(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) \cong \bigoplus_{k_1+\cdots+k_m=N} (\mathrm{Sym}_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^n)$$
$$\cong \bigoplus_{l_1+\cdots+l_m=N} (\mathrm{Sym}_q^{l_1}\mathbb{C}_q^m \otimes \cdots \otimes \mathrm{Sym}_q^{l_n}\mathbb{C}_q^m)$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_m)$-action f on the first term and an $\mathbf{U}_q(\mathfrak{sl}_n)$-action on the second. Howe: our $\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^n$ is the $\vec{k}$-weight space of this.

In particular, there is a functorial action

$$\Phi_m^\infty \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_2\text{-}fd\mathbf{Mod}$$

$$\vec{k} \mapsto \mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{sl}_2\text{-}fd\mathbf{Mod}}(\mathrm{Sym}_q^{\vec{k}}\mathbb{C}_q^2, \mathrm{Sym}_q^{\vec{l}}\mathbb{C}_q^2)$$

Howe: Φ_m^∞ is full. Or in words: all relations in $\mathfrak{sl}_2$-$fd\mathbf{Mod}$ follow from the (natural) ones in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ and the ones in the kernel of Φ_m^∞.

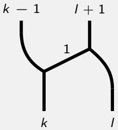
Let us copy-paste!

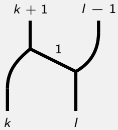
Theorem

There is a commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_m) & \xrightarrow{\quad \Phi_m^{\infty} \quad} & \mathfrak{sl}_2\text{-}fd\mathbf{Mod} \\ & {\scriptstyle \Upsilon_m} \searrow \qquad \nearrow {\scriptstyle \Gamma_{\mathbf{sym}}} & \\ & \mathbf{SymSp}(\mathfrak{sl}_2) & \end{array}$$

with

$\Upsilon_m(F_i 1_{\vec{k}}) \mapsto$ 

$\Upsilon_m(E_i 1_{\vec{k}}) \mapsto$ 

$\ker \Phi_m^{\infty}$ consists of "throwing certain tableaux away".

Ok, where is the catch?

So what is the difference between q-skew and q-symmetric Howe? This:

$$\bigwedge_q^N(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) \cong \textstyle\bigoplus_\lambda V_m(\lambda) \otimes V_n(\lambda^T)$$

and the sum runs over all tableaux λ that fit into an *mxn*-square (finitely many).

$$\mathrm{Sym}_q^N(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) \cong \bigoplus_\lambda V_m(\lambda) \otimes V_n(\lambda)$$

and the sum runs over all tableaux λ that fit into an $\min(m,n)x$anything-square (infinitely many).

Thus, because of "anything", we have to allow all possible labels $k \in \{0, 1, \ldots\}$. And because of $\min(m,n)$ we have to kill certain $\mathrm{End}_{\mathbb{C}_q}(V_m(\lambda))$'s for λ with too many rows. Latter gives the new, symmetric relations!

I do not have tenure. So I have to bore you a bit more.

Some additional remarks.

- The R-matrix braiding on $\mathfrak{sl}_2$-$fd\mathbf{Mod}$ and Lusztig's Weyl group braiding on $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ coincide again.
- As a consequence, on can obtain colored Jones polynomial without Jones-Wenzl projectors or infinite twists by a "MOY-like calculus".
- As a another consequence, the Reshetikhin-Turaev polynomials obtained from $\mathfrak{sl}_n$-$\mathbf{Mod}_{\wedge}$ and the colored Jones polynomials are (almost) "dual" to each other. The only difference are the $\mathrm{End}_{\mathbb{C}_q}(V_m(\lambda))$ one has to kill.
- This give a hint: categorify the colored Jones polynomial as Khovanov-Rozansky $\mathfrak{sl}_n$-homologies - without infinite twists or categorified Jones-Wenzl projectors.
- As a possible upshot: duality between Khovanov-Rozansky $\mathfrak{sl}_n$-homologies and colored Jones homologies (as predicted via HOMFLY-PT homology).

There is still much to do...

Thanks for your attention!