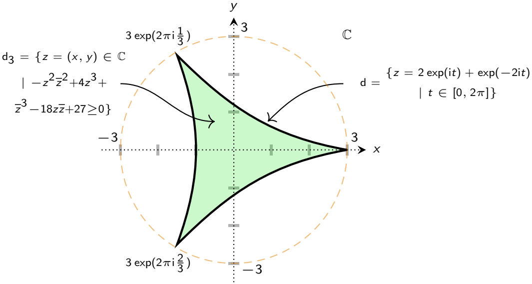
Generalizing zigzag algebras

Or: It's all about polynomials

Daniel Tubbenhauer



Joint work with Michael Ehrig, Marco Mackaay, Volodymyr Mazorchuk and Vanessa Miemietz

November 2018

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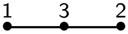
$\mathrm{A}_3 =$ 

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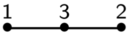
$\mathrm{A}_3 =$ 

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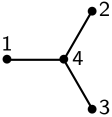
$$U_5(\mathrm{X}) = (\mathrm{X} - 2\cos(\tfrac{\pi}{6}))(\mathrm{X} - 2\cos(\tfrac{2\pi}{6}))\mathrm{X}(\mathrm{X} - 2\cos(\tfrac{4\pi}{6}))(\mathrm{X} - 2\cos(\tfrac{5\pi}{6}))$$

$\mathrm{D}_4 =$ 

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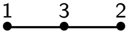
$\mathrm{A}_3 =$ 

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✓ for $e = 2$

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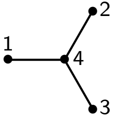
$\mathrm{D}_4 =$ 

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✓ for $e = 4$

Let $\boldsymbol{A} =$ … aph Γ.
Let U_{e+1} …

Cl … $\neq 0$.

$\mathrm{A}_3 =$ 1 … $\cos(\frac{3\pi}{4})\}$

U …

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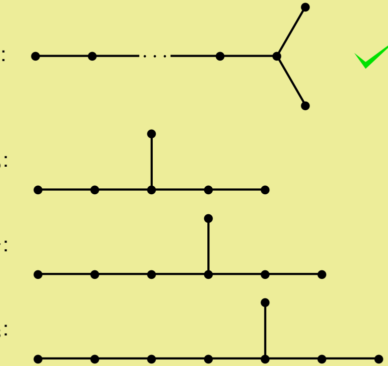
Smith ∼1969. The graphs solutions to (CP) are precisely ADE graphs for $e+2$ being (at most) the Coxeter number.



Type A_m: ✓ for $e = m-1$

Type D_m: ✓ for $e = 2m-4$

Type E_6: ✓ for $e = 10$

Type E_7: ✓ for $e = 16$

Type E_8: ✓ for $e = 28$

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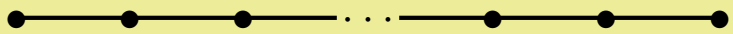
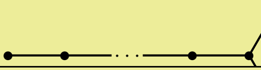
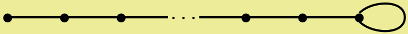
Type A_m:  ✓ for $e = m-1$

Type D_m:  ✓ for $e = 2m-4$

Fact. If Γ is allowed to have loops, then there is one extra family called tadpoles:

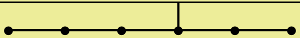


They usually do not play a role.

$A_3 =$ 1 … $\cos(\frac{3\pi}{4})\}$

Type E_6: … $e = 10$

U …

Type E_7:  ✓ for $e = 16$

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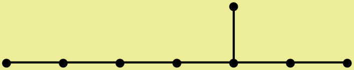
Type E_8:  ✓ for $e = 28$

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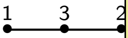
$\mathrm{A}_3 =$ 

$\cos(\tfrac{\pi}{4}), 0, 2\cos(\tfrac{3\pi}{4})\}$

Fact. $U_{e+1}(\boldsymbol{A})$ has negative entries for some e if and only if $\boldsymbol{A}$ is of type ADE.

This is a much stronger statement and the only proof I know uses categorification.

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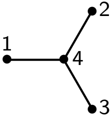
$\mathrm{D}_4 =$ 

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Zigzag algebras

Take the double graph $\Gamma_{\rightleftarrows}$ of Γ and add two loops $\alpha_{\mathtt{s}} = (\alpha_{\mathtt{s}})_{\mathtt{i}}$ and $\alpha_{\mathtt{t}} = (\alpha_{\mathtt{t}})_{\mathtt{i}}$ per vertex. Take its path algebra $\mathrm{R}(\Gamma_{\rightleftarrows})$.

Let $\mathrm{Z}_{\rightleftarrows} = \mathrm{Z}_{\rightleftarrows}(\Gamma)$ be the quotient of $\mathrm{R}(\Gamma_{\rightleftarrows})$ by:

(a) **Boundedness.** Any path involving three distinct vertices is zero.

(b) **The relations of the cohomology ring $\mathrm{H}^*(\mathrm{SL}(2)/\mathrm{B})$.** $\alpha_{\mathtt{s}} \circ \alpha_{\mathtt{t}} = \alpha_{\mathtt{t}} \circ \alpha_{\mathtt{s}}$, $\alpha_{\mathtt{s}} + \alpha_{\mathtt{t}} = 0$ and $\alpha_{\mathtt{s}} \circ \alpha_{\mathtt{t}} = 0$.

(c) **Zigzag.** $\mathtt{i}{\to}\mathtt{j}{\to}\mathtt{i} = \alpha_{\mathtt{s}} - \alpha_{\mathtt{t}}$ for $\mathtt{i}{-}\mathtt{j}$.

$\mathrm{Z}_{\rightleftarrows}$ is the zigzag algebra associated to Γ. It can be graded using the path length.

Example

Not important for today: This definition only works for more than three vertices.

Zigzag algebras

Take the double graph $\Gamma_{\rightleftarrows}$ of Γ and add two loops $\alpha_{\mathtt{s}} = (\alpha_{\mathtt{s}})_{\mathtt{i}}$ and $\alpha_{\mathtt{t}} = (\alpha_{\mathtt{t}})_{\mathtt{i}}$ per verte

$\Bbbk[\mathtt{x}]/(\mathtt{x}^2)$ is isomorphic to $\Bbbk[\alpha_{\mathtt{s}}, \alpha_{\mathtt{t}}]/(\alpha_{\mathtt{s}} + \alpha_{\mathtt{t}}, \alpha_{\mathtt{s}}\alpha_{\mathtt{t}})$ by $\mathtt{x} \mapsto \alpha_{\mathtt{s}} - \alpha_{\mathtt{t}}$.

We prefer this formulation, with loops in degree 2. Why? You will see later.

Let $Z_{\rightleftarrows} = Z_{\rightleftarrows}(\Gamma)$ be the quotient of $K(\Gamma_{\rightleftarrows})$ by:

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Take the double graph $\Gamma_{\rightleftarrows}$ of Γ and add two loops $\alpha_s = (\alpha_s)_i$ and $\alpha_t = (\alpha_t)_i$ per verte

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Let $Z_{\rightleftarrows} = Z_{\rightleftarrows}(\Gamma)$ be the quotient of $R(\Gamma_{\rightleftarrows})$ by:

(a) **Boundedne** zero.

One can define a kind of quasi-hereditary cover $Z^{C}_{\rightleftarrows}$ by killing x_i at a fixed set of vertices C.

(b) **The relatio** $\alpha_s \circ \alpha_t = \alpha_t \circ \alpha_s$, $\alpha_s + \alpha_t = 0$ and $\alpha_s \circ \alpha_t = 0$.

(c) **Zigzag.** $i \to j \to i = \alpha_s - \alpha_t$ for $i - j$.

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▸ Example

Zigzag algebras in mathematics

Zigzag algebras are around for many years. Here are some examples:

- **Wakamatsu & others $\sim$1980++.** Study of Artin algebras.
- **Huerfano–Khovanov $\sim$2000, Khovanov–Seidel $\sim$2000 & others.** Categorical braid group actions.
- **Implicit in the literature $<$2000, Huerfano–Khovanov $\sim$2000, Evseev–Kleshchev $\sim$2016 & others.** Finite groups in prime characteristic.
- **Implicit in the literature $<$2000, Stroppel $\sim$2003 & others.** Versions of category $\mathcal{O}$.
- **Implicit in the literature $<$2000, Qi–Sussan $\sim$2013, Andersen $\sim$2014 & others.** Representation theory of reductive groups in prime characteristic, quantum groups at roots of unity.
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But first, let us understand the zigzag algebras combinatorially.

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The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 2_{\mathsf{q}}, & \text{if } \mathtt{i}=\mathtt{j}, \quad \{\mathtt{i},\mathtt{x}_{\mathtt{i}}\} \text{ is a basis,} \\ \mathsf{q}, & \text{if } \mathtt{i}-\mathtt{j}, \quad \{\mathtt{i}\to\mathtt{j}\} \text{ is a basis,} \\ 0, & \text{else,} \quad \emptyset \text{ is a basis,} \end{cases}$$

where $\mathrm{qdim}(\_)$ denotes the graded dimension, and $2_{\mathsf{q}} = 1 + \mathsf{q}^2$.

The (left) projectives and simples:

$$\mathrm{P}_{\mathtt{i}} = \{\mathtt{i}, \mathtt{j}\to\mathtt{i}, \mathtt{x}_{\mathtt{i}} \mid \mathtt{i}-\mathtt{j}\} \quad \& \quad \mathrm{L}_{\mathtt{i}} = \{\mathtt{i}\}$$

The Loewy picture:

$$\mathrm{P}_{\mathtt{i}} = \begin{matrix} \mathtt{i} \\ \mathtt{j}\to\mathtt{i} \text{ (for } \mathtt{i}-\mathtt{j}) \\ \mathtt{x}_{\mathtt{i}} \end{matrix}$$

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$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 2_q, & \text{if } \mathtt{i}=\mathtt{j}, \quad \{\mathtt{i}, \mathtt{x_i}\} \text{ is a basis,} \\ q, & \text{if } \mathtt{i}-\mathtt{j}, \quad \{\mathtt{i}\rightarrow\mathtt{j}\} \text{ is a basis,} \\ 0, & \text{else,} \quad \emptyset \text{ is a basis,} \end{cases}$$

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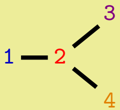
Example.

$$\boldsymbol{A}(\mathrm{D}_4) = \begin{pmatrix} 0&1&0&0 \\ 1&0&1&1 \\ 0&1&0&0 \\ 0&1&0&0 \end{pmatrix}$$



type D_4

The (left) projectives and simples:

$$\mathrm{P_i} = \{\mathtt{i}, \mathtt{j}\rightarrow ... \quad \& \quad \mathrm{L_i} = \{\mathtt{i}\}$$

The Loewy picture:

$$\mathrm{P_i} = \mathtt{j}\rightarrow\mathtt{i} \text{ (for } \mathtt{i}-\mathtt{j})$$

$$\mathtt{x_i}$$

The Cartan matrix

$$P_1 = \begin{matrix} 1 \\ 2 \to 1 \\ x_1 \end{matrix}$$

…$nz_{\rightleftarrows}(i,j)) = \begin{cases} 2_q, & \text{if } i = j, \quad \{i, x_i\} \text{ is a basis,} \\ q, & \text{if } i - j, \quad \{i \to j\} \text{ is a basis,} \\ 0, & \text{else,} \end{cases}$

where $q\dim(\_)$ denotes the grade… …nd $2_q =$

$$P_2 = \begin{matrix} 2 \\ 1 \to 2 \ \&\ 3 \to 2 \ \&\ 4 \to 2 \\ x_2 \end{matrix}$$

The (lef… …es and simples:

$$P_3 = \begin{matrix} 3 \\ 2 \to 3 \\ x_3 \end{matrix}$$

$P_i = \{i, j \to$… & $L_i = \{i\}$

Example.

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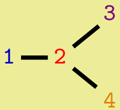

type D_4

The Loe…

$$P_4 = \begin{matrix} 4 \\ 2 \to 4 \\ x_4 \end{matrix}$$

$$P_i = j \to i \text{ (for } i - j)$$

$$x_i$$

The Cartan matrix

$$P_1 = \begin{matrix} 1 \\ 2 \to 1 \\ x_1 \end{matrix}$$

$\mathrm{nz}_{\rightleftarrows}$

$$P_1 = \begin{matrix} L_1 \\ L_2 \\ L_1 \end{matrix}$$

$$\begin{cases} 2_q, & \text{if } i = j, \quad \{i, x_i\} \text{ is a basis,} \\ q, & \text{if } i - j, \quad \{i \to j\} \text{ is a basis,} \\ 0, & \text{else,} \end{cases}$$

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$$P_2 = \begin{matrix} 2 \\ 1 \to 2 \;\&\; 3 \to 2 \;\&\; 4 \to 2 \\ x_2 \end{matrix}$$

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Example.

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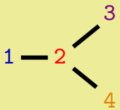

type D_4

$P_i = \{i, j \to i$[...] & $L_i = \{$

$$P_2 = \begin{matrix} L_2 \\ L_1 \;\&\; L_3 \;\&\; L_4 \\ L_2 \end{matrix}$$

The Loe[...]:

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$$x_i$$

The Cartan matrix

$\begin{cases} 2_q, & \text{if } i = j, \end{cases}$ $\{i, x_i\}$ is a basis,

q

where qdi

Thus, the Cartan matrix is

$$C(D_4) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The (left)

$$P_i = \{i, j{\to}i, x_i \mid i{-}j\} \quad \& \quad L_i = \{i\}$$

The Loewy picture:

$$P_i = \begin{matrix} i \\ j{\to}i \text{ (for } i{-}j) \\ x_i \end{matrix}$$

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where qdi

The (left)

P_i $\{i\}$

Fact. (Not hard to show.)
The Cartan matrix $\boldsymbol{C} = \boldsymbol{C}(Z_{\rightleftarrows})$ is

$$\boldsymbol{C} = 2\boldsymbol{I} + \boldsymbol{A}.$$

The Loewy picture:

$$P_i = \begin{array}{c} i \\ j \to i \\ x_i \end{array} \text{ (for } i - j\text{)}$$

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where qdi

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The Cartan matrix $C = C(Z_{\rightleftarrows})$ is

$$C = 2I + A.$$

The Loewy picture:

Fact. (Not hard to show.)
The graded Cartan matrix $C = C(Z_{\rightleftarrows})$ is

$$C = 2_q I + qA.$$

Algebraic properties of zigzag algebras

Theorem. $Z^{C}_{\rightleftarrows}$ is cellular if and only if Γ is a finite type A graph and $\mathtt{X} = \emptyset$ or $\mathtt{X} =$leaf. $Z^{C}_{\rightleftarrows}$ is relative cellular if and only if Γ is a finite type A graph and $\mathtt{X} = \emptyset$ or $\mathtt{X} =$leaf; or Γ is an affine type A graph and $\mathtt{X} = \emptyset$.

Theorem. $Z^{C}_{\rightleftarrows}$ is ▸ quasi-hereditary if and only if Γ is a finite type A graph and $\mathtt{X} =$leaf.

Theorem. $Z^{C}_{\rightleftarrows}$ is ▸ Koszul if and only if Γ is not a type ADE graph and $\mathtt{X} = \emptyset$.

Proof idea: Cellularity and quasi-hereditary

Main problem. It is not easy to show that an algebra is not cellular, since there are several choices involved which one could make.

Main idea. Use the numerical conditions to rule out most cases; treat the remaining cases by hand.

The cases of relative cellularity, quasi-hereditary and with vertex condition work basically in the same way - I omit details.

Step 1. Kill the majority of cases.

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Step 2. Kill the remaining infinite families.

Numerical condition. The Cartan matrix $\boldsymbol{C}$ of a cellular algebra satisfies $\boldsymbol{C} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}$.

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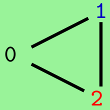

is the only remaining case, for which $\mathrm{Z}_{\rightleftarrows}$ is a 12-dimensional algebra.

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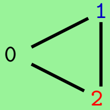

is the only remaining case, for which $\mathrm{Z}_{\rightleftarrows}$ is a 12-dimensional algebra.

Done.

Proof idea: koszulity

Main problem. Computing projective resolutions is hard.

Main idea 1. Get a numerical way to handle the projectives in some minimal projective resolution.

Main idea 2. Use a numerical condition to rule out the cases which are not Koszul.

Again, with vertex condition works similarly - I omit details.

Proof idea: koszulity

Step 1. Writing down candidates of projective resolutions.

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Numerical condition. The projectives turning up in the e^{th} step of a minimal projective resolution can be read off from the columns of $U_e(\boldsymbol{A})$. Example

Main idea 1 minimal projective resolution.

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Again, with ...

Neat consequence. A characterization of Dynkin diagrams.

Γ is a finite type ADE graph
if and only if
entries of $U_e(\boldsymbol{A})$ do not grow when $e \to \infty$.

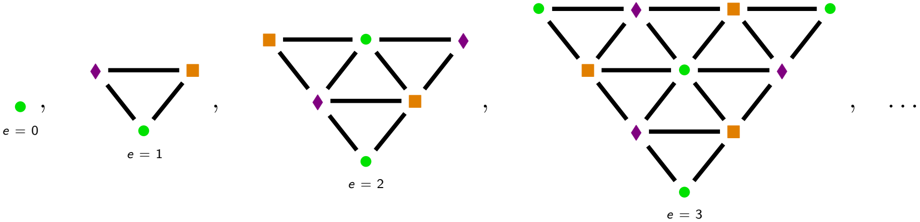
Γ is an affine type ADE graph
if and only if
entries of $U_e(\boldsymbol{A})$ grow linearly when $e \to \infty$.

Γ is neither finite nor affine type ADE graph
if and only if
entries of $U_e(\boldsymbol{A})$ grow exponentially when $e \to \infty$.

Admissible graphs

An unoriented, connected, simple graph Γ is called $\mathfrak{sl}_3$-admissible if it is tricolored and each edge is contained in a 2-simplex.

Example. The generalized type gA_e graphs, where $e \in \mathbb{Z}_{\geq 0}$:



We color our vertices green $\mathtt{g} = \{\mathtt{b}, \mathtt{y}\}$, orange $\mathtt{o} = \{\mathtt{r}, \mathtt{y}\}$ and purple $\mathtt{p} = \{\mathtt{b}, \mathtt{r}\}$.

It might be possible to relax these conditions, but we do not know for sure.
In particular, the explicit coloring can be avoided for zigzag algebras.

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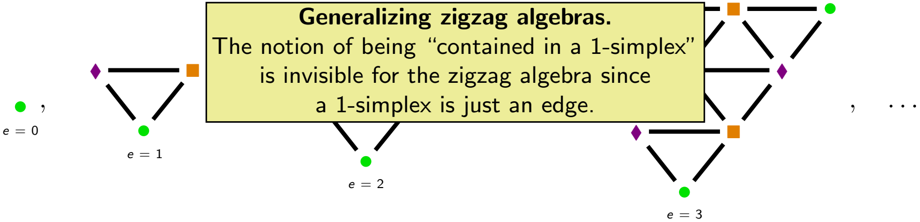

Generalizing zigzag algebras.
The notion of being "contained in a 1-simplex" is invisible for the zigzag algebra since a 1-simplex is just an edge.

We color our vertices green $\mathtt{g} = \{\mathtt{b}, \mathtt{y}\}$, orange $\mathtt{o} = \{\mathtt{r}, \mathtt{y}\}$ and purple $\mathtt{p} = \{\mathtt{b}, \mathtt{r}\}$.

Trihedral zigzag algebras

Take the double graph $\Gamma_{\rightleftarrows}$ of Γ and add three loops $\alpha_{\mathtt{b}} = (\alpha_{\mathtt{b}})_{\mathtt{i}}$, $\alpha_{\mathtt{r}} = (\alpha_{\mathtt{r}})_{\mathtt{i}}$ and $\alpha_{\mathtt{y}} = (\alpha_{\mathtt{y}})_{\mathtt{i}}$ per vertex; choose on of them per vertex.

Let $\mathrm{T}_{\rightleftarrows} = \mathrm{T}_{\rightleftarrows}(\Gamma)$ be the quotient of $\mathrm{R}(\Gamma_{\rightleftarrows})$ by:

(a) **Boundedness.** Paths involving vertices from two different 2-simplices are zero.

(b) **The relations of the cohomology ring** $\mathrm{H}^*(\mathrm{SL}(3)/\mathrm{B})$**.** $\alpha_{\mathtt{a}}\alpha_{\mathtt{b}} = \alpha_{\mathtt{b}}\alpha_{\mathtt{a}}$ for $a, b \in \{\mathtt{b}, \mathtt{r}, \mathtt{y}\}$, $\alpha_{\mathtt{b}} + \alpha_{\mathtt{r}} + \alpha_{\mathtt{y}} = 0$, $\alpha_{\mathtt{b}}\alpha_{\mathtt{r}} + \alpha_{\mathtt{b}}\alpha_{\mathtt{y}} + \alpha_{\mathtt{r}}\alpha_{\mathtt{y}} = 0$ and $\alpha_{\mathtt{b}}\alpha_{\mathtt{r}}\alpha_{\mathtt{y}} = 0$.

(c) **Sliding loops.** $\mathtt{i}\to\mathtt{j}\alpha_{\mathtt{i}} = -\alpha_{\mathtt{j}}\mathtt{i}\to\mathtt{j}$, $\mathtt{i}\to\mathtt{j}\alpha_{\mathtt{j}} = -\alpha_{\mathtt{i}}\mathtt{i}\to\mathtt{j}$ and $\mathtt{i}\to\mathtt{j}\alpha_{\mathtt{k}} = \alpha_{\mathtt{k}}\mathtt{i}\to\mathtt{j} = 0$.

(d) **Zigzag.** $\mathtt{i}\to\mathtt{j}\to\mathtt{i} = \alpha_{\mathtt{i}}\alpha_{\mathtt{j}}$.

(e) **Zigzig equals zag times loop.** $\mathtt{i}\to\mathtt{j}\to\mathtt{k} = \mathtt{i}\to\mathtt{k}\alpha_{\mathtt{i}} = -\alpha_{\mathtt{k}}\mathtt{i}\to\mathtt{k}$.

$\mathrm{T}_{\rightleftarrows}$ is the trihedral zigzag algebra associated to Γ. Its graded by path length.

▸ Example

Same as for the zigzag algebra: This definition only works for more than two 2-simplices.

Trihedral zigzag algebras

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One can also define a kind of quasi-hereditary cover $\mathrm{T}^{\mathsf{C}}_{\rightleftarrows}$.

(c) **Sliding loops.** $\mathtt{i}{\to}\mathtt{j}\alpha_{\mathtt{i}} = -\alpha_{\mathtt{j}}\mathtt{i}{\to}\mathtt{j}$, $\mathtt{i}{\to}\mathtt{j}\alpha_{\mathtt{j}} = -\alpha_{\mathtt{i}}\mathtt{i}{\to}\mathtt{j}$ and $\mathtt{i}{\to}\mathtt{j}\alpha_{\mathtt{k}} = \alpha_{\mathtt{k}}\mathtt{i}{\to}\mathtt{j} = 0$.

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Generalizing zigzag algebras.

Let $\mathrm{T}_{\rightleftarrows} = \mathrm{T}_{\rightleftarrows}(\Gamma)$ be the quotient of $\mathrm{R}(\Gamma_{\rightleftarrows})$ by:

(a) **Bou**… es are zero.

“Boundedness” is a direct generalization, where 1-simplex‘=’edge.

(b) **The relations of th**… (B). $\alpha_{\mathtt{a}}\alpha_{\mathtt{b}} = \alpha_{\mathtt{b}}\alpha_{\mathtt{a}}$ for $a, b \in \{\mathtt{b}, \mathtt{r}, \mathtt{y}\}$, $\alpha_{\mathtt{b}}$ … $= 0$ and $\alpha_{\mathtt{b}}\alpha_{\mathtt{r}}\alpha_{\mathtt{y}} = 0$.

“Flag” is a direct generalization.

(c) **Sliding loops.** $\mathtt{i}\!\to\!\mathtt{j}$ … and $\mathtt{i}\!\to\!\mathtt{j}\alpha_{\mathtt{k}} = \alpha_{\mathtt{k}}\mathtt{i}\!\to\!\mathtt{j} = 0$.

“Sliding loops” is a new relation.

(d) **Zigzag.** $\mathtt{i}\!\to\!\mathtt{j}\!\to$ …

$\mathtt{i}\!\to\!\mathtt{j}\!\to\!\mathtt{i} = \alpha_{\mathtt{i}}\alpha_{\mathtt{j}}$ generalizes $\mathtt{i}\!\to\!\mathtt{j}\!\to\!\mathtt{i} = \alpha_{\mathtt{s}} - \alpha_{\mathtt{t}}$.

(e) **Zigzig equals zag times loop.** $\mathtt{i}\!\to\!\mathtt{j}\!\to\!\mathtt{k} = \mathtt{i}\!\to\!\mathtt{k}\alpha_{\mathtt{i}} = -\alpha_{\mathtt{k}}\mathtt{i}\!\to\!\mathtt{k}$.

$\mathrm{T}_{\rightleftarrows}$ is the trihedr… path length.

“Zigzig equals zag times loop” is a new relation.

Example

Trihedral zigzag algebras in mathematics

Generalizing zigzag algebras does not work in all directions:

- Study of Artin algebras.
- Categorical braid group actions.
- Finite groups in prime characteristic.
- Versions of category $\mathcal{O}$.
- Representation theory of reductive groups in prime characteristic, quantum groups at roots of unity.
- In various places in categorification.

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The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 3_{q}!, & \text{if } \mathtt{i}=\mathtt{j}, \quad \text{the usual cohomology basis}, \\ q^{2}+q^{4}, & \text{if } \mathtt{i}-\mathtt{j}, \quad \{\mathtt{i}\to\mathtt{j}, \mathtt{i}\to\mathtt{j}\alpha_{\mathtt{a}}\} \text{ is a basis}, \\ 0, & \text{else}, \quad \emptyset \text{ is a basis}. \end{cases}$$

The volume elements are $\mathtt{x_i} = \alpha_{\mathtt{b}}^{2}\alpha_{\mathtt{r}} = -\alpha_{\mathtt{r}}^{2}\alpha_{\mathtt{b}}$ =etc.

The (left) projectives and simples:

$$\mathrm{P}_{\mathtt{i}} = \{\mathtt{i}, \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \alpha_{\mathtt{a}}^{2}, \alpha_{\mathtt{b}}^{2}, \mathtt{x_i}, \mathtt{j}\to\mathtt{i}, \mathtt{j}\to\mathtt{i}\alpha_{\mathtt{a}}, \mathtt{x_i} \mid \mathtt{i}-\mathtt{j}\} \quad \& \quad \mathrm{L}_{\mathtt{i}} = \{\mathtt{i}\}$$

The Loewy picture:

$$\mathrm{P}_{\mathtt{i}} = \begin{matrix} \mathtt{i} \\ \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \mathtt{j}\to\mathtt{i} \\ \alpha_{\mathtt{a}}^{2}, \alpha_{\mathtt{b}}^{2}, \mathtt{j}\to\mathtt{i}\alpha_{\mathtt{a}} \\ \mathtt{x_i} \end{matrix} \text{ (for } \mathtt{i}-\mathtt{j}\text{)}$$

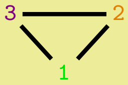
The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 3_q!, & \text{if } \mathtt{i}=\mathtt{j}, \qquad \text{the usual cohomology basis,} \\ q^2+q^4, & \text{if } \mathtt{i}-\mathtt{j}, \qquad \{\mathtt{i}\to\mathtt{j}, \mathtt{i}\to\mathtt{j}\alpha_{\mathtt{a}}\} \text{ is a basis,} \\ 0, & \text{else,} \qquad \emptyset \text{ is a basis.} \end{cases}$$

The volume elements are $\mathtt{x_i} = \alpha_{\mathtt{b}}^2\alpha_{\mathtt{r}} = -\alpha_{\mathtt{r}}^2\alpha_{\mathtt{b}}$ =etc.

Example.

$A(\mathrm{gA}_1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$



type gA_1

The (left) projectives and simples:

$$\mathrm{P}_{\mathtt{i}} = \{\mathtt{i}, \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \alpha_{\mathtt{a}}^2, \alpha_{\mathtt{b}}^2, \mathtt{x_i}, \ldots \mathtt{i} \mid \mathtt{i}-\mathtt{j}\} \quad \& \quad \mathrm{L}_{\mathtt{i}} = \{\mathtt{i}\}$$

The Loewy picture:

$$\mathrm{P}_{\mathtt{i}} = \begin{matrix} \alpha_{\mathtt{a}} \ldots \\ \alpha_{\mathtt{a}}^2, \alpha_{\mathtt{b}}^2, \mathtt{j}\to\mathtt{i}\alpha_{\mathtt{a}} \\ \mathtt{x_i} \end{matrix} \quad (\text{for } \mathtt{i}-\mathtt{j})$$

The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 3_q!, & \text{if } \mathtt{i}=\mathtt{j}, \quad \text{the usual cohomology basis,} \\ q^2+q^4, & \text{if } \mathtt{i}\text{—}\mathtt{j}, \quad \{\mathtt{i}\to\mathtt{j}, \mathtt{i}\to\mathtt{j}\alpha_{\mathtt{a}}\} \text{ is a basis,} \\ 0, & \text{else,} \quad \emptyset \text{ is a basis.} \end{cases}$$

The volume elements are $\mathtt{x_i} = \alpha_{\mathtt{b}}^2\alpha_{\mathtt{r}} = -\alpha_{\mathtt{r}}^2\alpha_{\mathtt{b}}$ =etc.

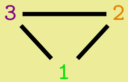
$$P_3 = \begin{matrix} 3 \\ \alpha_{\mathtt{b}} \;\&\; \alpha_{\mathtt{r}} \;\&\; 1\to 3 \;\&\; 2\to 3 \\ \alpha_{\mathtt{b}}^2 \;\&\; \alpha_{\mathtt{r}}^2 \;\&\; 1\to 3\alpha_{\mathtt{b}} \;\&\; 2\to 3\alpha_{\mathtt{r}} \\ \mathtt{x}_3 \end{matrix}$$

es:

$, \mathtt{x_i},$ … $_{\mathtt{i}} \mid \mathtt{i}\text{—}\mathtt{j}\}$ & $L_{\mathtt{i}} = \{\mathtt{i}\}$

Example.

$A(\mathrm{g}A_1) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$



type $\mathrm{g}A_1$

The Loewy picture:

$$P_{\mathtt{i}} = \begin{matrix} \alpha_{\mathtt{a}} \ldots \\ \alpha_{\mathtt{a}}^2, \alpha_{\mathtt{b}}^2, \mathtt{j}\to\mathtt{i}\alpha_{\mathtt{a}} \\ \mathtt{x_i} \end{matrix} \quad (\text{for } \mathtt{i}\text{—}\mathtt{j})$$

The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 3_q!, & \text{if } \mathtt{i}=\mathtt{j}, \quad \text{the usual cohomology basis,} \\ q^2+q^4, & \text{if } \mathtt{i}-\mathtt{j}, \quad \{\mathtt{i}\to\mathtt{j}, \mathtt{i}\to\mathtt{j}\alpha_{\mathtt{a}}\} \text{ is a basis,} \\ 0, & \text{else,} \quad \emptyset \text{ is a basis.} \end{cases}$$

The volume elements are $\mathtt{x}_\mathtt{i} = \alpha_\mathtt{b}^2\alpha_\mathtt{r} = -\alpha_\mathtt{r}^2\alpha_\mathtt{b}$ =etc.

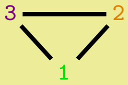
$$P_3 = \begin{matrix} 3 \\ \alpha_\mathtt{b} \;\&\; \alpha_\mathtt{r} \;\&\; 1\to3 \;\&\; 2\to3 \\ \alpha_\mathtt{b}^2 \;\&\; \alpha_\mathtt{r}^2 \;\&\; 1\to3\alpha_\mathtt{b} \;\&\; 2\to3\alpha_\mathtt{r} \\ \mathtt{x}_3 \end{matrix}$$

es:

$, \mathtt{x}_\mathtt{i},$ ... $\mathtt{i} \mid \mathtt{i}$-

Example.

$A(\mathrm{g}A_1) = \begin{pmatrix} 0&1&1 \\ 1&0&1 \\ 1&1&0 \end{pmatrix}$



type $\mathrm{g}A_1$

$$P_3 = \begin{matrix} L_3 \\ L_3 \;\&\; L_3 \;\&\; L_1 \;\&\; L_2 \\ L_3 \;\&\; L_3 \;\&\; L_1 \;\&\; L_2 \\ L_3 \end{matrix}$$

The Loewy picture:

$$P_\mathtt{i} = \begin{matrix} \alpha_\mathtt{a} \\ \alpha_\mathtt{a}^2, \alpha_\mathtt{b}^2, \mathtt{j}\to\mathtt{i}\alpha_\mathtt{a} \\ \mathtt{x}_\mathtt{i} \end{matrix} \quad (\text{for } \mathtt{i}-\mathtt{j})$$

The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 3_{\mathsf{q}}!, & \text{if } \mathtt{i}=\mathtt{j}, \quad \text{the usual cohomology basis,} \\ \mathsf{q}^2+\mathsf{q}^4, & \text{if } \mathtt{i}-\mathtt{j}, \quad \{\mathtt{i}\to\mathtt{j}, \mathtt{i}\to\mathtt{j}\alpha_{\mathtt{a}}\} \text{ is a basis,} \\ 0, & \text{else.} \quad \emptyset \text{ is a basis.} \end{cases}$$

The volum

Thus, the Cartan matrix is

$$\boldsymbol{C}(\mathrm{gA}_1) = \begin{pmatrix} 3! & 2 & 2 \\ 2 & 3! & 2 \\ 2 & 2 & 3! \end{pmatrix} = 2 \cdot \left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right)$$

The (left)

$$\mathrm{P}_{\mathtt{i}} = \{\mathtt{i}, \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \alpha_{\mathtt{a}}^2, \alpha_{\mathtt{b}}^2, \mathtt{x}_{\mathtt{i}}, \mathtt{j}\to\mathtt{i}, \mathtt{j}\to\mathtt{i}\alpha_{\mathtt{a}}, \mathtt{x}_{\mathtt{i}} \mid \mathtt{i}-\mathtt{j}\} \quad \& \quad \mathrm{L}_{\mathtt{i}} = \{\mathtt{i}\}$$

The Loewy picture:

$$\mathrm{P}_{\mathtt{i}} = \begin{matrix} \mathtt{i} \\ \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \mathtt{j}\to\mathtt{i} \\ \alpha_{\mathtt{a}}^2, \alpha_{\mathtt{b}}^2, \mathtt{j}\to\mathtt{i}\alpha_{\mathtt{a}} \\ \mathtt{x}_{\mathtt{i}} \end{matrix} \text{ (for } \mathtt{i}-\mathtt{j}\text{)}$$

The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 3_{\mathtt{q}}!, & \text{if } \mathtt{i}=\mathtt{j}, & \text{the usual cohomology basis,} \\ \mathtt{q}^2+\mathtt{q}^4, & \text{if } \mathtt{i}{-}\mathtt{j}, & \{\mathtt{i}{\to}\mathtt{j}, \mathtt{i}{\to}\mathtt{j}\alpha_{\mathtt{a}}\} \text{ is a basis,} \\ 0, & \text{else.} & \emptyset \text{ is a basis.} \end{cases}$$

Thus, the Cartan matrix is

$$\boldsymbol{C}(\mathrm{gA}_1) = \begin{pmatrix} 3! & 2 & 2 \\ 2 & 3! & 2 \\ 2 & 2 & 3! \end{pmatrix} = 2 \cdot \left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right)$$

The volum

The (left)

$$\mathrm{P}_{\mathtt{i}} = \{\mathtt{i}, \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \quad \quad \& \quad \mathrm{L}_{\mathtt{i}} = \{\mathtt{i}\}$$

Fact. (Not hard to show.)
The Cartan matrix $\boldsymbol{C} = \boldsymbol{C}(\mathrm{T}_{\rightleftarrows})$ is

$$\boldsymbol{C} = 2(3\boldsymbol{I} + \boldsymbol{A}).$$

The Loewy picture:

$$\mathrm{P}_{\mathtt{i}} = \begin{matrix} \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \mathtt{j}{\to}\mathtt{i} \\ \alpha_{\mathtt{a}}^2, \alpha_{\mathtt{b}}^2, \mathtt{j}{\to}\mathtt{i}\alpha_{\mathtt{a}} \\ \mathtt{x}_{\mathtt{i}} \end{matrix} \quad (\text{for } \mathtt{i}{-}\mathtt{j})$$

The Cartan matrix

$$\mathrm{qdim}(\mathrm{Hom}_{Z_{\rightleftarrows}}(\mathtt{i},\mathtt{j})) = \begin{cases} 3_{\mathtt{q}}!, & \text{if } \mathtt{i}=\mathtt{j}, \quad \text{the usual cohomology basis,} \\ \mathtt{q}^2+\mathtt{q}^4, & \text{if } \mathtt{i}-\mathtt{j}, \quad \{\mathtt{i}\to\mathtt{j}, \mathtt{i}\to\mathtt{j}\alpha_{\mathtt{a}}\} \text{ is a basis,} \\ 0, & \text{else.} \quad \emptyset \text{ is a basis.} \end{cases}$$

The volum

The (left)

Thus, the Cartan matrix is

$$\boldsymbol{C}(\mathrm{gA}_1) = \begin{pmatrix} 3! & 2 & 2 \\ 2 & 3! & 2 \\ 2 & 2 & 3! \end{pmatrix} = 2\cdot\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right)$$

$$\mathrm{P}_{\mathtt{i}} = \{\mathtt{i}, \alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \alpha^2 \ \alpha^2 \ \ldots\} \quad \& \quad \mathrm{L}_{\mathtt{i}} = \{\mathtt{i}\}$$

Fact. (Not hard to show.)
The Cartan matrix $\boldsymbol{C} = \boldsymbol{C}(\mathrm{T}_{\rightleftarrows})$ is

$$\boldsymbol{C} = 2(3\boldsymbol{I} + \boldsymbol{A}).$$

The Loewy picture:

$$\alpha_{\mathtt{a}}, \alpha_{\mathtt{b}}, \mathtt{j}\to\mathtt{i}$$

P (for i j)

Fact. (Not hard to show.)
The graded Cartan matrix $\boldsymbol{C} = \boldsymbol{C}(\mathrm{T}_{\rightleftarrows})$ is

$$\boldsymbol{C} = 2_{\mathtt{q}}(3_{\mathtt{q}}\boldsymbol{I} + \mathtt{q}^2\boldsymbol{A}).$$

Generalized Chebyshev polynomials

Observation. Let $L_{e\omega}$ be the $e+1$-dimensional irreducible representation of $\mathrm{SL}(2)$. We have the correspondence

$$L_1 \leftrightsquigarrow \mathsf{X} \quad \& \quad L_1^{\otimes k} \leftrightsquigarrow \mathsf{X}^k \quad \& \quad L_{e\omega} \leftrightsquigarrow U_e(\mathsf{X}).$$

Define a Chebyshev polynomial $U_e(\mathsf{X}_\omega)$ associated to any semisimple algebraic group G by the correspondence

$$L_{\omega_i} \leftrightsquigarrow \mathsf{X}_i \quad \& \quad L_{\omega_i}^{\otimes k} \leftrightsquigarrow \mathsf{X}_i^k \quad \& \quad L_{e_1\omega_1+\dots+e_r\omega_r} \leftrightsquigarrow U_e(\mathsf{X}_\omega).$$

where $L_{\omega_1}, \dots, L_{\omega_r}$ are the fundamental representations of G, $e = e_1 + \dots + e_r$ and $\mathsf{X}_\omega = \mathsf{X}_1, \dots, \mathsf{X}_r$.

Fact. The so-called multivariate Chebyshev polynomial $U_e(\mathsf{X}_\omega)$ comes up in the theory of orthogonal polynomials, and has roots and recurrence relations coming from the root datum of G only.

Generalized Chebyshev polynomials

Observation. Let $L_{e\omega}$ be the $e{+}1$-dimensional irreducible representation of $\mathrm{SL}(2)$. We have the correspondence

$$L_1 \leftrightsquigarrow \mathrm{X} \quad \& \quad L_1^{\otimes k} \leftrightsquigarrow \mathrm{X}^k \quad \& \quad L_{e\omega} \leftrightsquigarrow U_e(\mathrm{X}).$$

Example $G = \mathrm{SL}(2)$.
The usual Chebyshev polynomial – you have seen this before.

Define a Chebyshev polynomial $U_e(\mathrm{X}_\omega)$ associated to any semisimple algebraic group G by the correspondence

$$L_{\omega_i} \leftrightsquigarrow \mathrm{X}_i \quad \& \quad L_{\omega_i}^{\otimes k} \leftrightsquigarrow \mathrm{X}_i^k \quad \& \quad L_{e_1\omega_1+\dots+e_r\omega_r} \leftrightsquigarrow U_e(\mathrm{X}_\omega).$$

where $L_{\omega_1}, \dots, L_{\omega_r}$ are the fundamental representations of G, $e = e_1 + \dots + e_r$ and $\mathrm{X}_\omega = \mathrm{X}_1, \dots, \mathrm{X}_r$.

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Generalized Chebyshev polynomials

Observat ... n of $\mathrm{SL}(2)$. W ...

Example $G = \mathrm{SL}(3)$**.**

$\mathrm{SL}(3)$ has two fundamental representations $L_{1,0} = \mathrm{X}$ and $L_{0,1} = \mathrm{Y}$; the vector representation and its dual.

Moreover, we have irreducibles $L_{m,n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$.

We have the following Chebyshev-like recursion relations

$$U_{m,n}(\mathrm{X},\mathrm{Y}) = U_{n,m}(\mathrm{Y},\mathrm{X}),$$
$$\mathrm{X}U_{m,n}(\mathrm{X},\mathrm{Y}) = U_{m+1,n}(\mathrm{X},\mathrm{Y}) + U_{m-1,n+1}(\mathrm{X},\mathrm{Y}) + U_{m,n-1}(\mathrm{X},\mathrm{Y}),$$
$$\mathrm{Y}U_{m,n}(\mathrm{X},\mathrm{Y}) = U_{m,n+1}(\mathrm{X},\mathrm{Y}) + U_{m+1,n-1}(\mathrm{X},\mathrm{Y}) + U_{m-1,n}(\mathrm{X},\mathrm{Y}),$$

together with starting conditions for $e = 0, 1$. Example.

Define a ... gebraic group G ...

where L_{ω} ... $\cdots + e_r$ and $\mathrm{X}_\omega =$ ...

The roots of these polynomial are very cute.

Fact. The so-called ... $U_e(\mathrm{X}_\omega)$ comes up in the theory of orthogonal polynomials, and has roots and recurrence relations coming from the root datum of G only.

Generalized Chebyshev polynomials

Observation. Let $L_{e\omega}$ be the $e+1$-dimensional irreducible representation of $\mathrm{SL}(2)$. We have the correspondence

$$L_1 \leftrightsquigarrow \mathsf{X} \quad \& \quad L_1^{\otimes k} \leftrightsquigarrow \mathsf{X}^k \quad \& \quad L_{e\omega} \leftrightsquigarrow U_e(\mathsf{X}).$$

The $\mathrm{SL}(3)$ Chebyshev polynomial plays the same role for the trihedral zigzag algebras as the Chebyshev polynomials do for the zigzag algebras.

$$L_{\omega_i} \leftrightsquigarrow \mathsf{X}_i \quad \& \quad L_{\omega_i}^{\otimes k} \leftrightsquigarrow \mathsf{X}_i^k \quad \& \quad L_{e_1\omega_1+\cdots+e_r\omega_r} \leftrightsquigarrow U_e(\mathsf{X}_\omega).$$

where $L_{\omega_1}, \dots, L_{\omega_r}$ are the fundamental representations of G, $e = e_1 + \cdots + e_r$ and $\mathsf{X}_\omega = \mathsf{X}_1, \dots, \mathsf{X}_r$.

Fact. The so-called multivariate Chebyshev polynomial $U_e(\mathsf{X}_\omega)$ comes up in the theory of orthogonal polynomials, and has roots and recurrence relations coming from the root datum of G only.

Generalized Chebyshev polynomials

Observation. Let $L_{e\omega}$ be the $e{+}1$-dimensional irreducible representation of $\mathrm{SL}(2)$. We have the correspondence

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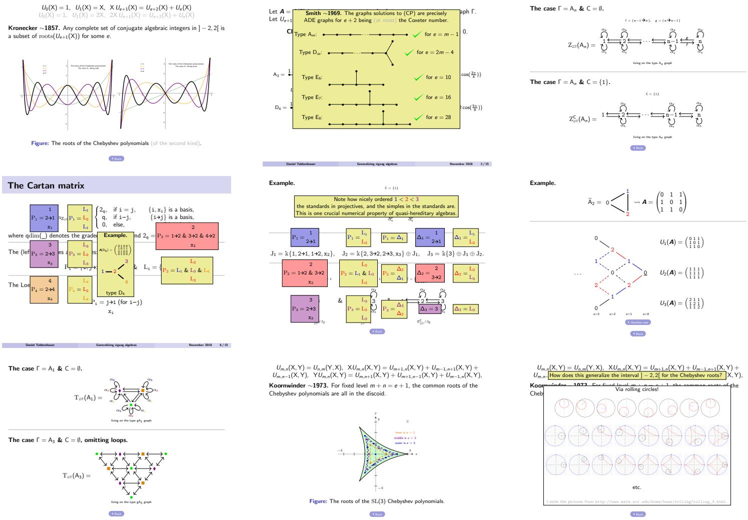
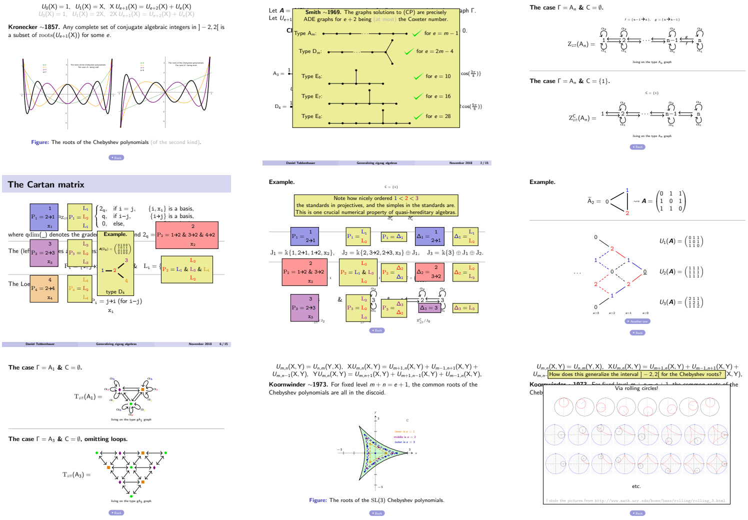
The $\mathrm{SL}(3)$ Chebyshev polynomial plays the same role for the trihedral zigzag algebras as the Chebyshev polynomials do for the zigzag algebras.

To wrap-up: What lies behind the horizon?
Zigzag algebras associated to root system;
generalizing the connection to modular representation theory of reductive groups.

Fact. The so-called multivariate Chebyshev polynomial $U_e(\mathrm{X}_\omega)$ comes up in the theory of orthogonal polynomials, and has roots and recurrence relations coming from the root datum of G only.



There is still much to do...



Thanks for your attention!

$$U_0(X) = 1, \quad U_1(X) = X, \quad X\, U_{e+1}(X) = U_{e+2}(X) + U_e(X)$$
$$U_0(X) = 1, \quad U_1(X) = 2X, \quad 2X\, U_{e+1}(X) = U_{e+2}(X) + U_e(X)$$

Kronecker ∼1857. Any complete set of conjugate algebraic integers in $]-2,2[$ is a subset of $\mathrm{roots}(U_{e+1}(X))$ for some e.

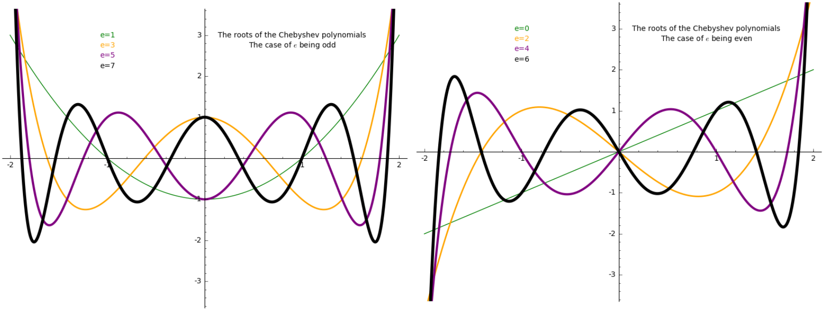


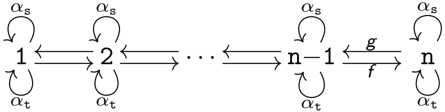
Figure: The roots of the Chebyshev polynomials (of the second kind).

Back

The case $\Gamma = A_n$ & $C = \emptyset$.

$f = (\mathtt{n-1} \to \mathtt{n}), \quad g = (\mathtt{n} \to \mathtt{n-1})$

$Z_{\rightleftarrows}(A_n) =$

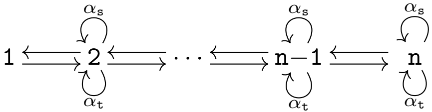


living on the type A_n graph

The case $\Gamma = A_n$ & $C = \{1\}$.

$C = \{1\}$

$Z^{C}_{\rightleftarrows}(A_n) =$



living on the type A_n graph

◂ Back

Definition (e.g. Cline–Parshall–Scott ~1988). A finite-dimensional algebra R is called quasi-hereditary if there exists a chain of ideals

$$0 = \mathrm{J}_0 \subset \mathrm{J}_1 \subset \cdots \subset \mathrm{J}_{k-1} \subset \mathrm{J}_k = \mathrm{R},$$

for some $k \in \mathbb{Z}_{\geq 1}$, such that the quotient $\mathrm{J}_l/\mathrm{J}_{l-1}$ is an hereditary ideal in $\mathrm{R}/\mathrm{J}_{l-1}$.

The point: Quasi-hereditary algebras have associated highest weight categories, i.e. they have simple, (co)standard Δ, indecomposable projective and tilting modules, all indexed by the same ordered set.

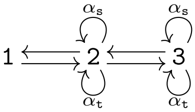
▸ Example

◂ Back

Example.

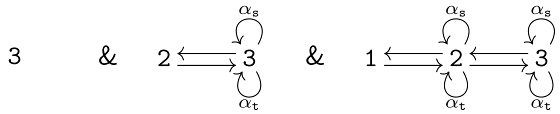
$C = \{1\}$

$$Z^{C}_{\rightleftarrows}(A_3) =$$



$$J_1 = \Bbbk\{1, 2{\to}1, 1{\to}2, x_2\}, \quad J_2 = \Bbbk\{2, 3{\to}2, 2{\to}3, x_3\} \oplus J_1, \quad J_3 = \Bbbk\{3\} \oplus J_1 \oplus J_2.$$

$\boldsymbol{C} = (1)$; det $= 1$ $\boldsymbol{C} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; det $= 1$ $\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$; det $= 1$



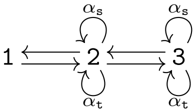
$Z^{C}_{\rightleftarrows}/J_2$ $Z^{C}_{\rightleftarrows}/J_1$ $Z^{C}_{\rightleftarrows}/J_0$

Back

Example.

$C = \{1\}$

$Z^{C}_{\rightleftarrows}(A_3) =$



$P_1 = \begin{matrix} 1 \\ 2\to 1 \end{matrix}$

$\Delta_1 = \begin{matrix} 1 \\ 2\to 1 \end{matrix}$

$$J_1 = \Bbbk\{1, 2\to 1, 1\to 2, x_2\}, \quad J_2 = \Bbbk\{2, 3\to 2, 2\to 3, x_3\} \oplus J_1, \quad J_3 = \Bbbk\{3\} \oplus J_1 \oplus J_2.$$

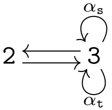
$P_2 = \begin{matrix} 2 \\ 1\to 2 \;\&\; 3\to 2 \\ x_2 \end{matrix}$

$\Delta_2 = \begin{matrix} 2 \\ 3\to 2 \end{matrix}$

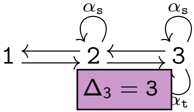
$= 1$ $\qquad$ $C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; det $= 1$ $\qquad$ $C = ($

$P_3 = \begin{matrix} 3 \\ 2\to 3 \\ x_3 \end{matrix}$

&



&



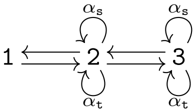
$\Delta_3 = 3$

$/J_2$ $\qquad$ $Z^{C}_{\rightleftarrows}/J_1$ $\qquad$ $Z^{C}_{\rightleftarrows}/J_0$

Back

Example.

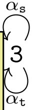
C = {1}

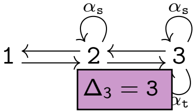
$Z^{C}_{\rightleftarrows}(A_3) =$ 

$P_1 = \begin{matrix} 1 \\ 2\to1 \end{matrix}$ $\quad P_1 = \begin{matrix} L_1 \\ L_2 \end{matrix}$ $\quad \Delta_1 = \begin{matrix} 1 \\ 2\to1 \end{matrix}$ $\quad \Delta_1 = \begin{matrix} L_1 \\ L_2 \end{matrix}$

$J_1 = \Bbbk\{1, 2\to1, 1\to2, x_2\}, \quad J_2 = \Bbbk\{2, 3\to2, 2\to3, x_3\} \oplus J_1, \quad J_3 = \Bbbk\{3\} \oplus J_1 \oplus J_2.$

$P_2 = \begin{matrix} 2 \\ 1\to2 \ \& \ 3\to2 \\ x_2 \end{matrix}$ $\quad P_2 = \begin{matrix} L_2 \\ L_1 \ \& \ L_3 \\ L_2 \end{matrix}$ $\quad C = ($ $\quad \Delta_2 = \begin{matrix} 2 \\ 3\to2 \end{matrix}$ $\quad \Delta_2 = \begin{matrix} L_2 \\ L_3 \end{matrix}$

$P_3 = \begin{matrix} 3 \\ 2\to3 \\ x_3 \end{matrix}$ & $P_3 = \begin{matrix} L_3 \\ L_2 \\ L_3 \end{matrix}$ 

& 

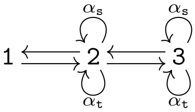
$\Delta_3 = 3$ $\quad \Delta_3 = L_3$

$/J_2$ $\quad Z^{C}_{\rightleftarrows}/J_0$

Back

Example.

$C = \{1\}$

$Z^C_{\rightleftarrows}(A_3) =$ 

$P_1 = \begin{matrix} 1 \\ 2\to 1 \end{matrix}$ $P_1 = \begin{matrix} L_1 \\ L_2 \end{matrix}$ $P_1 = \Delta_1$ $\Delta_1 = \begin{matrix} 1 \\ 2\to 1 \end{matrix}$ $\Delta_1 = \begin{matrix} L_1 \\ L_2 \end{matrix}$

$J_1 = \Bbbk\{1, 2\to 1, 1\to 2, x_2\}$, $J_2 = \Bbbk\{2, 3\to 2, 2\to 3, x_3\} \oplus J_1$, $J_3 = \Bbbk\{3\} \oplus J_1 \oplus J_2$.

$P_2 = \begin{matrix} 2 \\ 1\to 2 \ \& \ 3\to 2 \\ x_2 \end{matrix}$ $P_2 = \begin{matrix} L_2 \\ L_1 \ \& \ L_3 \\ L_2 \end{matrix}$ $P_2 = \begin{matrix} \Delta_2 \\ \Delta_1 \end{matrix}$ $\Delta_2 = \begin{matrix} 2 \\ 3\to 2 \end{matrix}$ $\Delta_2 = \begin{matrix} L_2 \\ L_3 \end{matrix}$

$P_3 = \begin{matrix} 3 \\ 2\to 3 \\ x_3 \end{matrix}$ & $P_3 = \begin{matrix} L_3 \\ L_2 \\ L_3 \end{matrix}$ $P_3 = \begin{matrix} \Delta_3 \\ \Delta_2 \end{matrix}$ $\Delta_3 = 3$ $\Delta_3 = L_3$

$/J_2$ $Z^C_{\rightleftarrows}/J_0$

Back

Example.

$C = \{1\}$

Note how nicely ordered $1 < 2 < 3$
the standards in projectives, and the simples in the standards are.
This is one crucial numerical property of quasi-hereditary algebras.

$P_1 = \begin{matrix} 1 \\ 2{\to}1 \end{matrix}$ $P_1 = \begin{matrix} L_1 \\ L_2 \end{matrix}$ $P_1 = \Delta_1$ $\Delta_1 = \begin{matrix} 1 \\ 2{\to}1 \end{matrix}$ $\Delta_1 = \begin{matrix} L_1 \\ L_2 \end{matrix}$

$$J_1 = \mathbb{k}\{1, 2{\to}1, 1{\to}2, x_2\}, \quad J_2 = \mathbb{k}\{2, 3{\to}2, 2{\to}3, x_3\} \oplus J_1, \quad J_3 = \mathbb{k}\{3\} \oplus J_1 \oplus J_2.$$

$P_2 = \begin{matrix} 2 \\ 1{\to}2 \;\&\; 3{\to}2 \\ x_2 \end{matrix}$ $P_2 = \begin{matrix} L_2 \\ L_1 \;\&\; L_3 \\ L_2 \end{matrix}$ $P_2 = \begin{matrix} \Delta_2 \\ \Delta_1 \end{matrix}$ $\Delta_2 = \begin{matrix} 2 \\ 3{\to}2 \end{matrix}$ $\Delta_2 = \begin{matrix} L_2 \\ L_3 \end{matrix}$

$P_3 = \begin{matrix} 3 \\ 2{\to}3 \\ x_3 \end{matrix}$ $P_3 = \begin{matrix} L_3 \\ L_2 \\ L_3 \end{matrix}$ $P_3 = \begin{matrix} \Delta_3 \\ \Delta_2 \end{matrix}$ $\Delta_3 = 3$ $\Delta_3 = L_3$

Back

Example.

C = {1}

Note how nicely ordered $1 < 2 < 3$
the standards in projectives, and the simples in the standards are.
This is one crucial numerical property of quasi-hereditary algebras.

α_t α_t

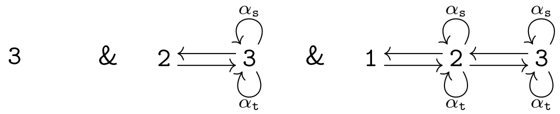
$J_1 = \Bbbk\{1, 2\to 1$ … $\{3\} \oplus J_1 \oplus J_2.$

The reciprocity:

$$\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boldsymbol{D}$ matrix encodes simples in standards.

$\boldsymbol{C} = (1)$; det $= 1$ $\quad$ $\boldsymbol{C} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; det $= 1$ $\quad$ $\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$; det $= 1$



$Z^{C}_{\rightleftarrows}/J_2$ $\quad$ $Z^{C}_{\rightleftarrows}/J_1$ $\quad$ $Z^{C}_{\rightleftarrows}/J_0$

Back

Example.

$C = \{1\}$

Note how nicely ordered 1 < 2 < 3
the standards in projectives, and the simples in the standards are.
This is one crucial numerical property of quasi-hereditary algebras.

α_t α_t

The reciprocity:

$$\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

D matrix encodes simples in standards.

$J_1 = \mathbb{k}\{1, 2\to1$ … $\{3\} \oplus J_1 \oplus J_2.$

$\boldsymbol{C} = (1)$; det $= 1$ $\quad$ $\boldsymbol{C} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; det $= 1$ $\quad$ $\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$; det $= 1$

α_s α_s α_s

Cellularity, roughly speaking, works very similarly, but ***D*** does not need to be a square matrix while in the relative case ***D*** does not even need to be a upper triangular matrix.

α_t α_t α_t

$Z^{C}_{\rightleftarrows}/J_2$ $\quad$ $Z^{C}_{\rightleftarrows}/J_1$ $\quad$ $Z^{C}_{\rightleftarrows}/J_0$

Back

A linear projective resolution of a graded module M of a positively graded algebra R is an exact sequence

$$\cdots \longrightarrow q^2 \mathrm{Q}_2 \longrightarrow q\mathrm{Q}_1 \longrightarrow \mathrm{Q}_0 \twoheadrightarrow \mathrm{M},$$

with graded projective R-modules $q^e \mathrm{Q}_e$ generated in degree e.

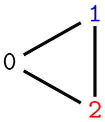
Definition (e.g. Priddy ∼1970). A finite-dimensional, positively graded algebra R is called Koszul if its degree 0 part is semisimple and each simple R-module admits a linear projective resolution.

The point: Koszul algebras have projective resolutions of simples which are as easy as possible.

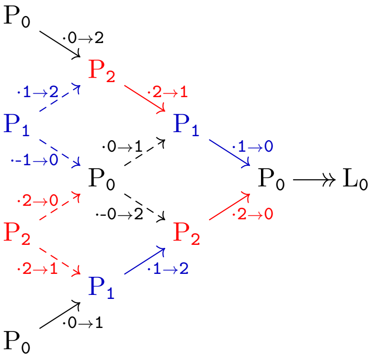
▸ Example

◂ Back

Example.

$$Z^{C=\emptyset}_{\rightleftarrows}(\widetilde{A}_2) =$$ 

From now I just draw the graphs.



0

$P_0 = 1{\to}0 \ \& \ 2{\to}0$

x_0

1

$P_1 = 0{\to}1 \ \& \ 2{\to}1,$

x_1

2

$P_2 = 0{\to}2 \ \& \ 1{\to}2$

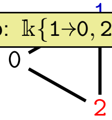
x_2

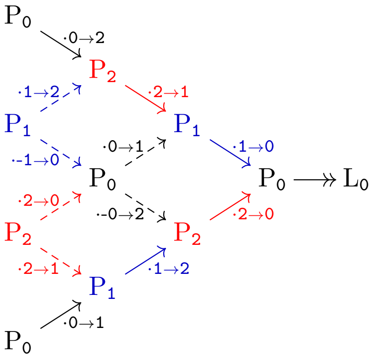
Back

Example.

Kernel in the first step: $\Bbbk\{1\to 0, 2\to 0, x_0\}$

$Z^{C=\emptyset}_{\rightleftarrows}(A_2) =$





$$0$$
$$P_0 = 1\to 0 \;\&\; 2\to 0$$
$$x_0$$
$$1$$
$$P_1 = 0\to 1 \;\&\; 2\to 1,$$
$$x_1$$
$$2$$
$$P_2 = 0\to 2 \;\&\; 1\to 2$$
$$x_2$$

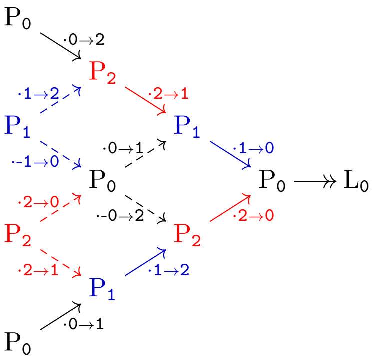
Back

Example.

Kernel in the first step: $\Bbbk\{1\to 0, 2\to 0, x_0\}$

$Z^{C=\emptyset}_{\rightleftarrows}(A_2) = 0$

Kernel in the second step: $\Bbbk\{2\to 1, x_1, 1\to 2, x_2\}$ and $\Bbbk\{0\to 1 - 0\to 2\}$.



0

$P_0 = 1\to 0 \;\&\; 2\to 0$

x_0

1

$P_1 = 0\to 1 \;\&\; 2\to 1,$

x_1

2

$P_2 = 0\to 2 \;\&\; 1\to 2$

x_2

Back

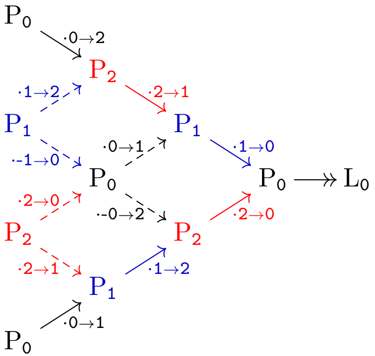
Example.

Kernel in the first step: $\Bbbk\{1{\to}0, 2{\to}0, x_0\}$

$Z^{C=\emptyset}_{\rightleftarrows}(A_2) = 0$

Kernel in the second step: $\Bbbk\{2{\to}1, x_1, 1{\to}2, x_2\}$ and $\Bbbk\{0{\to}1 - 0{\to}2\}$.

Kernel in the third step: $\Bbbk\{0{\to}2, x_2, 0{\to}1, x_1\}$ and $\Bbbk\{1{\to}2 - 1{\to}0\}$ and $\Bbbk\{2{\to}0 + 2{\to}1\}$.



$$\begin{matrix} 0 \\ P_0 = 1{\to}0 \ \& \ 2{\to}0 \\ x_0 \\ 1 \\ P_1 = 0{\to}1 \ \& \ 2{\to}1, \\ x_1 \\ 2 \\ P_2 = 0{\to}2 \ \& \ 1{\to}2 \\ x_2 \end{matrix}$$

Back

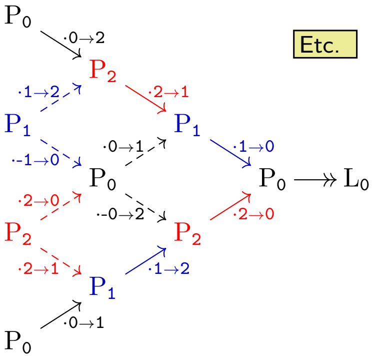
Example.

Kernel in the first step: $\Bbbk\{1{\to}0, 2{\to}0, x_0\}$

$Z^{C=\emptyset}_{\rightleftarrows}(A_2) = \; 0$

Kernel in the second step: $\Bbbk\{2{\to}1, x_1, 1{\to}2, x_2\}$ and $\Bbbk\{0{\to}1 - 0{\to}2\}$.

Kernel in the third step: $\Bbbk\{0{\to}2, x_2, 0{\to}1, x_1\}$ and $\Bbbk\{1{\to}2 - 1{\to}0\}$ and $\Bbbk\{2{\to}0 + 2{\to}1\}$.



Etc.

$$0$$
$$P_0 = 1{\to}0 \;\&\; 2{\to}0$$
$$x_0$$
$$1$$
$$P_1 = 0{\to}1 \;\&\; 2{\to}1,$$
$$x_1$$
$$2$$
$$P_2 = 0{\to}2 \;\&\; 1{\to}2$$
$$x_2$$

Example.

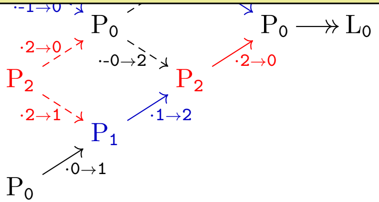
$$\mathbf{C} = \begin{pmatrix} 1+q^2 & q & q \\ q & 1+q^2 & q \\ q & q & 1+q^2 \end{pmatrix}$$

gives the cofactor matrix $\mathbf{C}^* = \begin{pmatrix} 1+q^2+q^4 & -q+q^2-q^3 & -q+q^2-q^3 \\ -q+q^2-q^3 & 1+q^2+q^4 & q \\ -q+q^2-q^3 & -q+q^2-q^3 & 1+q^2+q^4 \end{pmatrix}$

and the determinant $\det = 1 + 2q^3 + q^6$.

Then $1+q^2+q^4+2(-q+q^2-q^3)/(1+2q^3+q^6)$ 'taylored' gives

$$1 - 2q + 3q^2 - 4q^3 + 5q^4 - 6q^5 \pm \dots.$$



$P_1 = 0\to1$ & $2\to1$,

x_1

2

$P_2 = 0\to2$ & $1\to2$

x_2

Back

Example.

$$C = \begin{pmatrix} 1+q^2 & q & q \\ q & 1+q^2 & q \\ q & q & 1+q^2 \end{pmatrix}$$

gives the cofactor matrix $C^* = \begin{pmatrix} 1+q^2+q^4 & -q+q^2-q^3 & -q+q^2-q^3 \\ -q+q^2-q^3 & 1+q^2+q^4 & q \\ -q+q^2-q^3 & -q+q^2-q^3 & 1+q^2+q^4 \end{pmatrix}$

and the determinant $\det = 1 + 2q^3 + q^6$.

Then $1 + q^2 + q^4 + 2(-q + q^2 - q^3)/(1 + 2q^3 + q^6)$ 'taylored' gives

$$1 - 2q + 3q^2 - 4q^3 + 5q^4 - 6q^5 \pm \dots.$$

$\cdot$-1→0 $\quad \mathrm{P}_0 \qquad \mathrm{P}_0 \twoheadrightarrow \mathrm{L}_0 \qquad \mathrm{P}_1 = 0{\to}1\ \&\ 2{\to}1,$

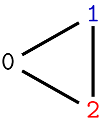
The numerical criterion for koszulness:

$\det^{-1} C^*$ has q-linear column sums.

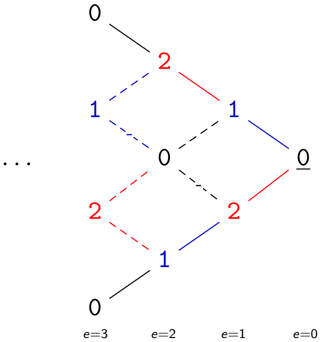
This is almost an if and only if: C encodes how projectives are filtered by simples. So, $\det^{-1} C^*$ encodes how simples are resolved by projectives.

Back

Example.

$\widetilde{A}_2 =$ 

$\rightsquigarrow \boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$



$$U_1(\boldsymbol{A}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

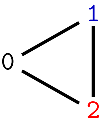
$$U_2(\boldsymbol{A}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$U_3(\boldsymbol{A}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

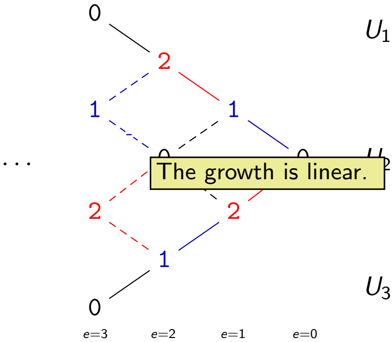
Another one

Back

Example.

$\widetilde{A}_2 =$ 

$\rightsquigarrow \boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$



$$U_1(\boldsymbol{A}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$U_2(\boldsymbol{A}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

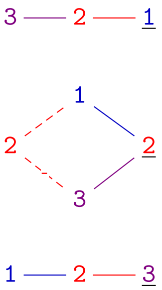
$$U_3(\boldsymbol{A}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Another one

Back

Example.

$$A_3 = 1 \,\text{——}\, 2 \,\text{——}\, 3 \;\rightsquigarrow\; \boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



?? $\quad U_1(\boldsymbol{A}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

?? $\quad U_2(\boldsymbol{A}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

?? $\quad U_3(\boldsymbol{A}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$e=3$ $\quad e=2$ $\quad e=1$ $\quad e=0$

Back

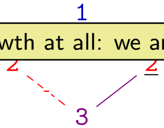
Example.

$$\mathrm{A}_3 = 1 \,\text{——}\, 2 \,\text{——}\, 3 \quad \rightsquigarrow \quad \boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

?? $\quad 3 \text{ — } 2 \text{ — } \underline{1} \qquad U_1(\boldsymbol{A}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

No growth at all: we are stuck in the 3^{th} step.

??



$U_2(\boldsymbol{A}) = \begin{pmatrix} \cdots \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

?? $\quad 1 \text{ — } 2 \text{ — } \underline{3} \qquad U_3(\boldsymbol{A}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$e{=}3 \qquad e{=}2 \qquad e{=}1 \qquad e{=}0$

Back

The inverses of the graded Cartan determinants.

$$\begin{aligned}
\mathrm{A}_n\colon & (1-\mathrm{q}^2)\textstyle\sum_{s=0}^{\infty}\mathrm{q}^{(2n+2)s}, && \text{gap}=2n-1,\\
\mathrm{D}_n, n\text{ even}\colon & (1-\mathrm{q}^2\pm\cdots+\mathrm{q}^{2n-4})\textstyle\sum_{s=0}^{\infty}(-1)^s(s+1)\mathrm{q}^{(2n-2)s}, && \text{gap}=1,\\
\mathrm{D}_n, n\text{ odd}\colon & (1-\mathrm{q}^2\pm\cdots-\mathrm{q}^{2n-4})\textstyle\sum_{s=0}^{\infty}\mathrm{q}^{(4n-4)s}, && \text{gap}=2n-1,\\
\mathrm{E}_6\colon & (1-\mathrm{q}^2+\mathrm{q}^4-\mathrm{q}^8+\mathrm{q}^{10}-\mathrm{q}^{12})\textstyle\sum_{s=0}^{\infty}\mathrm{q}^{24s}, && \text{gap}=11,\\
\mathrm{E}_7\colon & (1-\mathrm{q}^2+\mathrm{q}^4)\textstyle\sum_{s=0}^{\infty}(-1)^s\mathrm{q}^{18s}, && \text{gap}=13,\\
\mathrm{E}_8\colon & (1-\mathrm{q}^2+\mathrm{q}^4+\mathrm{q}^{10}-\mathrm{q}^{12}+\mathrm{q}^{14})\textstyle\sum_{s=0}^{\infty}(-1)^s\mathrm{q}^{30s}, && \text{gap}=15.
\end{aligned}$$

Observing now that the cofactor matrix has entries which are polynomials of degree $\leq 2n-2$, one is done. Type D_{2n} needs an extra argument along the same lines.

Back

Explicitly, for type A_3 we get

$$(1-q^2)(1+q^8+q^{16}+q^{24}+\dots) = 1-q^2+q^8-q^{10}+q^{16}-q^{18}+q^{24}-q^{26}+\dots.$$

$$\boldsymbol{A}^* = \begin{pmatrix} 1+q^2+q^4 & -q-q^3 & q \\ -q-q^3 & 1+q^2+q^4 & -q-q^3 \\ q & -q-q^3 & 1+q^2+q^4 \end{pmatrix}$$

Numerical resolutions are

$$1-q+q^2-0q^3+q^4-q^5+q^6-0q^7\pm\dots$$

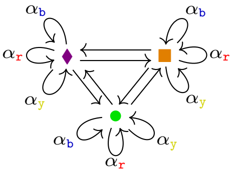
$$1-2q+q^2-0q^3+q^4-2q^5+q^6-0q^7\pm\dots$$

$$1-q+q^2-0q^3+q^4-q^5+q^6-0q^7\pm\dots$$

◂ Back

The case $\Gamma = A_1$ & $C = \emptyset$.

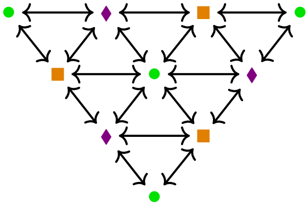
$T_{\rightleftarrows}(A_1) =$



living on the type gA_1 graph

The case $\Gamma = A_3$ & $C = \emptyset$, omitting loops.

$T_{\rightleftarrows}(A_3) =$



living on the type gA_3 graph

Back

Example. The first few $\mathrm{SL}(3)$ Chebyshev polynomials:

$e=0$	$U_{1,0}(X,Y)=X, \quad U_{0,1}(X,Y)=Y,$
$e=1$	$U_{2,0}(X,Y)=X^2-Y, \quad U_{1,1}(X,Y)=XY-1, \quad U_{0,2}(X,Y)=Y^2-X,$
$e=2$	$U_{3,0}(X,Y)=X^3-2XY+1, \quad U_{2,1}(X,Y)=X^2Y-Y^2-X,$ $U_{1,2}(X,Y)=XY^2-X^2-Y, \quad U_{0,3}(X,Y)=Y^3-2XY+1,$
$e=3$	$U_{4,0}(X,Y)=X^4-3X^2Y+Y^2+2X, \quad U_{3,1}(X,Y)=X^3Y-2XY^2-X^2+2Y,$ $U_{2,2}(X,Y)=X^2Y^2-X^3-Y^3,$ $U_{1,3}(X,Y)=XY^3-2X^2Y-Y^2+2X, \quad U_{0,4}(X,Y)=Y^4-3XY^2+X^2+2Y,$
$e=4$	$U_{5,0}(X,Y)=X^5-4X^3Y+3XY^2+3X^2-2Y, \quad U_{4,1}(X,Y)=X^4Y-3X^2Y^2-X^3+Y^3+4XY-1,$ $U_{3,2}(X,Y)=X^3Y^2-X^4-2XY^3+X^2Y+2Y^2-X, \quad U_{2,3}(X,Y)=X^2Y^3-Y^4-2X^3Y+XY^2+2X^2-Y,$ $U_{1,4}(X,Y)=XY^4-3X^2Y^2-Y^3+X^3+4XY-1, \quad U_{0,5}(X,Y)=Y^5-4XY^3+3X^2Y+3Y^2-2X.$

One usually considers them for one level $m+n=e+1$ together.

Back

$$U_{m,n}(\mathsf{X},\mathsf{Y}) = U_{n,m}(\mathsf{Y},\mathsf{X}), \;\; \mathsf{X}U_{m,n}(\mathsf{X},\mathsf{Y}) = U_{m+1,n}(\mathsf{X},\mathsf{Y}) + U_{m-1,n+1}(\mathsf{X},\mathsf{Y}) + U_{m,n-1}(\mathsf{X},\mathsf{Y}), \;\; \mathsf{Y}U_{m,n}(\mathsf{X},\mathsf{Y}) = U_{m,n+1}(\mathsf{X},\mathsf{Y}) + U_{m+1,n-1}(\mathsf{X},\mathsf{Y}) + U_{m-1,n}(\mathsf{X},\mathsf{Y}),$$

Koornwinder ∼1973. For fixed level $m + n = e + 1$, the common roots of the Chebyshev polynomials are all in the discoid.

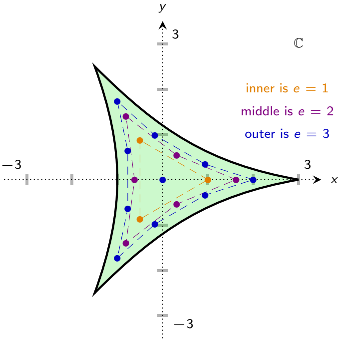


Figure: The roots of the $\mathrm{SL}(3)$ Chebyshev polynomials.

$U_{m,n}(X,Y) = U_{n,m}(Y,X), \quad XU_{m,n}(X,Y) = U_{m+1,n}(X,Y) + U_{m-1,n+1}(X,Y) +$

$U_{m,n-}$ How does this generalize the interval $]-2,2[$ for the Chebyshev roots? (X,Y),

Koornwinder ∼1973. For fixed level $m+n=e+1$, the common roots of the Chebyshev polynomials are all in the discoid.

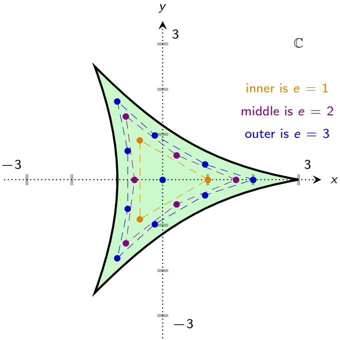


Figure: The roots of the $\mathrm{SL}(3)$ Chebyshev polynomials.

Back

$U_{m,n}(X,Y) = U_{n,m}(Y,X)$, $XU_{m,n}(X,Y) = U_{m+1,n}(X,Y) + U_{m-1,n+1}(X,Y) +$
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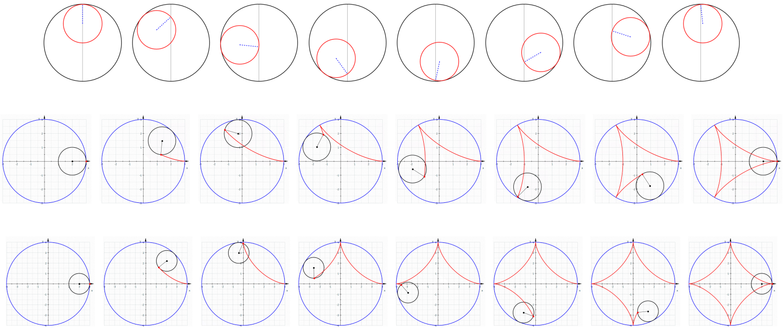
Cheby

Via rolling circles!



etc.

I stole the pictures from `http://www.math.ucr.edu/home/baez/rolling/rolling_3.html`.

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