LECTURE: CATEGORY THEORY

Disclaimer

Nobody is perfect, and I might have written or said something silly. If there is any doubt, then please check the references or contact me. All questions welcome!

What?
Consider the following question:

Why category theory? It is abstract nonsense and completely useless, right?

Well, there are many views and reasons to study category theory, and here is a short and biased list:

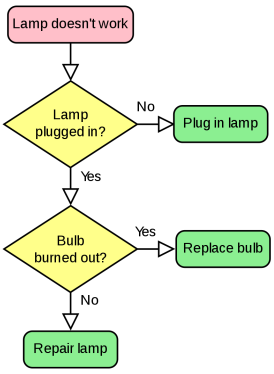
- **Organization of concepts.**
  The book [ML98] start with the sentence "Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows.". Similarly, quoting [AL91]: "In addition to its direct relevance to theoretical knowledge and current applications, category theory is often used as an (implicit) mathematical jargon rather than for its explicit notions and results. [...] In other words, many different formalisms and structures may be proposed for what is essentially the same concept; the categorical language and approach may simplify through abstraction, display the generality of concepts, and help to formulate uniform definitions."
  In other words, organizes many concepts in one language. It is like the bird's-eye view: by ignoring details, the bird's-eye view reveals hidden symmetries:

  This description is based on a very true quote in [Le14]: "Category theory takes a bird's eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level.".

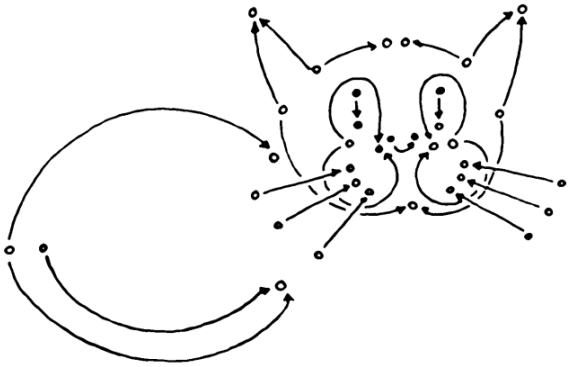
- **Organization of ideas.**
  Category theory is not just abstract nonsense, but can be applied in various parts of mathematics, the sciences and more generally. This aspect of category theory is not highlighted as often as it deserves to be. But times change and [FS19] starts with: "Category theory is becoming a central hub for all of pure mathematics. It is unmatched in its ability to organize and layer abstractions, to find commonalities between structures of all sorts, and to facilitate communication between different mathematical communities. But it has also been branching out into science, informatics, and industry. We believe that it has the potential to be a major cohesive force in the world, building rigorous bridges between disparate worlds, both theoretical and practical.".
  Nowadays category theory has entered for example computer science (think about Haskell), philosophy and logic (at least it is proposed, see e.g. [La17]), physics (see e.g. [BL11]) and

many more. Not so much as a really practical tool, but more like a flowchart:

Flowcharts



Categories



a) Flowcharts A symbolism that allows one to organize complicated facts

b) Category theory A symbolism that allows one to organize complicated facts

The flowchart description above is copied from Wikipedia, but it fits also perfectly well for category theory. In other words, category theory is not just a way to organize concepts, but also to organize ideas in general.

- **Beauty.**

  The book [AHS90] starts with a poem, describing category theory and many people's views on it spot on:

  There's a tiresome young man in Bay Shore.
  When his fiancée cried, 'I adore
  The beautiful sea',
  He replied, 'I agree,
  It's pretty, but what is it for?'

  Morris Bishop

  Nothing to add!

Who?

Fourth semester students in Mathematics interested in a mixture of (linear) algebra and discrete mathematics, but everyone is welcome.

Where and when?

- Time and date for the lecture.
  - ▷ Every Monday from 12:00–14:00.
  - ▷ Online.
  - ▷ First lecture: Monday 21.Feb.2022. Last lecture: Monday 09.May.2022.
- Time and date for the tutorials.
  - ▷ Every Friday from 12:00–14:00.
  - ▷ Online.
  - ▷ First tutorial: Friday 25.Feb.2022. Last tutorial: Friday 13.May.2022.

Material for the lecture

- The lecture is a mix of various sources. The main source is [ML98], and the lecture follows the list of topics presented therein. However, the lecture takes a different perspective compared to [ML98] and potentially reading either of [AHS90], [AL91], [FS19], [Le14], [Mi14], [Ri16] or [Si11] should be beneficial. These are also used for the lecture, but [ML98] is the only reference given explicitly in the list below.

- Website www.dtubbenhauer.com/lecture-ct-2022.html
- Prerecorded lectures on the "What is...category theory?" playlist here: www.youtube.com/c/VisualMath/playlists
- Exercise sheets are available on the course website.

Schedule and some details.

1. The beginnings – What is...category theory?
   - **Speaker and date.** Daniel Tubbenhauer, 21.Feb.2022, 12:00–14:00.
   - **Plan.** After some organizational preliminaries are addressed, this talk will be a collection of motivating example, none of which will be explained in a formal matter, but rather in an intuitive way. The talk also defines a category in the standard and in an object-free way.
   - **YouTube.** Videos 1 and 2 on the "What is...category theory?" playlist.
   - **Literature.** This talk is not following any explicit literature, but is rather an overview of what will follow.

2. Diagrams in categories – Commuting and alike.
   - **Speaker and date.** Daniel Tubbenhauer, 28.Feb.2022, 12:00–14:00.
   - **Plan.** This talk starts by introducing or recalling commutative diagrams, which are one of the main players in category theory. This is done without functors, but in a rather graph-theoretical fashion. A focus will also be put on properties invariant under certain operations on categories such as duality.
   - **YouTube.** Videos 3, 4 and 5 on the "What is...category theory?" playlist.
   - **Literature.** A (restricted) collection of the concepts introduced in [ML98, Chapter I and II], but in different order.

3. Functors I – The basics about functors.
   - **Speaker and date.** Daniel Tubbenhauer, 07.Mar.2022, 12:00–14:00.
   - **Plan.** The concept of a functor is fundamental for category theory. This talk is an example-focused introduction to functors and their basic properties.
   - **YouTube.** Videos 6 and 7 on the "What is...category theory?" playlist.
   - **Literature.** Parts of [ML98, Chapter I], but explained differently.

4. Functors II – Natural transformations and equivalence.
   - **Speaker and date.** Daniel Tubbenhauer, 14.Mar.2022, 12:00–14:00.
   - **Plan.** The talk is about functor categories, which includes natural transformations, the notion of equivalence and also various constructions involving functors such as inclusions asa subcategory.
   - **YouTube.** Videos 8, 9, 10 and 11 on the "What is...category theory?" playlist.
   - **Literature.** Again, this is a (restricted) collection of the concepts introduced in [ML98, Chapter I and II], but in different order.

5. Yoneda – Yoneda lemma and Yoneda embedding.
   - **Speaker and date.** Daniel Tubbenhauer, 21.Mar.2022, 12:00–14:00.

- **Plan.** The Yoneda lemma and its consequences is the topic of this talk. This needs some preparation and also includes some of the upshots such as the Yoneda embedding. We also briefly discuss concrete categories.
- **YouTube.** Videos 12 and 13 on the "What is...category theory?" playlist.
- **Literature.** This lecture is loosely following [ML98, Sections III.1 and III.2].

6. Limits I – Examples of limits.
   - **Speaker and date.** Daniel Tubbenhauer, 28.Mar.2022, 12:00–14:00.
   - **Plan.** We discuss the standard and most enlightling examples of limits such as initial and terminal objects, products and coproducts etc.
   - **YouTube.** Videos 14, 15, 16 and 17 on the "What is...category theory?" playlist.
   - **Literature.** Parts of [ML98, Chapters III and V].

7. Limits II – Universal properties and limits abstractly.
   - **Speaker and date.** Daniel Tubbenhauer, 04.Apr.2022, 12:00–14:00.
   - **Plan.** The main topic of this talk are limits defined abstractly via indexing categories. We will see how the examples from the previous lecture fit into this definition. Related concepts such as complete categories and continuous functors are also discussed.
   - **YouTube.** Videos 18, 19 and 20 on the "What is...category theory?" playlist.
   - **Literature.** As before, parts of [ML98, Chapters III and V].

8. Adjoint functors I – The algebraic approach.
   - **Speaker and date.** Daniel Tubbenhauer, 11.Apr.2022, 12:00–14:00.
   - **Plan.** Adjoint functors are arguably one of the most important concepts in category theory. This talk is devoted to explain them carefully along many examples.
   - **YouTube.** Videos 22, 23 and 24 on the "What is...category theory?" playlist.
   - **Literature.** Parts of [ML98, Chapter IV].

9. Adjoint functors II – The diagrammatic approach.
   - **Speaker and date.** Daniel Tubbenhauer, 18.Apr.2022, 12:00–14:00.
   - **Plan.** Adjoint functors can also be motivated diagrammatically, and this is what is explained in this talk. To this end, the classical string diagrammatic is explained.
   - **YouTube.** Videos 25 and 26 on the "What is...category theory?" playlist.
   - **Literature.** Diagrammatic notation is not used in [ML98], so no explicit reference for this talk.

10. Monoids I – Monads and their modules.
    - **Speaker and date.** Daniel Tubbenhauer, 25.Apr.2022, 12:00–14:00.
    - **Plan.** Monads are special cases, but very important special cases, of how monoids show up in category theory. This concept as well as representations (or modules) of monads, called algebras, are explained in this talk.
    - **YouTube.** Videos 27, 28, 29 and 30 on the "What is...category theory?" playlist.
    - **Literature.** Part (a very limited collection actually) of [ML98, Chapter VI].

11. Monoids II – Monoidal categories.
    - **Speaker and date.** Daniel Tubbenhauer, 02.May.2022, 12:00–14:00.
    - **Plan.** Monoidal categories categorify monoids, and the purpose of this talk is to introduce them algebraically and diagrammatically. To this end, the string diagrammatics is beefed-up.
    - **YouTube.** Videos 31, 32 and 33 on the "What is...category theory?" playlist.
    - **Literature.** [ML98, Chapter VII], but with the caveat that diagrammatic notation is not used in [ML98].

12. Whats next? – Some outlook including diagrammatics.
    - **Speaker and date.** Daniel Tubbenhauer, 09.May.2022, 12:00–14:00.
    - **Plan.** As a conclusion of the lecture, this talk will explain what potential further directions category theory has to offer. An example are 2-categories and more diagrammatic calculi related to low-dimensional topology.
    - **YouTube.** Videos 34+ on the "What is...category theory?" playlist.
    - **Literature.** Partially material beyond [ML98].

References

[AL91] A. Asperti, G. Longo. Categories, types, and structures. An introduction to category theory for the working computer scientist. Foundations of Computing Series. MIT Press, Cambridge, MA, 1991.

[AHS90] J. Adámek, H. Herrlich, G.E. Strecker. Abstract and concrete categories: the joy of cats. Reprint of the 1990 original [Wiley, New York; MR1051419]. Repr. Theory Appl. Categ. No. 17 (2006), 1–507. URL: http://katmat.math.uni-bremen.de/acc/acc.pdf

[BL11] J.C. Baez, A.D. Lauda. A prehistory of n-categorical physics. Deep beauty, 13–128. Cambridge Univ. Press, Cambridge, 2011. URL: https://arxiv.org/abs/0908.2469

[FS19] B. Fong, D.I. Spivak. An invitation to applied category theory. Seven sketches in compositionality. Cambridge University Press, Cambridge, 2019. xii+338 pp. URL: https://arxiv.org/abs/1803.05316

[La17] E. Landry et al. Categories for the working philosopher. Edited by Elaine Landry. Oxford University Press, Oxford, 2017. xiv+471 pp.

[Le14] T. Leinster. Basic category theory. Cambridge Studies in Advanced Mathematics, 143. Cambridge University Press, Cambridge, 2014. viii+183 pp. URL: https://arxiv.org/abs/1612.09375

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[Mi14] B. Milewski. Category Theory for Programmers. Collected from the series of blog posts. URL: https://bartoszmilewski.com/2014/10/28/category-theory-for-programmers-the-preface/ URL: https://github.com/hmemcpy/milewski-ctfp-pdf

[Ri16] E. Riehl. Category Theory in Context. Dover Publications, 2016. URL: https://math.jhu.edu/~eriehl/context/

[Si11] H. Simmons. An introduction to category theory. Cambridge University Press, Cambridge, 2011. x+226 pp.

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