Topology – week 10

Math3061

Daniel Tubbenhauer, University of Sydney

Words for surfaces

A polygonal decomposition for a surface that has one face can be encoded in a word

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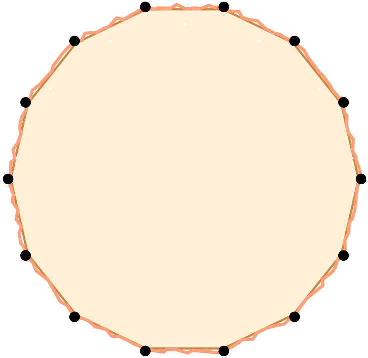

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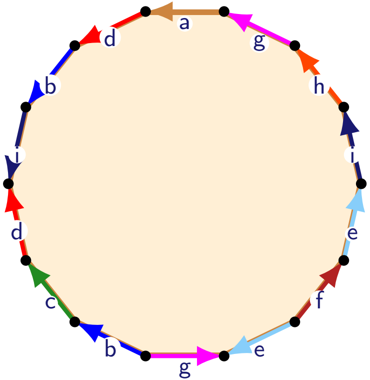

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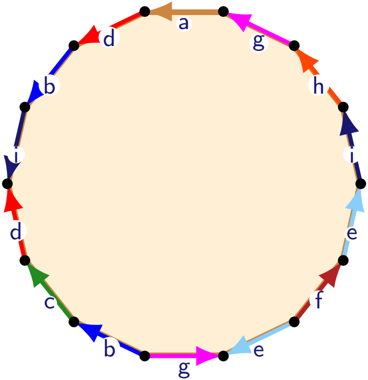

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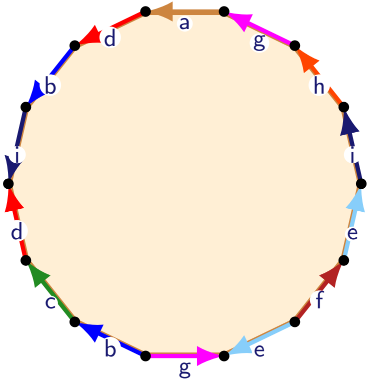

- write x for an edge pointing anticlockwise
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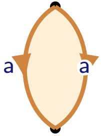
- We always read the word in anticlockwise order

- $S^2 =$ 

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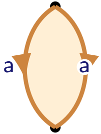
- $S^2 =$ 

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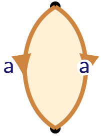
- $S^2 =$ 

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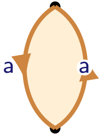
- $\mathbb{P}^2 =$ 

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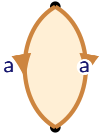
- $S^2 =$ 

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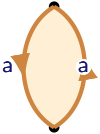
- $\mathbb{P}^2 =$ 

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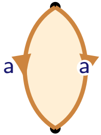
- $S^2 =$ 

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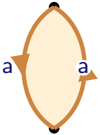
- $\mathbb{P}^2 =$ 

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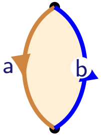
- $\mathbb{D}^2 =$ 

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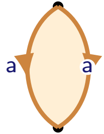
- $S^2 =$ 

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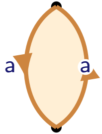
- $\mathbb{P}^2 =$ 

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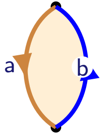
- $\mathbb{D}^2 =$ 

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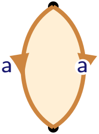
- $S^2 =$ 

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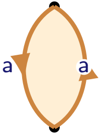
- $\mathbb{P}^2 =$ 

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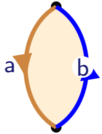
- $\mathbb{D}^2 =$ 

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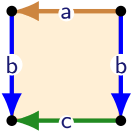
- $\mathbb{A} =$ 

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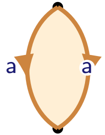
- $S^2 =$ 

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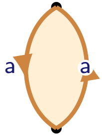
- $\mathbb{P}^2 =$ 

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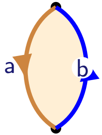
- $\mathbb{D}^2 =$ 

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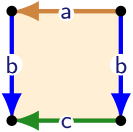
- $\mathbb{A} =$ 

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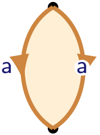
- $S^2 =$ 

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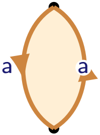
- $\mathbb{P}^2 =$ 

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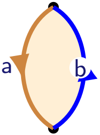
- $\mathbb{D}^2 =$ 

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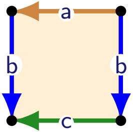
- $\mathbb{A} =$ 

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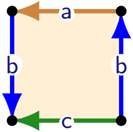
- $\mathbb{M} =$ 

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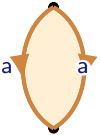
- $S^2 =$ 

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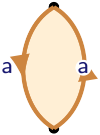
- $\mathbb{P}^2 =$ 

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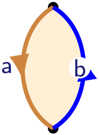
- $\mathbb{D}^2 =$ 

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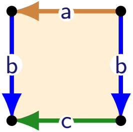
- $\mathbb{A} =$ 

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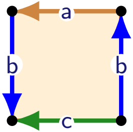
- $\mathbb{M} =$ 

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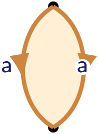
- $S^2 =$ 

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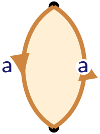
- $\mathbb{P}^2 =$ 

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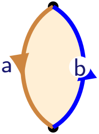
- $\mathbb{D}^2 =$ 

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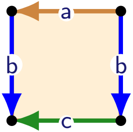
- $\mathbb{A} =$ 

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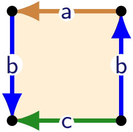
- $\mathbb{M} =$ 

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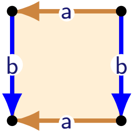
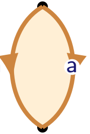
- $\mathbb{T} =$ 

- $S^2 =$ 

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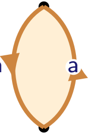
- $\mathbb{P}^2 =$ 

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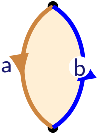
- $\mathbb{D}^2 =$ 

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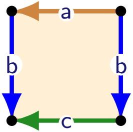
- $\mathbb{A} =$ 

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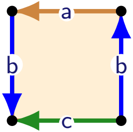
- $\mathbb{M} =$ 

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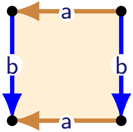
- $\mathbb{T} =$ 

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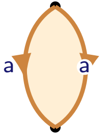
- $S^2 =$ 

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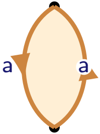
- $\mathbb{P}^2 =$ 

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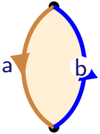
- $\mathbb{D}^2 =$ 

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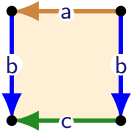
- $\mathbb{A} =$ 

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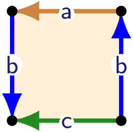
- $\mathbb{M} =$ 

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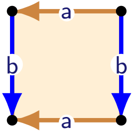
- $\mathbb{T} =$ 

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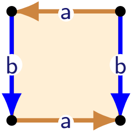
- $\mathbb{K} =$ 

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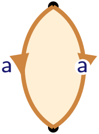
- $S^2 =$ 

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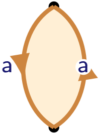
- $\mathbb{P}^2 =$ 

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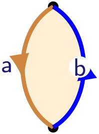
- $\mathbb{D}^2 =$ 

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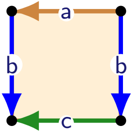
- $\mathbb{A} =$ 

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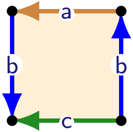
- $\mathbb{M} =$ 

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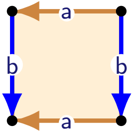
- $\mathbb{T} =$ 

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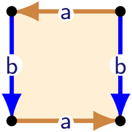
- $\mathbb{K} =$ 

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Properties of words

- Words encode orientability
  - Orientable: $\ldots a \ldots \overline{a} \ldots$ or $\ldots \overline{a} \ldots a \ldots$
  - Non-orientable: $\ldots a \ldots a \ldots$ or $\ldots \overline{a} \ldots \overline{a} \ldots$

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- The word of a surface is well-defined only up to cyclic permutation and reversing the direction of any edge

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  - Non-orientable: $\ldots a \ldots a \ldots$ or $\ldots \overline{a} \ldots \overline{a} \ldots$
- Words give a compact and easily readable way of describing surfaces
- Words can be read in clockwise or anticlockwise order
  (we always read in anticlockwise order)
- The word of a surface is well-defined only up to cyclic permutation and reversing the direction of any edge

Example The following words are all words for the torus $\mathbb{T}$:

$$\begin{array}{cccc} a\,b\,\overline{a}\,\overline{b} & b\,\overline{a}\,\overline{b}\,a & \overline{a}\,\overline{b}\,a\,b & \overline{b}\,a\,b\,\overline{a} \\ a\,\overline{b}\,\overline{a}\,b & \overline{b}\,\overline{a}\,b\,a & \overline{a}\,b\,a\,\overline{b} & b\,a\,\overline{b}\,\overline{a} \end{array}$$

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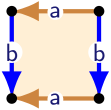
$$a\,b\,\overline{a}\,\overline{b} \qquad b\,\overline{a}\,\overline{b}\,a \qquad \overline{a}\,\overline{b}\,a\,b \qquad \overline{b}\,a\,b\,\overline{a}$$
$$a\,\overline{b}\,\overline{a}\,b \qquad \overline{b}\,\overline{a}\,b\,a \qquad \overline{a}\,b\,a\,\overline{b} \qquad b\,a\,\overline{b}\,\overline{a}$$

- The word of a surface can be used to give generators and relations for the first homotopy group of the surface — this generalises independent cycles and are beyond the scope of this unit

Standard words for closed orientable surfaces

- Connected sums of tori: $\#^t\mathbb{T}$
  - $\mathbb{T} =$ 

$= a\ b\ \overline{a}\ \overline{b}$

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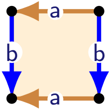
  - $\mathbb{T} =$ 

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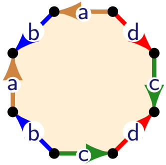
  - $\#^2\mathbb{T} =$ 

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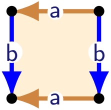
  - $\mathbb{T} =$ 

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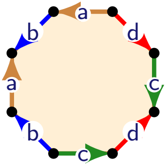
  - $\#^2\mathbb{T} =$ 

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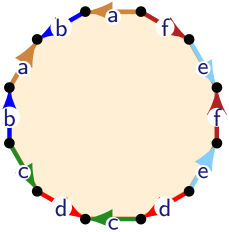
  - $\#^3\mathbb{T} =$ 

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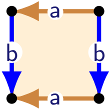
  - $\mathbb{T} =$ 

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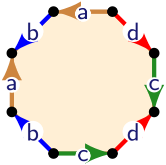
  - $\#^2\mathbb{T} =$ 

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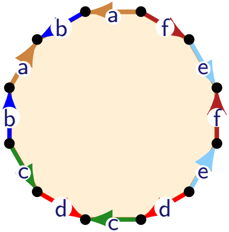
  - $\#^3\mathbb{T} =$ 

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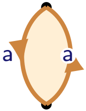
  - $\ldots \quad \#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \ldots\ a_t\ b_t \overline{a_t}\ \overline{b_t}$

Words for closed non-orientable surfaces

- Connected sums of projective plans $\#^p\mathbb{P}^2$
  - $\mathbb{P}^2 =$ 

$= a\ a$

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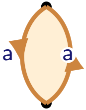
  - $\mathbb{P}^2 =$ 

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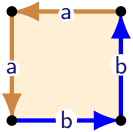
  - $\#^2\mathbb{P}^2 =$ 

$= a\ a\ b\ b$

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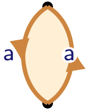
  - $\mathbb{P}^2 =$ 

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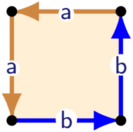
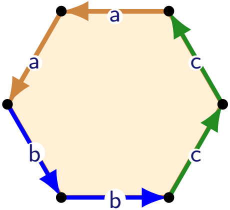
  - $\#^2\mathbb{P}^2 =$ 

$= a\ a\ b\ b$
  - $\#^3\mathbb{P}^2 =$ 

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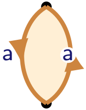
  - $\mathbb{P}^2 =$ 

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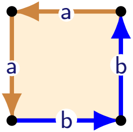
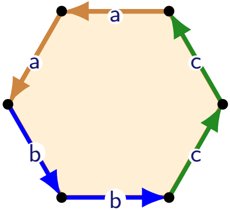
  - $\#^2\mathbb{P}^2 =$ 

$= a\ a\ b\ b$
  - $\#^3\mathbb{P}^2 =$ 

$= a\ a\ b\ b\ c\ c$
  - $\ldots \quad \#^p\mathbb{P}^2 = a_1\ a_1\ a_2\ a_2\ \ldots\ a_p\ a_p$

Standard words for surfaces with boundary

- $\#^d \mathbb{D}^2$

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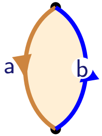
  - $\mathbb{D}^2 =$  $= a\ b$

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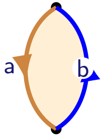
  - $\mathbb{D}^2 =$ 

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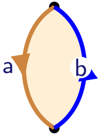
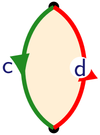
  - $\#^2 \mathbb{D}^2 =$ 

$\#$ 

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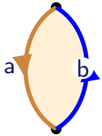
  - $\mathbb{D}^2 =$ 

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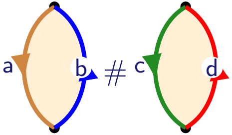
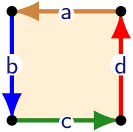
  - $\#^2\mathbb{D}^2 =$ 

$\overset{?}{\cong}$ 

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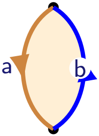
  - $\mathbb{D}^2 =$ 

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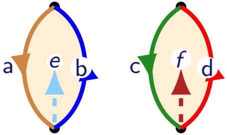
  - $\#^2\mathbb{D}^2 =$ 

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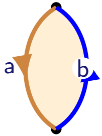
  - $\mathbb{D}^2 =$ 

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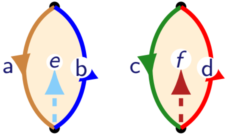
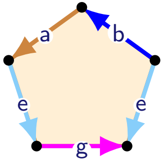
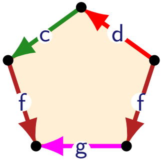
  - $\#^2\mathbb{D}^2 =$ 

$\cong$ 

$\#$ 

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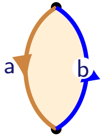
  - $\mathbb{D}^2 =$ 

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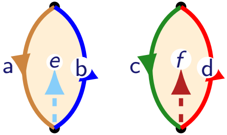
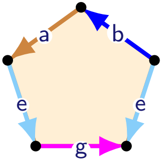
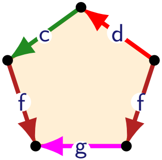
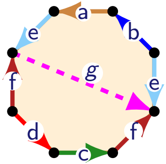
  - $\#^2\mathbb{D}^2 =$ 

 $\cong$ 

 $\#$ 

$\cong$ 

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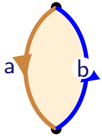
  - $\mathbb{D}^2 =$ 

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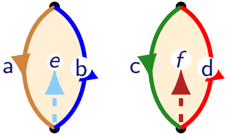
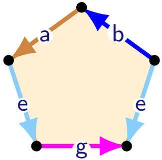
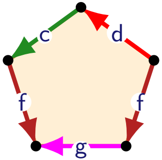
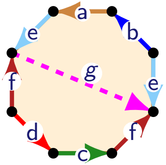
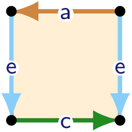
  - $\#^2\mathbb{D}^2 =$ 

$\cong$ 

$\#$ 

$\cong$ 

$\cong$ 

$= a\ e\ c\ \overline{e}$

Standard words for surfaces with boundary

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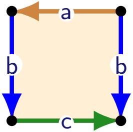
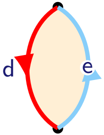
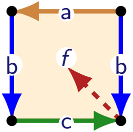
Standard words for surfaces with boundary

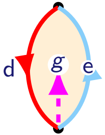
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2$

Standard words for surfaces with boundary

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

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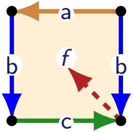
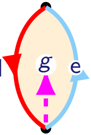
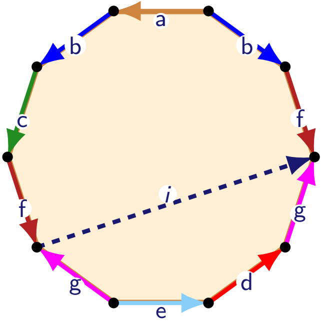
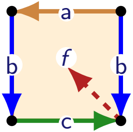
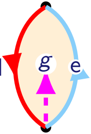
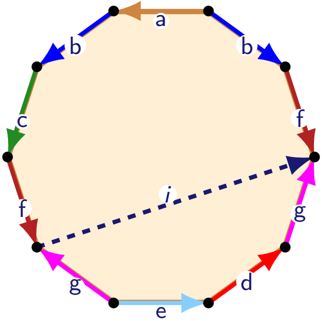
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \,\#\, \mathbb{D}^2 \cong$ 

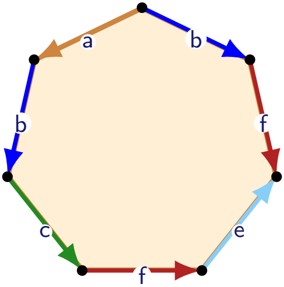
$\#$ 

$\cong$ 

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

$\cong$ 

$\cong$ 

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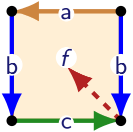
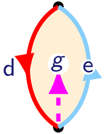
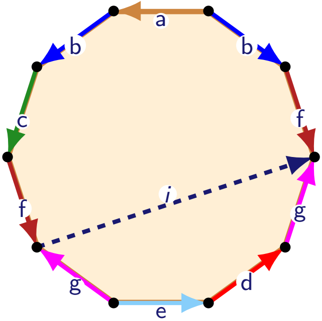
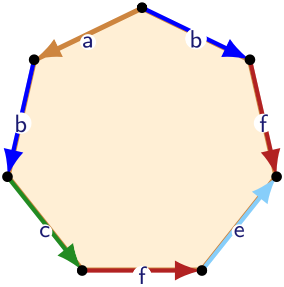
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

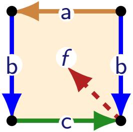
$\#$ 

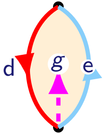
$\cong$ 

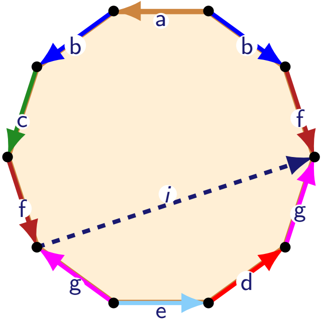
$\cong$ 

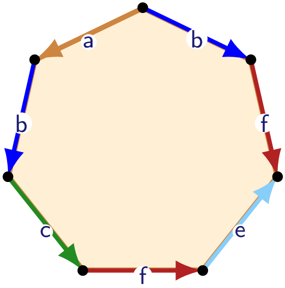
$= a\ b\ c\ f\ e\ \overline{f}\ \overline{b}$

Standard words for surfaces with boundary

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

$\cong$ 

$\cong$ 

$= a\ b\ c\ f\ e\ \overline{f}\ \overline{b}$

- $\#^d\mathbb{D}^2 = a_1\ b_1\ a_2\ b_2\ \ldots\ b_{d-1} a_d\ \overline{b}_{d-1}\ \ldots \overline{b}_2\ \overline{b}_1$

Words to surfaces

What standard surface is given by the word $a\ d\ b\ \overline{c}\ e\ \overline{c}\ \overline{f}\ d\ h\ g\ e\ \overline{a}\ h$?

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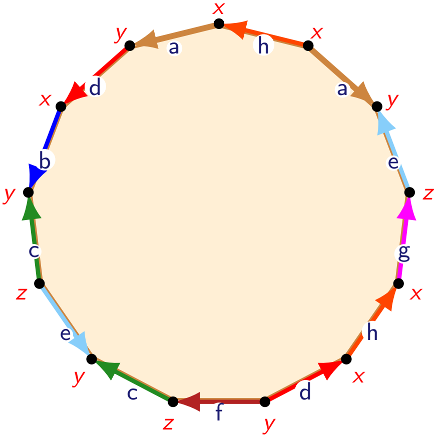

Free edges
$x \xrightarrow{b} y$
$x \xrightarrow{g} z$
$y \xrightarrow{f} z$

$\Longrightarrow$ $d = 1$ and $\chi(S) = 3 - 8 + 1 = -4$

$\Longrightarrow$ $S \cong \mathbb{D}^2 \# \#^5 \mathbb{P}^2$

$\Longrightarrow$ $S = a\ b\ b\ c\ c\ d\ d\ e\ e\ f\ f$

The vertex-degree equation revisited

When we looked at graphs we proved the vertex-degree equation:

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Answer Yes and no!

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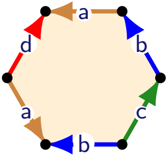

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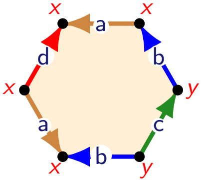

The degree of a vertex

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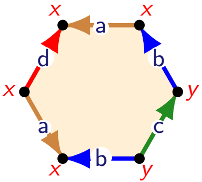

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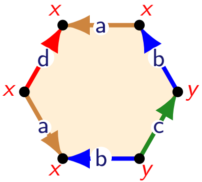

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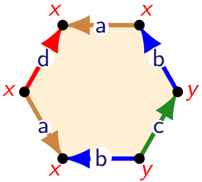

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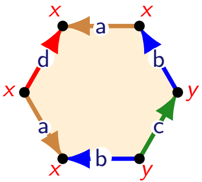

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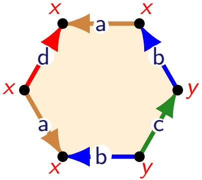

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The vertex-degree equation holds using either identified or non-identified edges and vertices because in both cases the degree of a vertex is defined to be the number of incident edges to the vertex

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Therefore, we have two degree-vertex equations:

- The graph degree-vertex equation where we do not identify edges and vertices in S
- The surface degree-vertex equation where we do identify edges and vertices in S

The degree of a face

Let $S = (V, E, F)$ be a surface with polygonal decomposition

Let $f \in F$ be a face of S. The degree of f is

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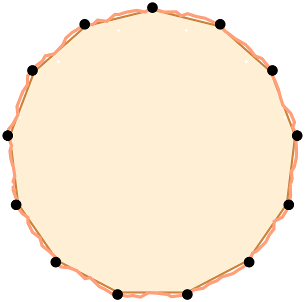

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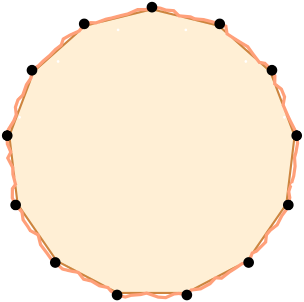

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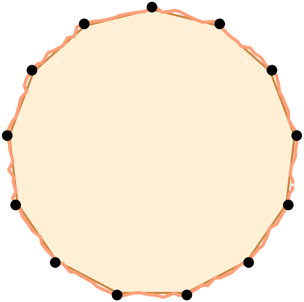

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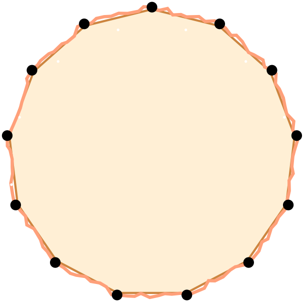

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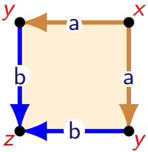
Question How are $\sum \deg(f)$ and $2|E|$ related?

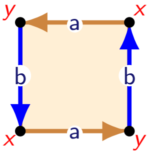
Face degrees of basic surfaces

In all cases $\deg(\text{face}) = 4$ as there are 4 non-identified edges

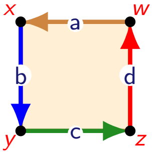
- Sphere

$S^2 \cong$ 

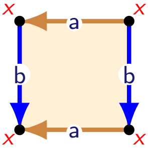
- Projective plane

$\mathbb{P}^2 \cong$ 

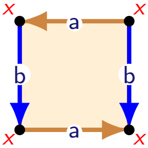
- Disk

$\mathbb{D}^2 \cong$ 

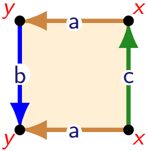
- Torus

$\mathbb{T} \cong$ 

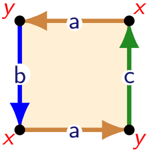
- Klein bottle

$\mathbb{K} \cong$ 

- Annulus

$\mathbb{A} \cong$ 

- Möbius band

$\mathbb{M} \cong$ 

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Remark To use this formula we need to know the number of identified edges in the polygonal decomposition

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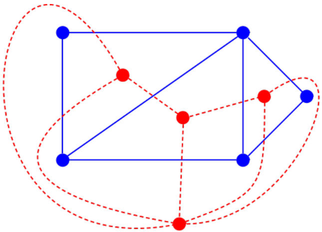
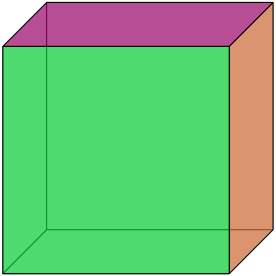


The dual of the cube



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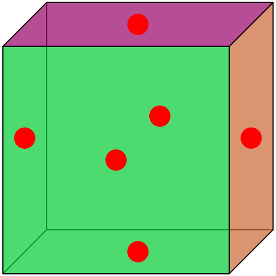

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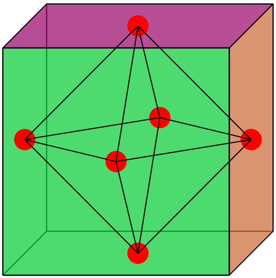

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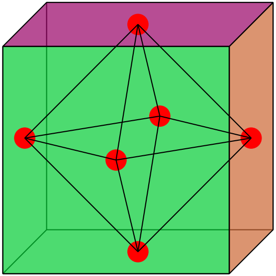

$\Longrightarrow$ the dual surface to the cube is the octahedron

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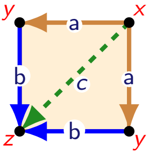
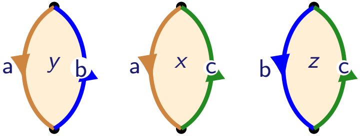
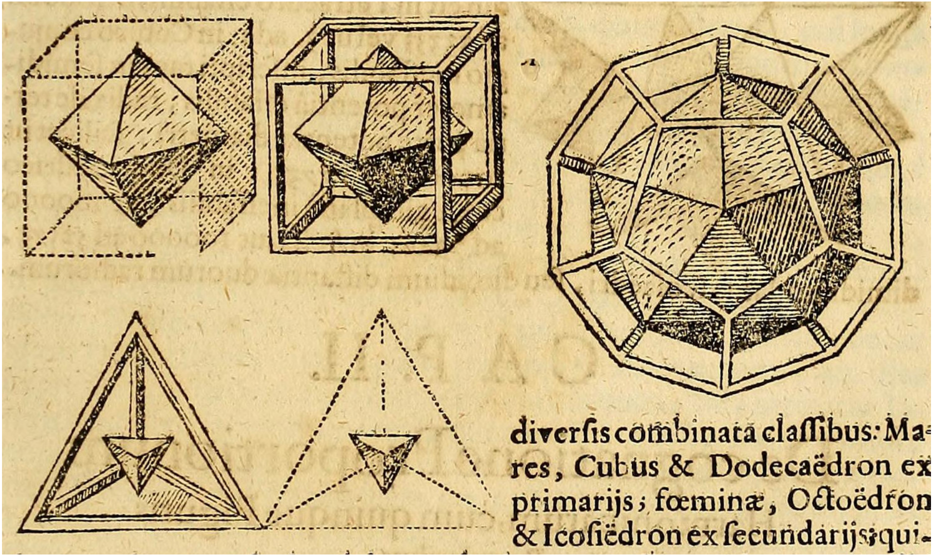
Example

$S^2 =$ 

$(S^2)^* =$ 

We will see better examples when we look at Platonic solids


diverſis combinata claſſibus: Ma-
res, Cubus & Dodecaëdron ex
primarijs; fœminæ, Octoëdron
& Icoſiëdron ex ſecundarijs; qui-

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- If $e, e' \in E$ then the paths $F(e)$ and $F(e')$ can intersect only at the images of their endpoints

Planar graphs

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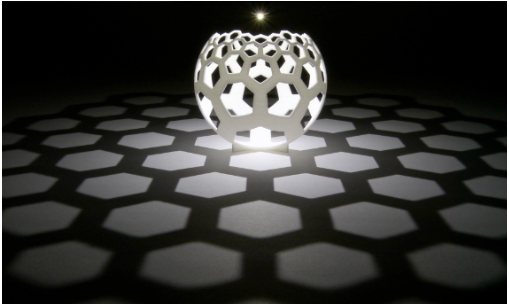
Proof Stereographic projection! (Move G away from ∞.)



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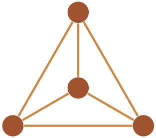

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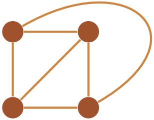

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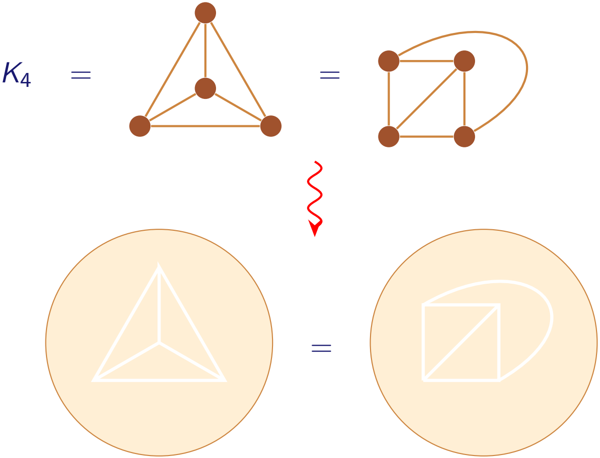

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Remark The argument cheats slightly because we are implicitly assuming that the edges are “nice” curves. This allows us to side-step issues connected with the Jordan curve theorem

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Proof Use the previous theorem or argue by induction on $|E|$

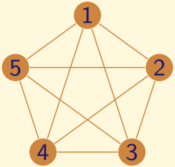
Case 1 G is a tree

Combine $|V| - |E| = 1$ (previous lectures) and that there is only one face

Case 2 G is not a tree

By $\chi(S^2) = 2$ and the previous theorem

Proposition

The complete graph $K_5 =$  *is not planar*

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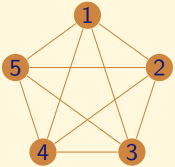
The complete graph $K_5 =$  *is not planar*

Proof Assume that K_5 is planar with $|F|$ faces

We have $|V| = 5$ and $|E| = 10$, so $2 = |V| - |E| + |F| \quad \Longrightarrow \quad |F| = 7$

Let's count the number of faces in this polygonal decomposition differently

- The faces correspond to cycles in K_5
- Every face has at least 3 edges, so by the degree-face equation

$$\Longrightarrow \quad 2|E| = \sum_{f \in F} \deg(f) \geq 3|F|$$

$$\Longrightarrow \quad 2|E| = 20 \geq 21 = 3|F| \qquad \lightning\lightning\lightning$$

Hence, the complete graph K_5 is not planar

Planarity of complete graphs

Corollary

The complete graph K_n *is planar if and only if* $1 \leq n \leq 4$

Planarity of complete graphs

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Proof

K_5 sits in K_n for $n \geq 5$, and the previous theorem applies

Planarity of bipartite graphs

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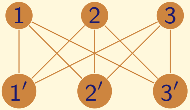
The bipartite graph $K_{3,3} =$ 

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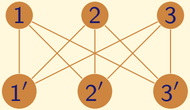
The bipartite graph $K_{3,3} =$  *is not planar*

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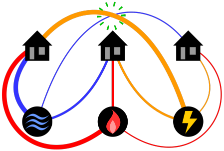

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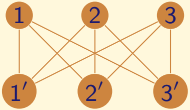
The bipartite graph $K_{3,3} =$  *is not planar*

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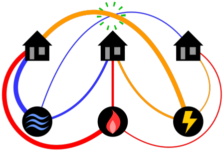

Theorem (Kuratowski)

Let G be a graph. Then G if planar if and only if it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$

The proof is out of the scope of this unit!

Platonic solids

A Platonic solid is a surface that has a polygonal decomposition that is constructed using regular n-gons of the same shape and size such that the same number of polygons meet at every vertex

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	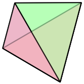	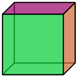	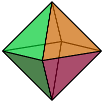	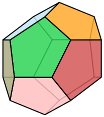	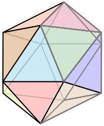
n	3	4	3	5	3
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	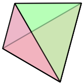	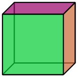	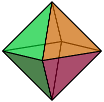	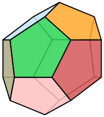	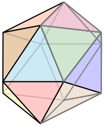
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	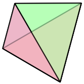	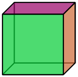	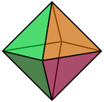	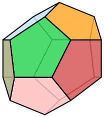	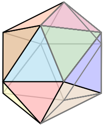
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- Can we understand them as polygonal decompositions of the sphere?

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We require $p \geq 3$, $n \geq 3$ and $|E| \geq 2$

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$\implies$ $2 = \chi(S^2) = |V| - |E| + |F| = \frac{2|E|}{p} - |E| + \frac{2|E|}{n}$

$\implies$ $\frac{1}{2} + \frac{1}{|E|} = \frac{1}{p} + \frac{1}{n}$

$\implies$ $\frac{1}{p} + \frac{1}{n} = \frac{1}{2} + \frac{1}{|E|} > \frac{1}{2}$

We require $p \geq 3$, $n \geq 3$ and $|E| \geq 2$

The equations above give:

$$|E| = \left(\frac{1}{p} + \frac{1}{n} - \frac{1}{2}\right)^{-1}, \ |V| = \frac{2|E|}{p} \text{ and } |F| = \frac{2|E|}{n}$$

Classification of Platonic solids

Theorem

The complete list of Platonic solids is:

p	n	$\frac{1}{p}+\frac{1}{n}$	$e=\left(\frac{1}{p}+\frac{1}{n}-\frac{1}{2}\right)^{-1}$	$v=\frac{2e}{p}$	$f=\frac{2e}{n}$	*Platonic solid*
3	3	$\frac{2}{3}$	6	4	4	*Tetrahedron*
3	4	$\frac{7}{12}$	12	8	6	*Cube*
3	5	$\frac{8}{15}$	30	20	12	*Dodecahedron*
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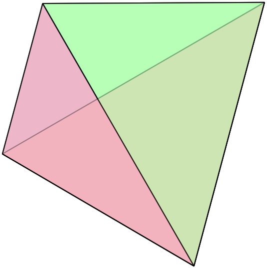
Proof Since $\frac{1}{p}+\frac{1}{n}>\frac{1}{2}$ and $p,n\geq 3$ we get $n<6$ since $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$
Case-by-case we then get the above values for p,n as the only possible values for Platonic solids.

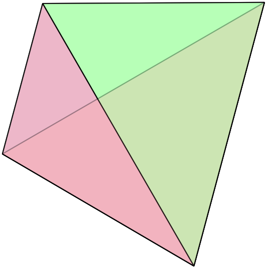
To prove existence we need to actually construct them

Proof Continued Their construction is well-known:

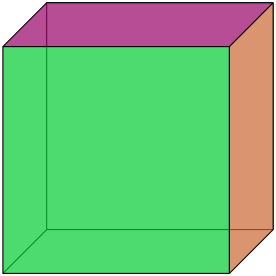
Dual tetrahedron = tetrahedron

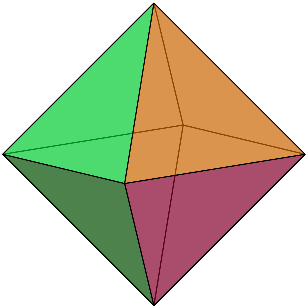
There is a symmetry in the Platonic solids given by $(p, n) \leftrightarrow (n, p)$. This corresponds to taking the dual surface



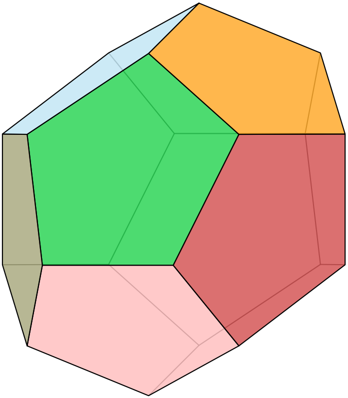


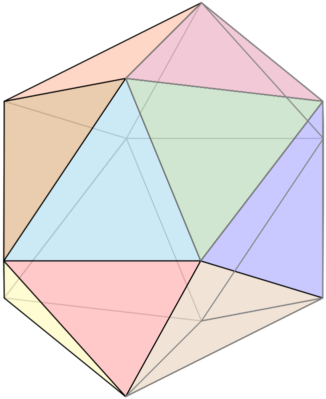
Cube and octahedron





Dodecahedron and icosahedron





Platonic soccer balls

Here are two dodecahedral decompositions of S^2

Soccer ball

Example A ball is made by gluing together triangles and octagons so that each octagon is connected to four non-touching triangles. Determine the number of octagons and triangles used

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Let there be $|V|$ vertices, $|E|$ edges and $|F|$ faces

Write $|F| = o + t$, where $o = \#$octagons and $t = \#$triangles

$$\implies \quad 2 = |V| - |E| + o + t$$

We have:

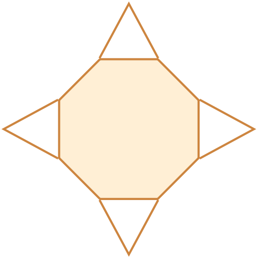
- vertex-degree equation: $3|V| = 2|E|$
- face-degree equation: $2|E| = 3t + 8o$
- Every octagon meets 4 triangles,

$\implies \quad 3t = 4o \quad \implies \quad 2|E| = 12o$

$\implies \quad 2 = o\big(4 - 6 + 1 + \frac{4}{3}\big) = \frac{o}{3}$

$\implies \quad o = 6$ and $t = 8$

$\implies \quad |E| = 36$ and $|V| = 24$



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As with the Platonic solids, we have only shown that if such a surfaces exists then there are 6 octagons, 8 triangles, 24 vertices and 36 edges but we have not shown that such a surface exists!

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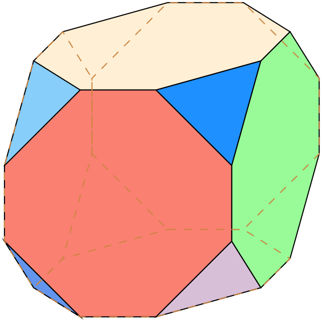
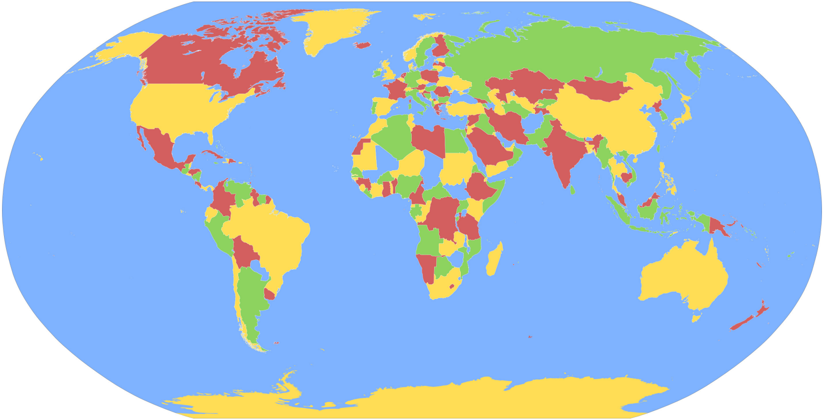
In fact, this surface does exist and it can be constructed by cutting triangular corners off a cube



Coloring maps

Question

How many different colors do you need to color a map so that adjacent countries have different colors?



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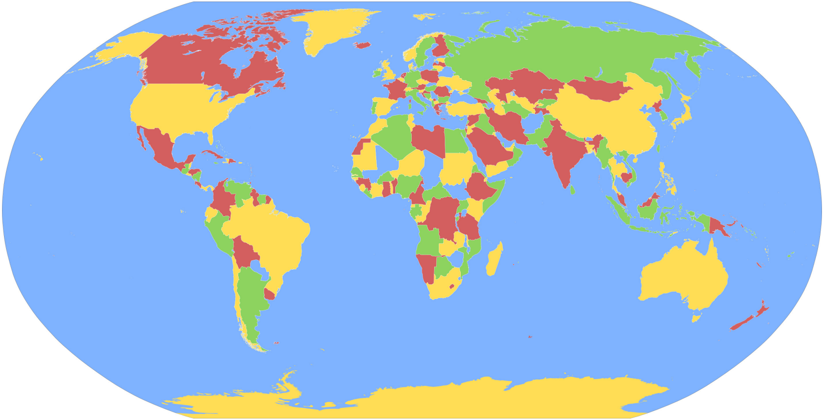

A map is a polygonal decomposition. The answer to this question involves the same ideas we used to understand Platonic solids

Chromatic number of (connected – assumed from now on) surfaces

Let $P = (V, E, F)$ be a polygonal decomposition of a surface S

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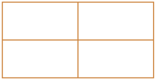

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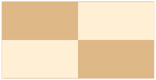

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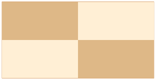

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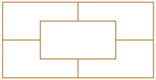

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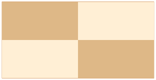

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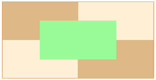

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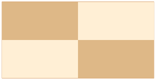

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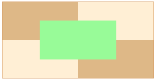

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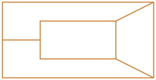

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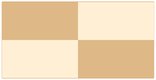

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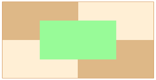

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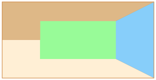

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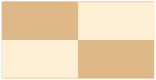

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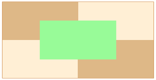

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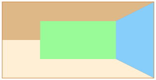

$C_P(\mathbb{D}^2) = 4$

$\Longrightarrow \quad C(\mathbb{D}^2) \geq 4$

For maps of the world we are most interested in $C(\mathbb{D}^2) = C(S^2)$

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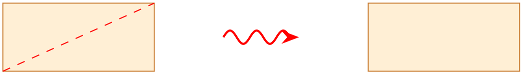

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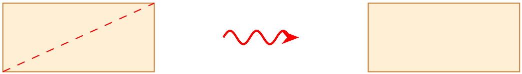

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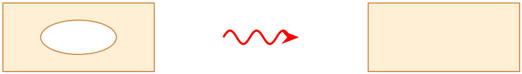

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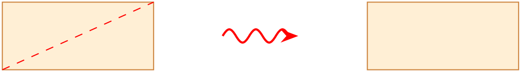

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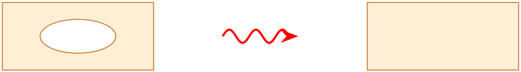

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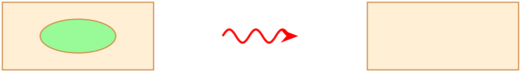

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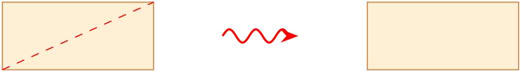

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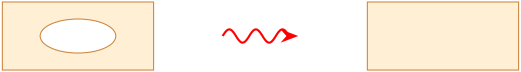

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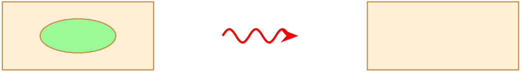

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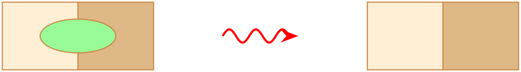

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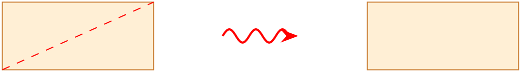

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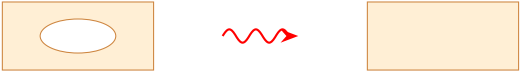

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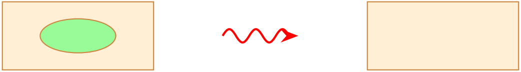

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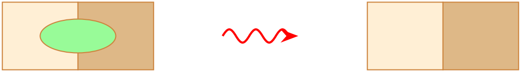

These assumptions are purely for convenience because, in each case, we can colour these maps using the same number of colours

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Moreover,

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Understanding map colourings

The basic idea is to use the Euler characteristic and the degree-vertex and degree-face equations to understand colourings

Let $M = (V, E, F)$ be a map on a surface S. Set

- $\partial_V = \frac{2|E|}{|V|}$, the average vertex-degree
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By definition, $\partial_V|V| = 2|E| = \partial_F|F|$

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Remark For a Platonic solid that is made from n-gons with p polygons meeting at each vertex we have $\partial_V = p$ and $\partial_F = n$

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Suppose that M is a map on a closed surface S. Then

$$\partial_F = \left(1 - \frac{\chi(S)}{|F|}\right) \Big/ \left(\frac{1}{2} - \frac{1}{\partial_V}\right)$$

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Maps on sphere and projective planes

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Let M *be a map on* S^2 *or* $\mathbb{P}^2$*. Then* $\partial_F < 6$

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2. If the average face degree $\partial_F < 6$ then there must be at least one face f with $\deg(f) \le 5$
   This observation will be important when we prove the Five color theorem (not quite the four color theorem)