Talk 2: First instance for an Auslander-Reiten quiver.

1. Kernels, Cokernels, Exact Sequences

recall (linalg): $f: V \to V'$ linear map
$\ker(f) = \{v \in V \mid f(v) = 0\}$ subspace of V
$\operatorname{coker}(f) = V'/\operatorname{im} f = \{v' + f(V) \mid v' \in V'\}$ quotient space of V'

We generalize these concepts to representations:

let Q be a quiver, and let $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $M' = (M_i', \varphi_\alpha')_{i \in Q_0, \alpha \in Q_1}$ be two representations of Q. Furthermore let $f = (f_i)_{i \in Q_0} : M \to M'$ be a morphism of representations. Recall that each f_i is a linear map from the vector space M_i to the vector space M_i'.

For each vertex $i \in Q_0$, let $L_i = \ker f_i$, and for each arrow $i \xrightarrow{\alpha} j$ in Q_1, let $\psi_\alpha : L_i \to L_j$ be the restriction of φ_α to L_i, that is, $\psi_\alpha(x) = \varphi_\alpha(x)$ for all $x \in L_i$.

Definition: The representation $\ker f = (L_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is called the kernel of f.

remark: The inclusions $\operatorname{incl}_i : \ker f_i \hookrightarrow M_i$ induce an injective morphism of representations:

$$(\operatorname{incl}_i)_{i \in Q_0} : \ker f \hookrightarrow M.$$

For each vertex $i \in Q_0$, let $N_i = \operatorname{coker} f_i = M_i'/f_i(M_i)$ and for each arrow $i \xrightarrow{\alpha} j$ in Q_1, define $\chi_\alpha : N_i \to N_j$ by

$$\chi_\alpha(m_i' + f_i(M_i)) = \varphi_\alpha'(m_i') + f_j(M_j) \quad \forall m_i' \in M_i'.$$

Definition: The representation $\operatorname{coker} f = (N_i, \chi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is called the cokernel of f.

remark: The projections $\operatorname{proj}_i : M_i' \twoheadrightarrow \operatorname{coker} f_i$ induce a surjective morphism of representations:

$$(\operatorname{proj}_i)_{i \in Q_0} : M' \twoheadrightarrow \operatorname{coker} f.$$

Definition: A representation L is called <u>subrepresentation</u> of a representation M if there is an injective morphism $i: L \hookrightarrow M$. In this situation, the <u>quotient representation</u> M/L is defined to be the cokernel of i.

Theorem (1st isom. thm.): If $f: M \to N$ is a morphism of representations, then
$$\operatorname{im} f \cong M/\ker f.$$
(proof admitted)

Next we introduce the notion of exact sequences which will be fundamental for the rest of the lecture.

Definition: A sequence of morphisms $L \xrightarrow{f} M \xrightarrow{g} N$ is called <u>exact at M</u> if $\operatorname{im} f = \ker g$.
A sequence of morphisms $\cdots \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \cdots$ is called <u>exact</u> if it is exact at every M_i.

Definition: A <u>short exact sequence</u> is an exact sequence of the form:
$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$
Note that the sequence is short exact iff:
- f is injective
- $\operatorname{im} f = \ker g$
- g is surjective

Example: Let $f: M \to N$ be a morphism in $\operatorname{rep} Q$. Then the sequence
$$0 \to \ker f \xrightarrow{u} M \xrightarrow{f} N \xrightarrow{p} \operatorname{coker} f \to 0$$
is <u>exact</u>, and the sequence
$$0 \to \ker f \xrightarrow{u} M \xrightarrow{p} M/\ker f \to 0$$
is <u>short exact</u>.
(note: u is the inclusion and p is the projection, see previous remarks)

Example: Let Q be the quiver $1 \to 2$, and consider the three representations:

$S(2)$ $\quad (0 \to k)$

M $\quad (k \xrightarrow{1} k)$

$S(1)$ $\quad (k \to 0)$

Then
$$0 \to S(2) \xrightarrow{f} M \xrightarrow{g} S(1) \to 0$$
$$0 \to S(2) \xrightarrow{f'} S(1) \oplus S(2) \xrightarrow{g'} S(1) \to 0$$
are short exact sequences.

$f = (f_1, f_2) = (0, 1)$
$g = (g_1, g_2) = (1, 0)$
$f' = (f_1', f_2') = (0, 1)$
$g' = (g_1', g_2') = (1, 0)$

<u>Definition:</u> A morphism $f: L \to M$ is called a <u>section</u> if there exists a morphism $h: M \to L$ s.t. $h \circ f = 1_L$.
A morphism $g: M \to N$ is called a <u>retraction</u> if there exists a morphism $h: N \to M$ s.t. $g \circ h = 1_N$.

<u>Definition:</u> We say that a short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ <u>splits</u>, if f is a section.

<u>Example:</u> In the previous example the second short exact sequence splits, because the morphism:

$$h': S(1) \oplus S(2) \xrightarrow{(0,1)} S(2) \text{ verifies } h' \circ f' = 1_{S(2)}.$$

The first sequence does not split, since there is no non zero morphism from M to $S(2)$; hence f cannot be a section.

<u>Proposition:</u> Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in rep$Q$.
Then:
(a) f is a section iff g is a retraction
(b) If f is a section, then $\operatorname{im} f$ ($= \ker g$) is a direct summand of M (i.e. $\exists M'$ s.t. $\operatorname{im} f \oplus M' = M$).

<u>Sketch of the proof:</u>

(a) <u>"$\Rightarrow$"</u>: · Suppose that f is a section. Then $\exists h \in \operatorname{Hom}(M,L)$ s.t. $h \circ f = 1_L$.
· <u>to show</u>: $\exists h' \in \operatorname{Hom}(N,M)$ s.t. $g \circ h' = 1_N$.
→ define $h': N \to M$
$$n \mapsto h'(n) = m - f \circ h(m)$$
→ show that h' does not depend on the choice of m.
→ show that h' is a morphism

$$\begin{tikzcd}
L_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & L_j \arrow[d, "f_j"] \\
M_i \arrow[r, "\varphi_\alpha'"] \arrow[d, "g_i"'] \arrow[u, bend left=60, "h_i"] & M_j \arrow[d, "g_j"] \arrow[u, bend right=60, "h_j"'] \\
N_i \arrow[r, "\varphi_\alpha''"] \arrow[u, bend left=60, "h_i'"] & N_j \arrow[u, bend right=60, "h_j'"']
\end{tikzcd}$$

<u>to show:</u> $\varphi_\alpha' \circ h_i'(n_i) = h_j' \circ \varphi_\alpha''(n_i)$

(note: diagram commutes with respect to f, g, h)

<u>"$\Leftarrow$"</u>: same principle as above
→ define $h: M \to L$
$$m \mapsto h(m) = l$$
where $l \in L$ s.t. $f(l) = m - h' \circ g(m)$

(6) to show: $M = \mathrm{im}\, h' \oplus \ker g$

$$(M_i, \varphi_\alpha')_{i\in Q_0, \alpha\in Q_1} = \left(\mathrm{im}\, h_i' \oplus \ker g_i, \begin{bmatrix} \varphi_\alpha'|_{\mathrm{im}\, h_i'} & 0 \\ 0 & \varphi_\alpha'|_{\ker g_i} \end{bmatrix}\right)_{i\in Q_0, \alpha\in Q_1}$$

<u>Corollary:</u> If the sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is split exact, then $M \cong L \oplus N$.
(follows from the previous proposition and the 1st isom. thm.)

2. Hom functors

We now want to introduce the Hom functors and study their effect on short exact sequences.

<u>Definition (categories):</u> Let $\mathcal{C}, \mathcal{C}'$ be two k-categories.
A <u>covariant functor</u> (resp. a <u>contravariant functor</u>) $F: \mathcal{C} \to \mathcal{C}'$ is a mapping that associates
- to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{C}'$ and
- to each morphism $f: X \to Y$ in $\mathcal{C}$ a morphism $F(f): F(X) \to F(Y)$ in $\mathcal{C}'$ (resp. $F(f): F(Y) \to F(X)$)

such that
$$F(1_X) = 1_{F(X)} \quad \text{and} \quad F(g\circ f) = F(g)\circ F(f) \quad \text{for all objects } X$$
$$(\text{resp. } F(g\circ f) = F(f)\circ F(g))$$
and for all morphisms f and g in $\mathcal{C}$.

<u>two very important functors:</u> Hom functors $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$
(X arbitrary fixed object in category $\mathcal{C}$)

$\mathrm{Hom}(X,-): \mathcal{C} \to$ category of k-vector spaces

$$Y \longmapsto \mathrm{Hom}(X,Y)$$

$$(f: Y\to Z) \longmapsto f_*: \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$
$$g \longmapsto f_*(g) = f\circ g$$

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f\circ g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

The map f_* is called the <u>pushforward</u> of f.

$$\mathrm{Hom}(-,X): \mathcal{C} \to \text{category of } k\text{-vector spaces}$$

$$Y \mapsto \mathrm{Hom}(Y,X)$$

$$(f: Y \to Z) \mapsto f^*: \mathrm{Hom}(Z,X) \to \mathrm{Hom}(Y,X)$$
$$g \mapsto f^*(g) = g \circ f$$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Z \arrow[d, "g"] \\ & X \end{tikzcd}$$

The map f^* is called the <u>pullback</u> of f.

Applying the Hom functors to short exact sequences of representations yields new exact sequences of vector spaces.

<u>Theorem</u> (1.10/1.13): Let Q be a quiver and $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ (resp. $L \xrightarrow{f} M \xrightarrow{g} N \to 0$) be a sequence in rep Q. Then this sequence is exact iff for every representation $X \in \mathrm{rep}\, Q$, the sequence
$$0 \to \mathrm{Hom}(X,L) \xrightarrow{f_*} \mathrm{Hom}(X,M) \xrightarrow{g_*} \mathrm{Hom}(X,N)$$
(resp. $0 \to \mathrm{Hom}(N,X) \xrightarrow{g^*} \mathrm{Hom}(M,X) \xrightarrow{f^*} \mathrm{Hom}(L,X)$) is exact.

<u>Corollary</u> (1.11/1.14): A sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in rep Q is split exact iff for every $X \in \mathrm{rep}\, Q$, the sequence
$$0 \to \mathrm{Hom}(X,L) \xrightarrow{f_*} \mathrm{Hom}(X,M) \xrightarrow{g_*} \mathrm{Hom}(X,N) \to 0$$
(resp. $0 \to \mathrm{Hom}(N,X) \xrightarrow{g^*} \mathrm{Hom}(M,X) \xrightarrow{f^*} \mathrm{Hom}(L,X) \to 0$) is exact.

<u>remark</u>: If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ splits, then $0 \to \mathrm{Hom}(X,L) \xrightarrow{f_*} \mathrm{Hom}(X,M) \xrightarrow{g_*} \mathrm{Hom}(X,N) \to 0$ splits too. Indeed, $g \circ h = 1_N \Rightarrow g_* \circ h_* = 1_{\mathrm{Hom}(X,N)}$.

<u>remark</u>: If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ does not split, then f^* and g_* are not always surjective.

<u>Example</u>: In a previous example we saw that the short exact sequence $0 \to S(2) \xrightarrow{f} M \xrightarrow{g} S(1) \to 0$ is non-split.
Taking $X = S(1)$ and applying $\mathrm{Hom}(S(1), -)$, we get a morphism $g_*: \mathrm{Hom}(S(1), M) \to \mathrm{Hom}(S(1), S(1))$ which is not surjective since $\mathrm{Hom}(S(1), M) = 0$ and $\mathrm{Hom}(S(1), S(1)) \cong k$.

3. First Examples of Auslander-Reiten quivers

> **Goal of Representation theory**
> study representations and morphisms in rep Q
> and (new) exact sequences in rep Q.

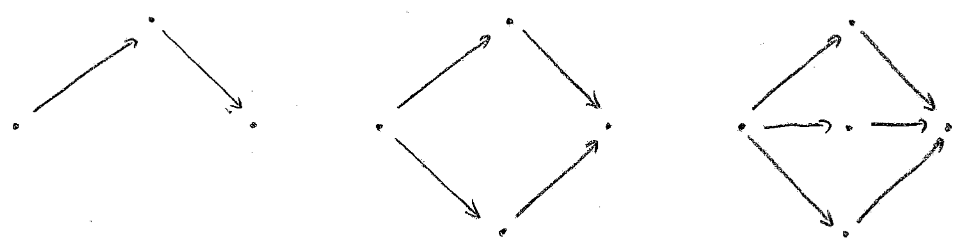
(see previous lecture)

The Auslander-Reiten quiver is a good first approximation of rep Q. In the case where the number of isoclasses of indecomposable representations is finite, the A-R quiver even provides complete information about rep Q.
In this section, we give a sneak preview of Auslander-Reiten quivers (more details later in the semester).

Let Q be a quiver. The A-R quiver of Q is a new quiver Γ_Q whose vertices are the isoclasses of indecomposable representations and whose arrows are given by the so-called irreducible morphisms ($\rightsquigarrow$ an irred. morphism between two indecomp. repres. is a morphism that does not factor nontrivially through another representation)

Recall that we can build any representation out of indecomposable ones; thus the vertices of the A-R quiver represent the building blocks for the representations.
The arrows of the A-R quiver, the irreducible morphisms, can be thought of the building blocks for morphisms in the sense that many (but in general not all!) morphisms are compositions of irreducible morphisms.

We also want to study short exact sequences of representations. As with morphisms, many of them (but in general not all!) are obtained by gluing together the so-called almost split sequences (details later in the semester). The almost split sequences are represented in the A-R quiver as meshes:



__Example:__ let Q be the quiver $1 \to 2$. There are precisely three indecomposable representations:

$$\begin{array}{ccc} S(2) & M & S(1) \\ 0 \to k & k \xrightarrow{1} k & k \to 0 \end{array}$$

We have seen in the previous lecture (Schiffler: example 1.3) that $\mathrm{Hom}(S(1), M) = 0$, $\mathrm{Hom}(S(1), S(2)) = 0$, $\mathrm{Hom}(M, S(2)) = 0$, $\mathrm{Hom}(M, S(1)) \cong k$, $\mathrm{Hom}(S(2), M) \cong k$, $\mathrm{Hom}(S(2), S(1)) = 0$
and we conclude that there is only one non-split short exact sequence with indecomposable repres. at the end points:

$$0 \to S(2) \to M \to S(1) \to 0$$

which is actually an almost split sequence. Thus the A-R quiver consists of three vertices, two arrows and one mesh:

$$\begin{tikzcd}[column sep=small] & M \arrow[dr] & \\ S(2) \arrow[ur] & & S(1) \end{tikzcd}$$

__Shorthand notation:__

$Q_0 = \{1, 2, \ldots, n\}$ set of vertices
$M = (M_i, \varphi_\alpha)$ indecomposable representation
$\dim M = (d_1, d_2, \ldots, d_n)$ dimension vector

- digit i appears exactly d_i times
- if for $i \xrightarrow{\alpha} j$ the corresponding map $\varphi_\alpha : M_i \to M_j$ is non zero, then the digit i is placed above the digit j

In the example above we can picture the representation M by $\begin{smallmatrix}1\\2\end{smallmatrix}$, meaning that $M_1 = k$, $M_2 = k$ and the arrow is going downward from 1 to 2 and carries the identity map, i.e.

$$\begin{tikzcd}[column sep=small] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\ 2 \arrow[ur] & & 1 \end{tikzcd}$$

__Example:__ let Q be the quiver $1 \to 2 \leftarrow 3$. There are precisely six isoclasses of indecomposable representations:

$$\begin{array}{ccc} S(2) & P(1) & P(3) \\ 0 \to k \leftarrow 0 & k \xrightarrow{1} k \leftarrow 0 & 0 \to k \xleftarrow{1} k \\ I(2) & S(1) & S(3) \\ k \xrightarrow{1} k \xleftarrow{1} k & k \to 0 \leftarrow 0 & 0 \to 0 \leftarrow k \end{array}$$

symbolic notation:
$S(2) = 2$, $P(1) = \begin{smallmatrix}1\\2\end{smallmatrix}$, $P(3) = \begin{smallmatrix}3\\2\end{smallmatrix}$, $I(2) = \begin{smallmatrix}1 & 3\\ & 2\end{smallmatrix}$, $S(1) = 1$, $S(3) = 3$

There are three almos split sequences:

$$0 \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 0$$

$$0 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 3 \to 0$$

$$0 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 1 \to 0$$

and the Auslander-Reiten quiver is of the form:

$$\begin{tikzcd}
 & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & & 1 \\
2 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] & & 3
\end{tikzcd}$$

let us point out that there are two further non-split short exact sequences with indecomposable end terms:

$$0 \to 2 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 3 \to 0$$

$$0 \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0$$

each of which can be obtained by "gluing the meshes" of two almost split sequences in the A-R quiver.

<u>Problem 1.3</u> (problem section, Schiffler, chapter 1)

Write M as a direct sum of the indecomposable representations listed in the previous example, where M is the representation

$$M \qquad k^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}} k^3 \xleftarrow{\begin{bmatrix}1\\0\\0\end{bmatrix}} k$$

Then $M = I(2) \oplus P(1) \oplus S(2)$, as

$$\underbrace{\underbrace{(k \xrightarrow{1} k \xleftarrow{1} k) \oplus (k \xrightarrow{1} k \longleftarrow 0)}_{(k^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} k)} \oplus (0 \longrightarrow k \longleftarrow 0)}_{k^2 \xrightarrow{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}} k^3 \xleftarrow{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}} k}$$

which is isomorphic to the given repr. M

$$k^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}} k^3 \xleftarrow{\begin{bmatrix}1\\0\\0\end{bmatrix}} k$$