Topology – week 7

Math3061

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Technicalities

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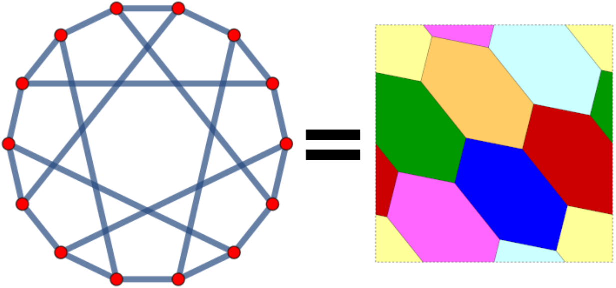
- I apologize in advances for any typos or other errors on these slides!
- I'd be grateful for any corrections that you send to me or post on Ed



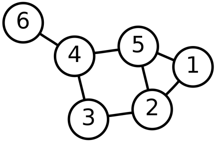
Topology

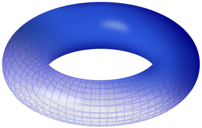
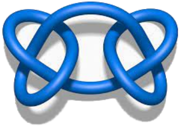
Unit outline

Topology is the study of properties of spaces that are preserved by continuous deformation

We will study:

- Graphs
- Surfaces
- Knots



These are all "topological" objects and we will study all of them by using invariants and by approximating them with graphs

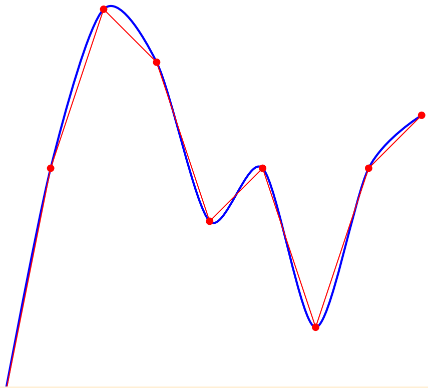
- In topology we are allowed to bend and stretch
- We are not allowed to cut, tear or join surfaces together

Underlying theme in this unit

In this course we want to understand curves and surfaces *but* we allow ourselves to wiggle and stretch the curves and surfaces

Thinking about an arbitrary curve or surface in space is hard, and this is even before we allow them to be continuously deformed

One of the key ideas that we will use is that we can approximate curves and surfaces using graphs



Topological equivalences

Topologically, a square and a circle are the same

Topologically, a cube and a sphere are the same

We will see in more detail why these are the same later

...as well as looking at more exotic surfaces

A torus is the same as a coffee mug

Source https://en.wikipedia.org/wiki/Topology

Graphs

A (finite) graph is an ordered pair $G = (V, E)$, where:

- V is a non-empty finite set of vertices
- E is a finite multiset of edges, which are unordered pairs of vertices

The difference between a set and a multiset is that multisets can have repeated entries

Examples

- $V = \{1, 2, 3\}$ and $E = \big\{\{1, 1\}, \{1, 1\}, \{1, 1\}, \{2, 3\}\big\}$
- $V = \{a, b, c, d\}$ and $E = \big\{\{a, a\}, \{a, b\}, \{a, b\}, \{a, c\}, \{a, d\}\big\}$

Graphs in the plane

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vertices = distinct points in $\mathbb{R}^2$, edges = curves between the points

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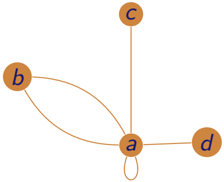
Examples

- $V = \{1, 2, 3\}$ and $E = \{\{1, 1\}, \{1, 1\}, \{1, 1\}, \{2, 3\}\}$

(not connected!)

- $V = \{a, b, c, d\}$ and $E = \{\{a, a\}, \{a, b\}, \{a, b\}, \{a, c\}, \{a, d\}\}$



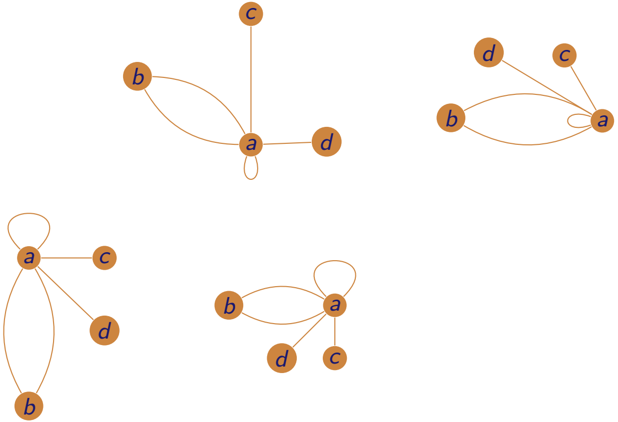
(connected!)

As shown, we allow loops and duplicate edges

Warning: drawings can be misleading

Drawings of graphs are useful pictorial aids, but be careful:
There are many ways to draw the same graph so we always need to check that whatever are doing does not depend on how the graph is drawn!

Here are four different ways to draw the same graph



Standard graphs

Path graphs P_n, for $n \geq 1$ (also called line graphs)

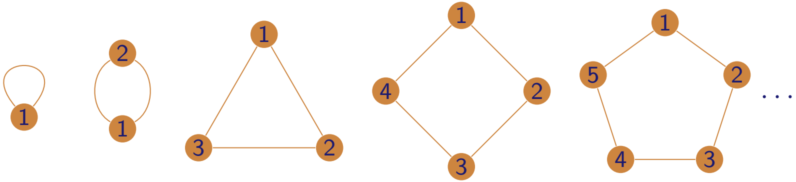
Vertex set $V = \{1, 2, \dots, n\}$

Edge set $E = \big\{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\big\}$

Cyclic graphs C_n, for $n \geq 1$

Vertex set $V = \{1, 2, \dots, n\}$

Edge set the multiset $E = \big\{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\big\}$

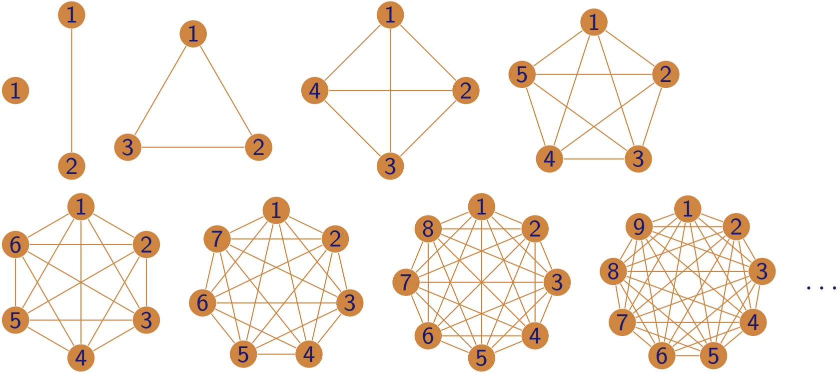


Standard graphs...

Complete graphs K_n, for $n \geq 1$

Vertex set $V = \{1, 2, \ldots, n\}$

Edge set $E = \{ \{i, j\} \mid 1 \leq i < j \leq n \}$



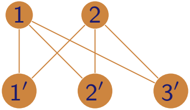
Standard graphs...

Complete bipartite graphs $K_{n,m}$, for $n, m \geq 1$

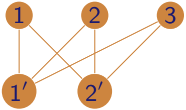
Vertex set $V = \{1, 2, \ldots, n, 1', 2', \ldots, m'\}$

Edge set $E = \left\{ \{i, j'\} \,\middle|\, 1 \leq i \leq n, 1 \leq j \leq m \right\}$

$K_{2,3}$



$K_{3,2}$



Directed graphs

We sometimes use directed graphs where we care about the orientation of the edges.

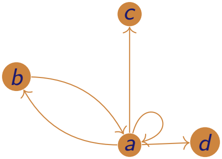
Formally, a (finite) directed graph is an ordered pair (V, E), where:

- V is a finite set of vertices
- E is a finite multiset of directed edges, or ordered pairs of vertices

Examples

- $V = \{1, 2, 3\}$ and $E = \big\{(1,1),(1,1),(1,1),(2,3)\big\}$

- $V = \{a, b, c, d\}$ and $E = \big\{(a,a),(a,b),(b,a),(a,c),(a,d)\big\}$



Subgraphs

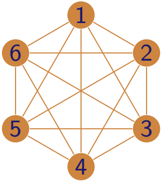
A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ such that

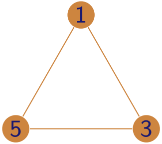
$$W \subseteq V \text{ and } F \subseteq E$$

If (V, E) is a graph and $W \subseteq V$ then the full subgraph of (V, E) with vertex set W is the graph (W, F) with $F = \left\{ \{w, w'\} \in E \,\middle|\, w, w' \in W \right\}$

That is, the full subgraph of $G = (V, E)$ with vertex set W is the subgraph of G that contains every edge in G that connects vertices in W.

Example The full subgraph of K_6 with vertex set $W = \{1, 3, 5\}$ is:





That is, $F = \big\{\{1, 3\}, \{3, 5\}, \{5, 1\}\big\}$

Clearly, (W, F) is "the same" as the cyclic graph C_3

...but what does it mean for graphs to be "the same "?

Isomorphic graphs

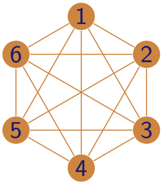
Two graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic, written $G \cong H$, if there is a bijection $f : V \longrightarrow W$ such that the induced map on edges, which sends an edge $\{v, v'\} \in E$ to $\{f(v), f(v')\}$, is also a bijection.

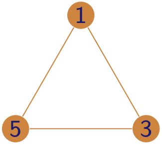
Notice that if $f : G \longrightarrow H$ is a graph isomorphism then:

- if $\{v, v'\} \in E$ is an edge of G then $\{f(v), f(v')\} \in F$ is an edge of H
- Every edge $\{w, w'\} \in F$ can be written uniquely as $\{f(v), f(v')\}$

Examples

- $G \cong H$ if and only if $H \cong G$
- $K_{n,m} \cong K_{m,n}$
- The full subgraph of K_6 with vertex set $W = \{1, 3, 5\}$ has edge set $F = \{\{1, 3\}, \{3, 5\}, \{5, 1\}\}$





Claim $(W, F) \cong C_3$

For example, define f by

$f(1) = 1$,
$f(3) = 2$, and
$f(5) = 3$

Subgraphs of complete graphs

Proposition

Let $G = (V, E)$ be a graph on n vertices that has no loops and no duplicated edges. Then G is isomorphic to a subgraph of K_n.

Proof

Write $V = \{v_1, v_2, \ldots, v_n\}$.

Let $N = \{1, 2, \ldots, n\}$ be the vertex set of K_n and let
$$E_n = \{\, \{i, j\} \mid 1 \le i < j \le n \,\}$$
be its edge set.

Define $H = (N, E_V)$ to be the subgraph of K_n with
$$E_V = \{\, \{i, j\} \mid \{v_i, v_j\} \in E \,\}.$$

Then the map $f : N \longrightarrow V$ given by $f(i) = v_i \in V$ is a graph isomorphism.

Planar graphs

A planar graph is a graph that can be drawn in the $\mathbb{R}^2$ in such a way that no edges cross.

This gives a planar embedding of the graph

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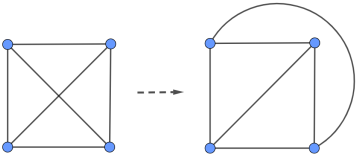
This gives a planar embedding of the graph

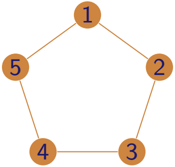
Examples

- Graphs can have planar embeddings and other non-planar realizations
- Every path graph P_n is planar
- Every cyclic graph C_n is planar



Planar graph (K_4)

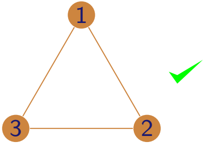
Planar embedding of K_4

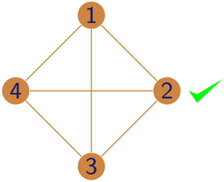


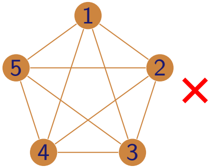
Complete graphs are rarely planar

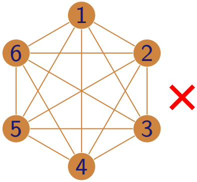
- K_1 

- K_2 

- K_3 

- K_4 

- K_5 

- K_6 

Graph embeddings in $\mathbb{R}^3$

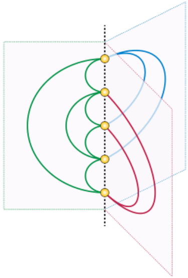
Theorem

Every graph can be embedded (i.e. without edge crossings) in $\mathbb{R}^3$

Moral Graphs are "low dimensional" objects

Proof First, loops and duplicate edges are easy to treat, so we ignore them. Next, use a book embedding:

A 3-page embedding of K_5:



In general, one can embed K_n into a book with $\lceil n/2 \rceil$ pages. Since every graph is a subgraph of some K_n, so we are done since books $\subset \mathbb{R}^3$

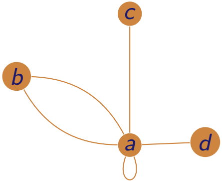
The degree of a vertex

Let $G = (V, E)$ be a graph. The degree of a vertex $v \in V$ is

$$\deg(v) = \#\Big\{\text{number of edges in } E \text{ that have } v \text{ as an endpoint}\Big\}$$

Examples

- 

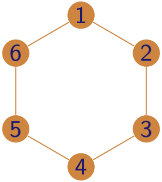
deg(1)=3
- 

deg(a)=5
- P_n 

deg(4)=2

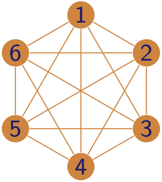
Degrees of vertices in standard graphs; examples

- C_n



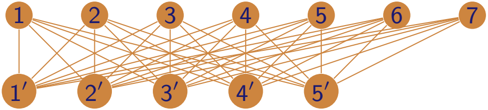
deg(4)=2

- K_n



deg(4)=5

- $K_{n,m}$



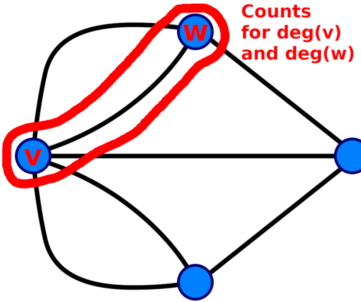
deg(4)=5

Proposition (Vertex-degree equation = handshaking lemma)

Let $G = (V, E)$ be a finite graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof If I shake your hand, then you shake mine: every edges is adjacent to two vertices, hence each edges contributes twice



The handshaking lemma

Proposition (Vertex-degree equation = handshaking lemma)

Let $G = (V, E)$ be a finite graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof

Strictly speaking, we would use induction on $|E|$:

There is nothing to show if there is no edge, and if $|E| > 0$ remove any edge e use induction for $E' = E \setminus \{e\}$, and add e using the previous observation

Let $G = (V, E)$ be a graph. The Euler characteristic of G is the integer

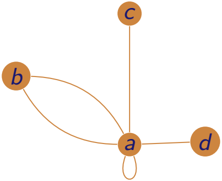
$$\chi(G) = |V| - |E|$$

Moral

$\chi(G) = \#(\text{degree } 0 \text{ components of } G) - \#(\text{degree } 1 \text{ components of } G)$

Examples

- 

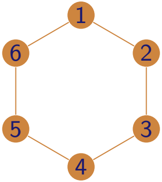
$\chi(G) = -1$
- 

$\chi(G) = -1$
- P_n 

$\chi(G) = 1$

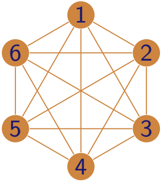
The Euler characteristic of standard graphs

- C_n



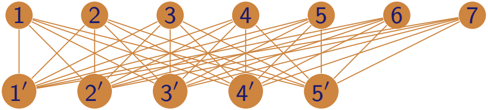
$\chi(G) = 0$

- K_n



$\chi(G) = n - \frac{1}{2}n(n-1) = -\frac{1}{2}n(n-3)$

- $K_{n,m}$



$\chi(G) = n + m - nm$

Subdividing graphs

Let $G = (V, E)$. A subdivision of G is any graph $\dot{G}$ that is obtained from G by successively replacing V with $V \cup \{u\}$, for $u \notin V$, and E with $E \cup \big\{\{v, u\}, \{u, w\}\big\} \setminus \big\{\{v, w\}\big\}$, for an edge $\{v, w\} \in E$

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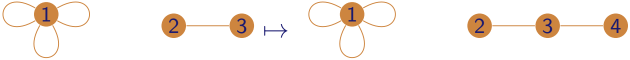
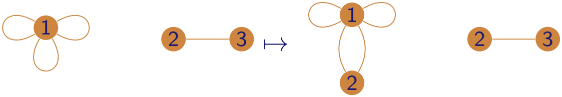
That is, we successively replace an edge 

with 

Examples

- 

- 

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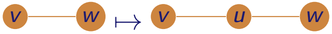
Proposition

Let $\dot{G}$ be a subdivision of G. Then $\chi(\dot{G}) = \chi(G)$

Proof

The operation

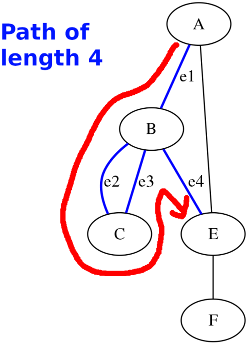


clearly increases V and E by one, so their difference does not change.

Let $G = (V, E)$ be a graph and $v, w \in V$. A path in G of length n from v to w is a sequence of vertices $v = v_0, v_1, \ldots, v_n = w$ such that $\{v_i, v_{i+1}\} \in E$, for $0 \leq i < n$.

That is, the path looks like 

Example



Connectivity in graphs

Observations

- Every vertex is a path of length 0
- A path can pass through any edge zero or more times
- A path can go through any vertex zero or more times
- A path $P = (v = v_0 \to v_2 \to \cdots \to v_n = w)$ of length n in a graph is the same as a graph homomorphism (not nes. an iso) $f : P_{n+1} \longrightarrow G$ with $f(i) = v_{i-1}$, for $1 \leq i \leq n+1$

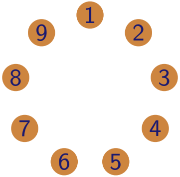
A graph is connected if there is a path between any two vertices

The connected components of a graph G are the maximal connected subgraphs of G. That is, $H = (W, F)$ is a connected component of $G = (V, E)$ if H is
connected and $\{v, w\} \in F$ whenever $\{v, w\} \in E$ and $w \in W$

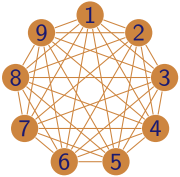
Example

Not connected, two connected components

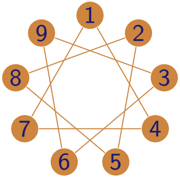
- A fully “disconnected” graph:



- A fully connected graph:



- Three connected components



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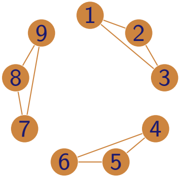
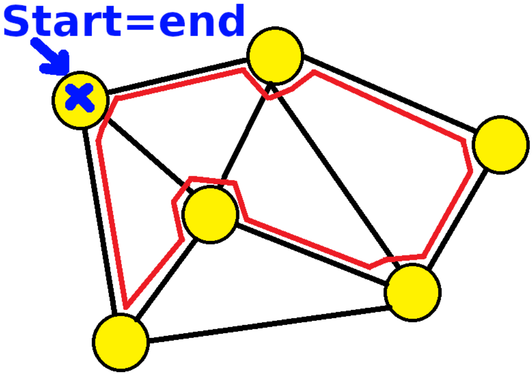


Circuits

A circuit or cycle in G is a path from any vertex to itself

Example



Observations about circuits

Observations

- Every vertex is a circuit of length 0
- A circuit can pass through any edge zero or more times
- A circuit can go through any vertex zero or more times
- A circuit $P = (v = v_0 \to v_2 \to \cdots \to v_n = v)$ of length n in a graph is the same as a graph homomorphism (not nes. an iso) $f : C_n \longrightarrow G$ with $f(i) = v_i$, for $0 \le i \le n$
- "Inefficient circuits" backtrack over the same edges and vertices
- We will soon see that the Euler characteristic is closed related to the number of "reduced" circuits in a graph

Contractible circuits

A circuit $v = v_0 \to v_1 \to \cdots \to v_n = v$ is contractible if it contains two consecutive repeated edges $\{v_i, v_{i+1}\} = \{v_{i+1}, v_{i+2}\}$, for some $0 \leq i \leq n-2$

Reduced circuits

A circuit is reduced if it is not contractible

Notice that every circuit $v = v_0 \to v_1 \to \cdots \to v_n = v$ can be replaced with a reduced circuit by successively deleting the repeated edges

$$v_i \to v_{i+1} \to v_{i+2} = v_i.$$

Observations

- Reduced circuits are "efficient" in the sense that they do not backtrack
- A reduced circuit of length n is not necessarily isomorphic to the cycle graph C_{n+1} because it could, for example, be a figure 8 graph

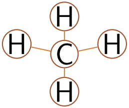
Leaves and trees

A non-trivial circuit is a reduced circuit of length $n > 0$

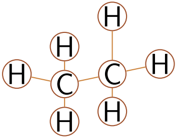
A tree is a connected graph that has no non-trivial circuits

Examples

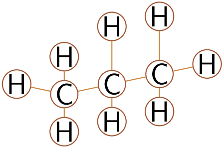
- Saturated hydrocarbons



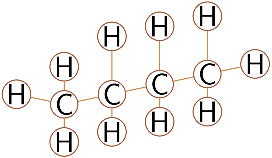
Methane



Ethane

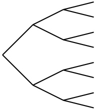


Propane

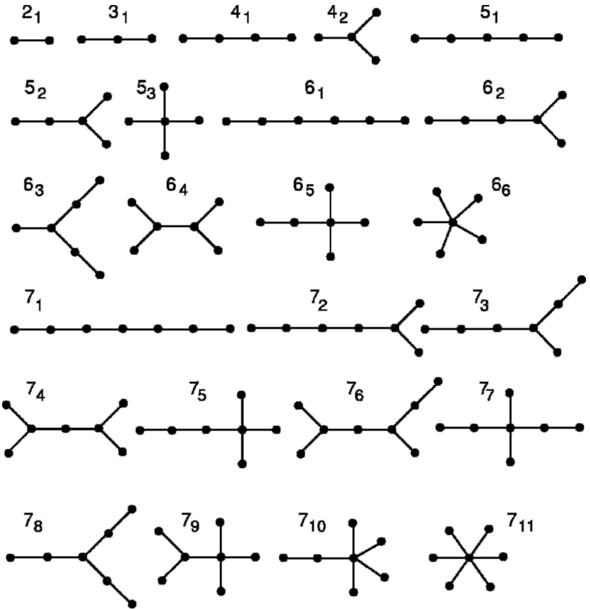


Butane

- A tournament tree



A catalog of small (connected) trees



Trees have leaves

If T is a tree then a leaf in T is any vertex of degree 1

Theorem

Let T be a tree with at least one edge. Then T has at least two leaves.

Remark This result provides an inductive tool for proving facts about trees because removing a leaf gives a tree with one less edge and vertex

Proof Take a longest reduced path P in T, then both endpoints of P are leaves

Why? Say the endpoints are v and w. WLOG suppose v is not a leaf; then v has at least two neighbors and one of them is not in P. (Otherwise we would have a circuit.) Thus one can make P longer. Contradiction

The Euler characteristic of a tree

Theorem

Suppose that T is a tree. Then $\chi(T) = 1$

Proof Argue by induction on the number of edges $|E|$

For $|E|$ small use the previous table.

Otherwise, remove one leave (which exists by the previous statement). The resulting tree has $\chi(T) = 1$, and adding the leave back increases V and E by one, so χ remains constant

Number of edges and vertices in a tree

Corollary

Suppose that $T = (V, E)$ is a tree. Then $|V| = |E| + 1$.

Proof By the previous statement

Proposition

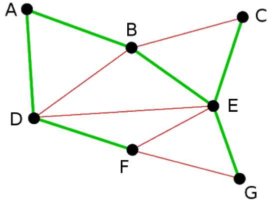
Suppose that $G = (V, E)$ is a connected graph.
Then G has a subgraph $T = (V, F)$ (same vertices) that is a tree

Proof We remove edges to break circuits

(Formally, use induction on the number of nontrivial circuits of G)

A spanning tree of G is any subgraph T of G that is a tree and has the same set of vertices as G

Example



Spanning trees continued

Proposition

Suppose that $G = (V, E)$ is a connected graph.
Then G has a spanning tree $T = (V, F)$ (same vertices)

Proof Remove edges from nontrivial circuit of G to break them; the result is a spanning tree

(Formally, use induction on the number of nontrivial circuit of G)

An upper bound on $\chi(G)$

Corollary

Suppose that G is a connected graph. Then $\chi(G) \leq 1$ with equality if and only if G is a tree.

Proof By the previous statements G has more edges than any of its spanning trees, hence, $\chi(\text{span tree}) = 1$ implies the corollary

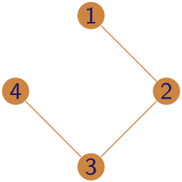
Corollary

Let G be a connected graph. The number of independent cycles (defined via example on the next slide) in G is $1 - \chi(G)$

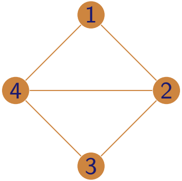
Proof By the previous statements

Independent cycles

Examples



$\chi = 1$, no independent cycles



$\chi = -1$, two independent cycles

We have $\{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\} =$
$\{\{1,2\},\{2,4\},\{1,4\}\} + \{\{2,3\},\{3,4\},\{2,4\}\} \bmod 2$

Remark It is possible to construct a vector space of "cycles" that has dimension $1 - \chi(G)$, which shows that the number of independent cycles makes sense. This is beyond the scope of this course.

Topology – week 8

Math3061

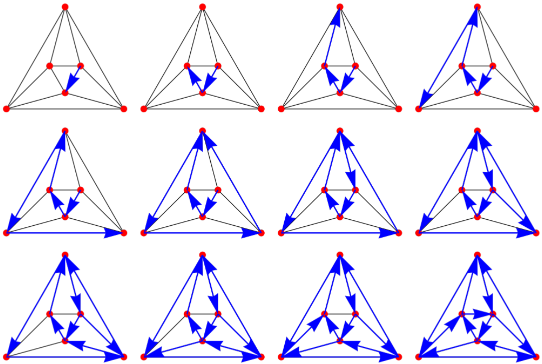
Daniel Tubbenhauer, University of Sydney

Eulerian circuits and graphs

A Eulerian circuit is a circuit that passes through every edge exactly once

A graph is Eulerian if it has a Eulerian circuit

Example

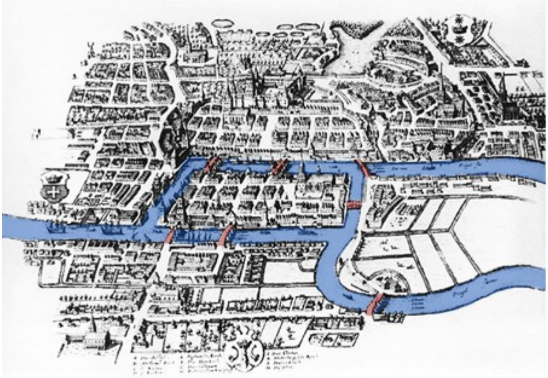


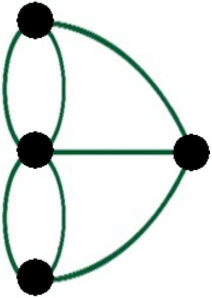
Warning Eulerian graphs do not need to be connected because they may have vertices of degree 0!

Finding Eulerian circuits

In 1736 Euler asked when graphs have Eulerian circuits (without having this terminology)

The motivation was that they wanted to know if it was possible to walk around the city of Königsberg crossing each bridge exactly once





In answering this question Euler laid the foundations of graph theory

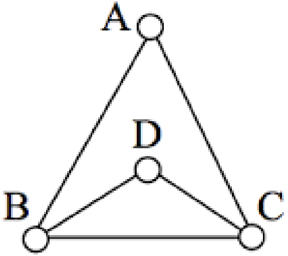
Classifying Eulerian graphs

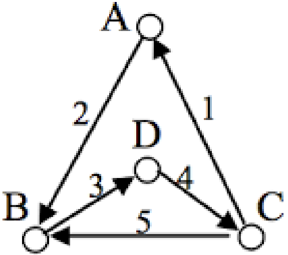
Theorem

Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree

Proof

Assume that there is at least one vertex v of odd degree. Since we want to visit every edge exactly once we will eventually get stuck in v or another vertex of odd degree while trying to create an Eulerian cycle. Hence, G can not have an Eulerian cycle





Classifying Eulerian graphs

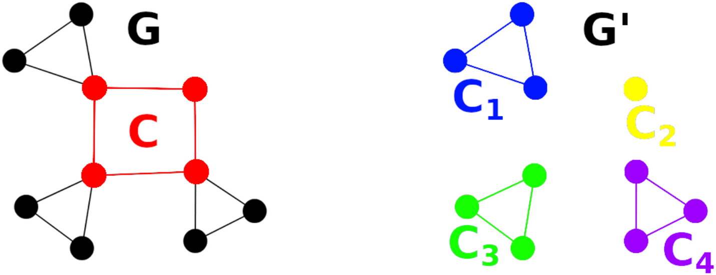
Proof continued

Conversely, if every vertex has even degree, then G is not a tree so contains some circuit C. If C is an Euler circuit we are done, and if not remove all edges of C from G. The resulting (potentially disconnected) graph G' has still even degrees for all of its vertices but fewer edges than G

So we can argue by induction on the number of edges (the base case has no edges and is thus clear), and inductively we can assume that the connected components of G' have Euler circuits $C_1, \ldots, C_n$

Proof continued

We piece C and $C_1, \ldots, C_n$ together into an Euler cycle: we walk along C and whenever we hit a vertex of C_i we take a detour over C_i



Eulerian paths

A Eulerian path is a path that is not a circuit and which passes through every edge exactly once

Corollary

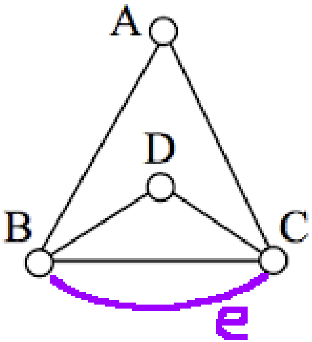
Let $G = (V, E)$ be a connected graph that is not Eulerian. Then G has a Eulerian path if and only if it has exactly two vertices of odd degree

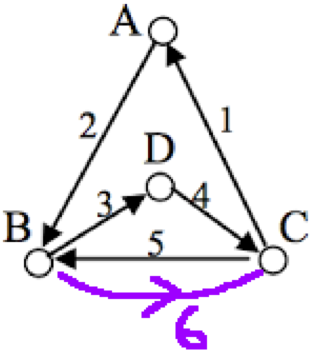
Proof

Only vertices of odd degree can be a start or an end vertex, so we need precisely two of them (all other must be of even degree by the same argument as before)

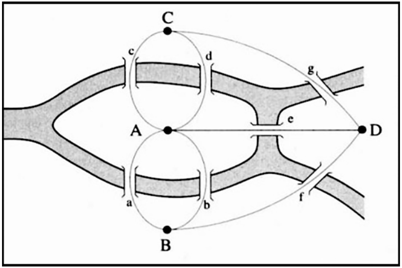
Proof continued

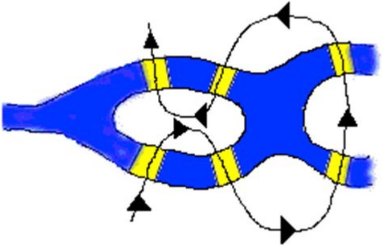
Conversely, if v and w are the two vertices of even degree, then we put an additional edge e between them. We get a graph $G' = G \cup \{e\}$ and the previous theorem gives us an Euler circuit C in G'. Then $C \setminus \{e\}$ is an Euler path





What about Königsberg?





There is no Eulerian circuit since all vertices have odd degree

There is no Eulerian path since all vertices have odd degree

Solution: Destroy bridge e ;-)

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Remarks

- Homeomorphism is the higher dim analog of isomorphism for graphs
  We treat two spaces as being "equal" if they are homeomorphic
- The maps f and g are both bijections with continuous inverses
- We have $X \cong X$
- If $X \cong Y$, then $Y \cong X$
- If $X \cong Y$ and $Y \cong Z$, then $X \cong Z$

Examples of homeomorphisms

Proposition

If $a < b$ and $c < d$, then $[a, b] \cong [c, d]$

Proof

Define maps $f : [a, b] \longrightarrow [c, d]; x \mapsto c + \frac{d-c}{b-a}(x - a)$
$$g : [c, d] \longrightarrow [a, b]; x \mapsto a + \frac{b-a}{d-c}(x - c)$$

Exercise Show that $(a, b) \cong (c, d)$ and $(a, b] \cong (c, d] \overset{!!!}{\cong} [a, b) \cong [c, d)$

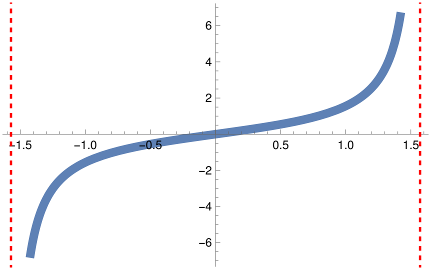
Proposition

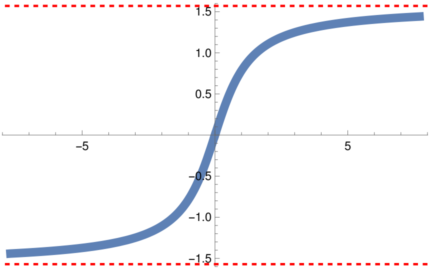
If $a < b$, then $(a, b) \cong \mathbb{R}$

Proof It is enough to show that $(-\frac{\pi}{2}, \frac{\pi}{2}) \cong \mathbb{R}$

Proof continued

Homeomorphisms are given by $f(x) = \tan(x)$ and $g(x) = \tan^{-1}(x)$





Proposition

$$\square \cong \bigcirc = S^1 \qquad \textit{We show that} \quad \Diamond \cong \bigcirc$$

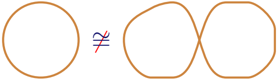
Proof

The square is $\{\,(x,y) \mid |x|+|y|=1\,\}$ and $S^1 = \{\,(x,y) \mid x^2+y^2=1\,\}$

Define: $f : \square \longrightarrow S^1; (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

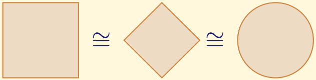
$g : S^1 \longrightarrow \square; (x,y) \mapsto \left(\frac{x}{|x|+|y|}, \frac{y}{|x|+|y|}\right)$

Note that



For free we see that the square and disk are homeomorphic:

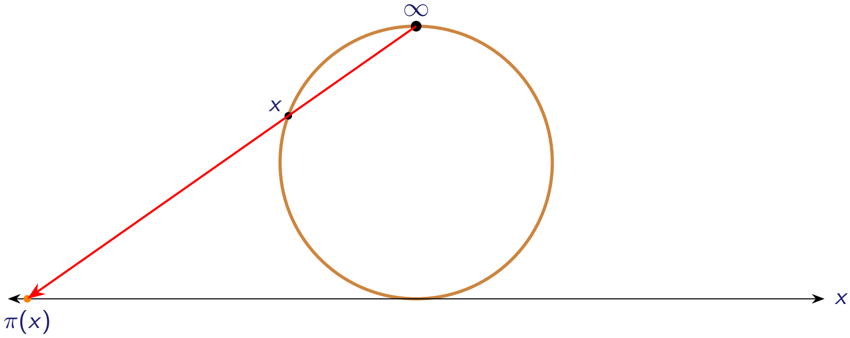
Corollary



Stereographic projection in two dimensions

Think of the north pole of the circle S^1 as ∞

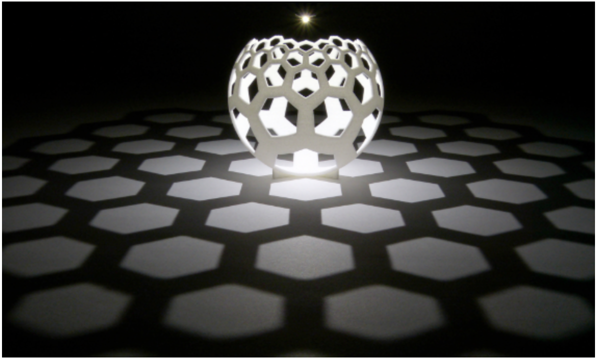
Stereographic projection gives a homeomorphism $\pi\colon S^1 \setminus \{\infty\} \to \mathbb{R}$:



Stereographic projection in three dimensions

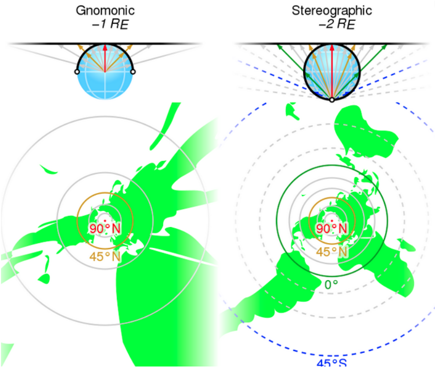
Think of the north pole of the circle S^2 as ∞

Stereographic projection gives a homeomorphism $\pi \colon S^2 \setminus \{\infty\} \to \mathbb{R}^2$:



Maps

Stereographic projection is used to draw maps:



Other projections are also used such as gnomonic projections, conic projections and the Mercator projection, which is a cylindrical projection

Now that we have seen homeomorphisms we are ready to define surfaces

Surfaces — informal definition

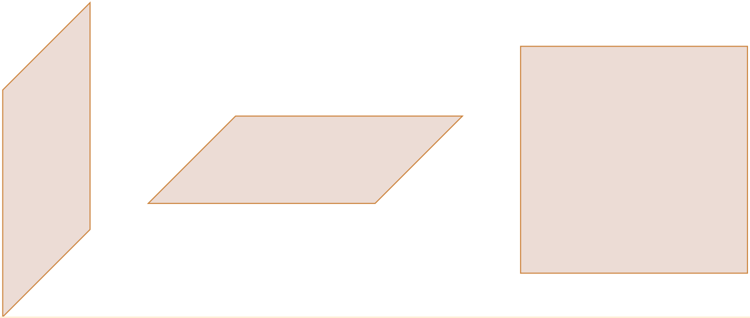
Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $f(x, y) = z$ / alternatively to a disc

Here "locally" means that we can find a "local neighborhood" of every point where the function looks like the plane $f(x, y) = z$ / a disc

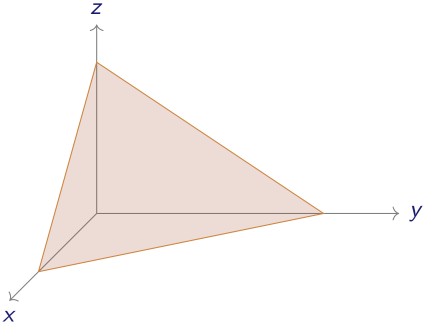
Examples

- A standard xyz-plane in $\mathbb{R}^3$

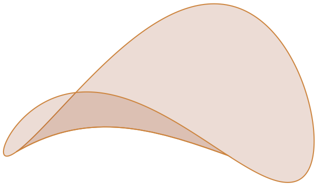


Surfaces — examples...

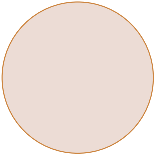
- Non-standard planes in $\mathbb{R}^3$



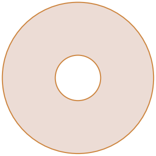
- Curved surfaces in $\mathbb{R}^3$



- A disk $\mathbb{D}^2$



- An annulus $\mathbb{A} \cong$ cylinder



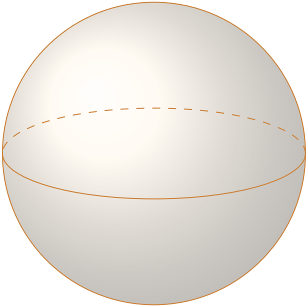
$\cong$

Strictly speaking, these are not surfaces according to our definition because they have a boundary, whereas planes in $\mathbb{R}^2$ do not have boundaries.

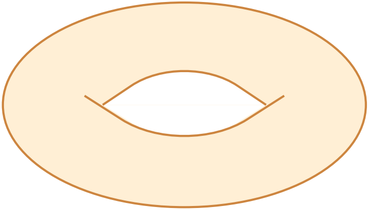
Our rigorous definition of a surface will allow surfaces with boundaries

Surfaces — examples...

- A sphere S^2



- A torus $\mathbb{T}$



Surfaces — real world examples...

- A sphere $S^2 \cong$ soccer ball

- A torus $\mathbb{T} \cong$ swim ring

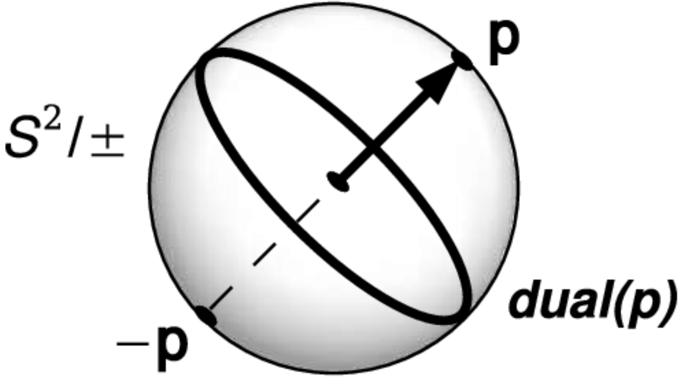
Surfaces — real world example...

- Here is a surface with boundary:

The patches are examples of neighborhoods which are discs

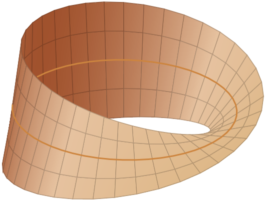
Surfaces — examples...

- The real projective plane $\mathbb{P}^2 = S^2/\text{antipode}$

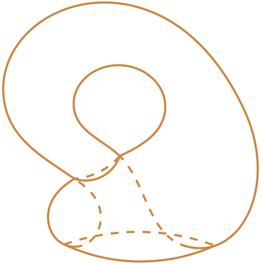


We will see other ways to describe $\mathbb{P}^2$ later

- A Möbius band, or Möbius strip, $\mathbb{M}$



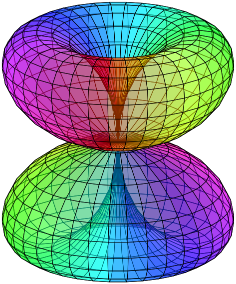
- A Klein bottle $\mathbb{K}$, also Klein surface



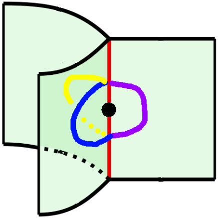
This is a three dimensional "shadow" of a four dimensional object

Surfaces — non-examples

- This is not a surface because of the cusp at the origin



- This is not a surface because the indicated point has not a disc neighborhood



Identification spaces

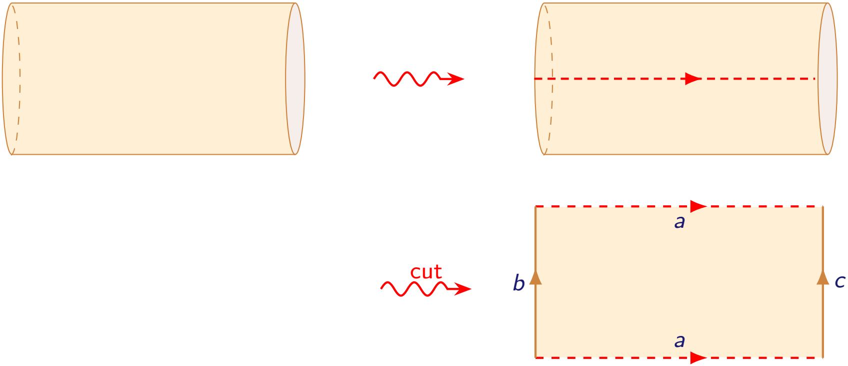
A partition of a surface $S \subseteq \mathbb{R}^m$ is a collection $X_1, \dots, X_r$ of subsets of S such that $S = X_1 \cup X_2 \cup \cdots \cup X_r$

The space S is an identification space for $Y \subseteq \mathbb{R}^n$ if there exists a continuous surjective map $f : S \longrightarrow Y$

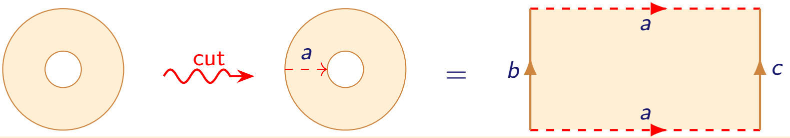
Note $Y = f(X_1) \cup f(X_2) \cup \cdots \cup f(X_r)$ and that the map f implicitly identifies the points in $f(X_{i_1}) \cap \cdots \cap f(X_{i_s})$, for $1 \leq i_1, \dots, i_s \leq r$

This makes is possible to understand Y in terms of, often, easier spaces $X_1, \dots, X_r$, which we think of as covering Y like a patchwork quilt

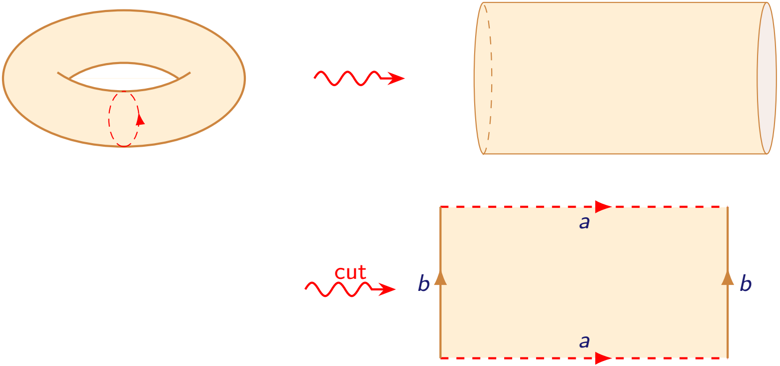
Identification space for a cylinder



That is, the cylinder is the identification space obtained by identifying the top and bottom edges of a suitably sized rectangle

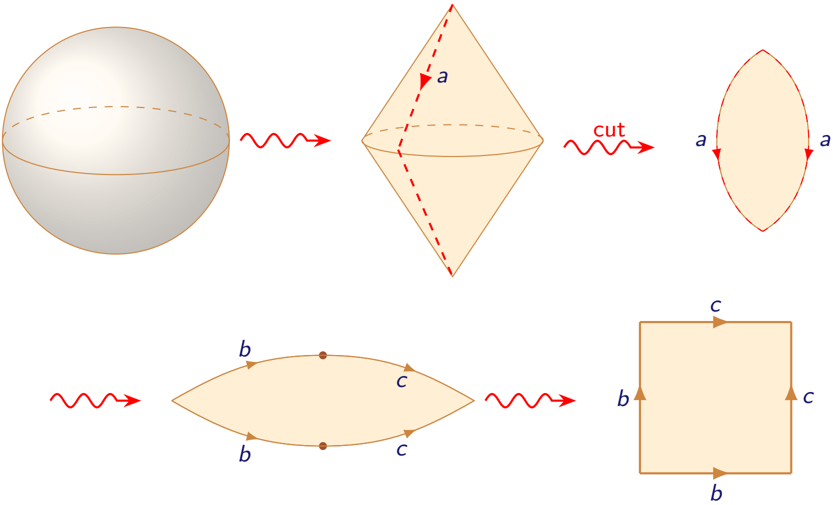
$\mathbb{A} =$ 

Identification space for a torus



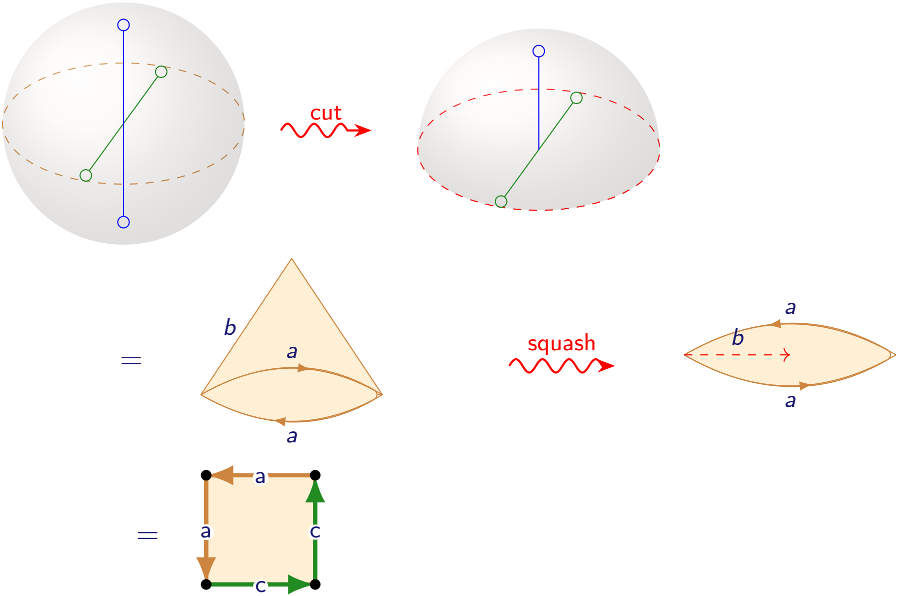
So, the torus $\mathbb{T}$ is obtained by identifying the top and bottom, and the left and right, edges of a rectangle

Identification space for a sphere

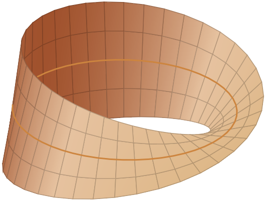


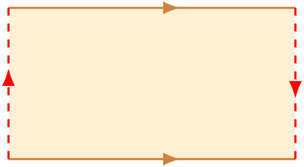
The sphere S^2 is obtained by identifying adjacent sides of a rectangle, or a 2-gon (a polygon with two sides)

Identification space for the projective plane $\mathbb{P}^2$

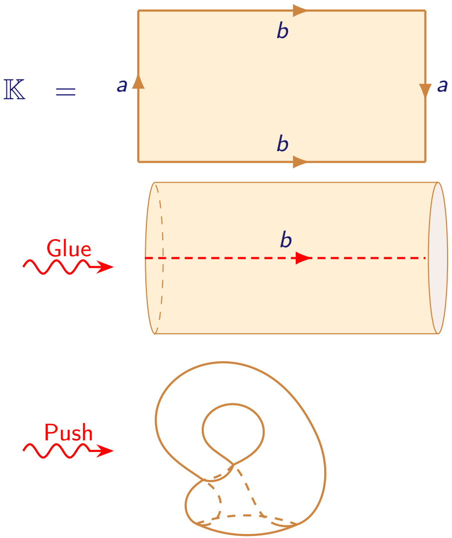


Identification space for a Möbius strip





The Klein bottle is defined to be the identification space



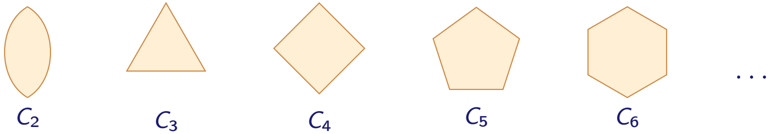
It is not clear how we to do the last step in $\mathbb{R}^3$ and, in fact, we can't!

Polygons in $\mathbb{R}^n$

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

A polygon is an embedding of the cyclic graph C_m into $\mathbb{R}^2$, together with its face, such that such that the vertices of C_m map to distinct points in $\mathbb{R}^2$ and the images of the edges do not intersect in $\mathbb{R}^2$

$\implies$ The image of C_n in $\mathbb{R}^2$ is homeomorphic to the closed disc $\mathbb{D}^2$



Remarks

- The image of C_m in $\mathbb{R}^2$ is an m-gon, or a polygon with m sides
- Polygons are surfaces in $\mathbb{R}^2$. They are different from cyclic graphs because they have vertices, edges and one face
- The graph C_2 has only one edge. When working with surfaces we think of C_2 as having two edges so that its image in $\mathbb{R}^2$ is a 2-gon

Surfaces and polygonal decompositions

Definition

A surface S is an identification space in $\mathbb{R}^n$ that is obtained by gluing together polygons along their edges in such a way that at most two edges meet along any edge
The polygons give a polygonal decomposition of the surface S

Remarks

- A surface is an identification space where we identify pairs of edges in polygons. Informally, a surface is a patchwork quilt of polygons
- This essentially agrees with our earlier definition of surfaces because every polygon is homeomorphic to a closed disc $\mathbb{D}^2$ so, locally, surfaces look like planes / like discs
- A surface can have many seemingly different polygonal decompositions
- A surface with a polygonal decomposition has vertices, edges and faces
- We sometimes write $S = (V, E, F)$, where V is the vertex set, edge set E, and face set F

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

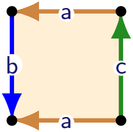
- Label all of the edges with letters: $a, b, c, \ldots$
- Use the same color for edges that have the same label
- Fix a direction of every edge (this is important!)

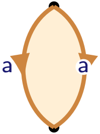
Remarks

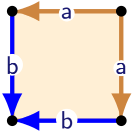
- Identifying edges implicitly identifies vertices
- Colouring the edges is not strictly necessary but makes it easier to see how the edges are identified in the polygonal decomposition
- You do not need to color the edges in your work, but you can if you want to
- It is important to give the correct orientation, or direction, for the paired edges because changing the direction of a paired edge will usually change the surface
- When doing surgery always double check that you do not accidentally change the orientation of an edge

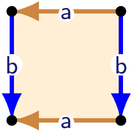
Examples of polygonal decompositions

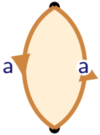
We have already seen that:

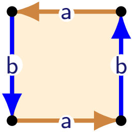
- Annulus $\mathbb{A} \cong$ 

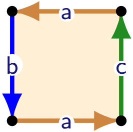
- Sphere $S^2 \cong$ 

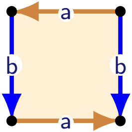
$\cong$ 

- Torus $\mathbb{T} \cong$ 

- Projective plane $\mathbb{P}^2 \cong$ 

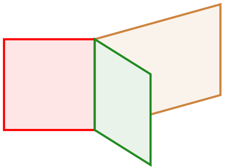
$\cong$ 

- Möbius strip $\mathbb{M} \cong$ 

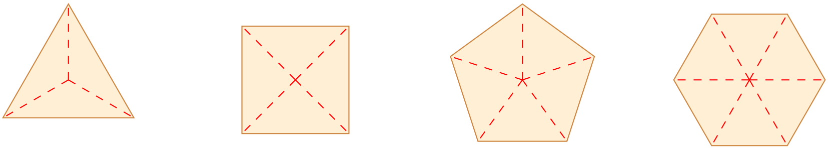
- Klein bottle $\mathbb{K} \cong$ 

Important facts about polygonal decompositions

- Every polygon is homeomorphic to a closed disk $\mathbb{D}^2$
- At most two polygons meet in any edge, so



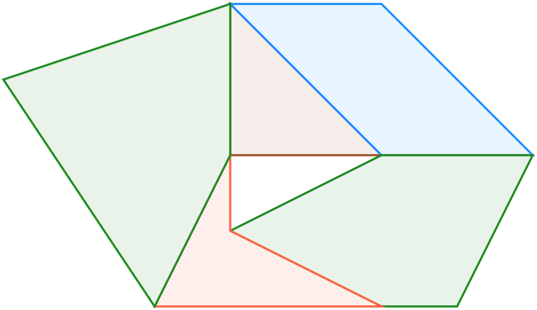
  is not polygonal decomposition of a surface
- Any polygonal decomposition can be replaced with one that only uses 3-gons:



$\Longrightarrow$ Iterating this process, shows that any surface has infinitely many different polygonal decompositions!

Important facts about polygonal decompositions...

- Every connected surface has a polygonal decomposition with one polygon — with identified edges
  (A polygonal surface is connected if the underlying graph is connected)



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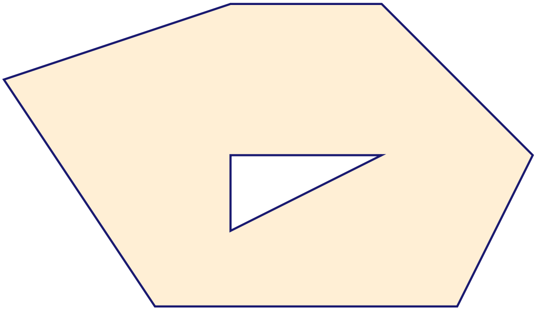

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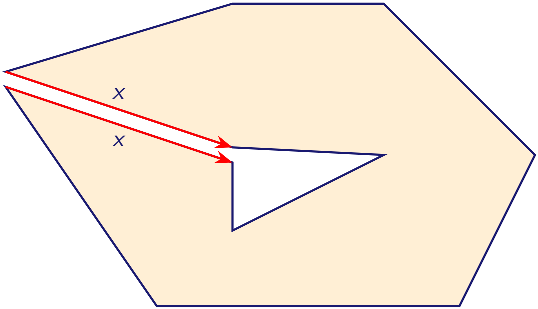
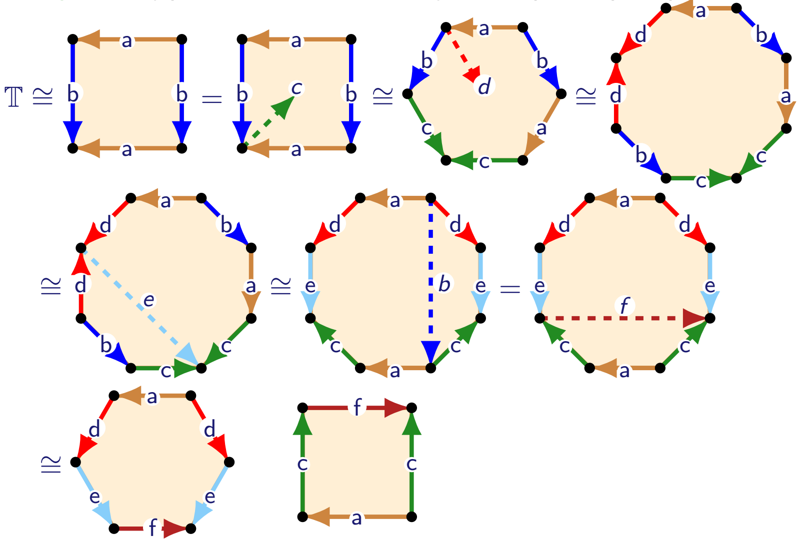


- We have to check that what we are doing does not depend on the choice of polygonal decomposition

Surgery: cutting and gluing

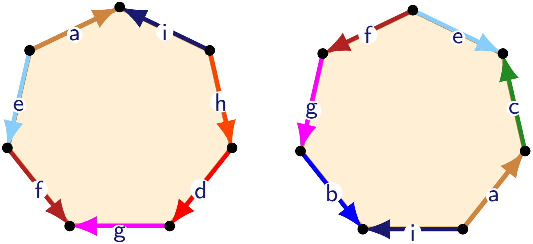
Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing



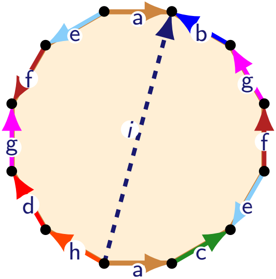
We want an easy way to identify surfaces from polygonal decompositions

Example surface

Exercise Can we describe the following surface?



$=$



Answer Not yet! First we need more language and technology.

Free and paired edges and the boundary

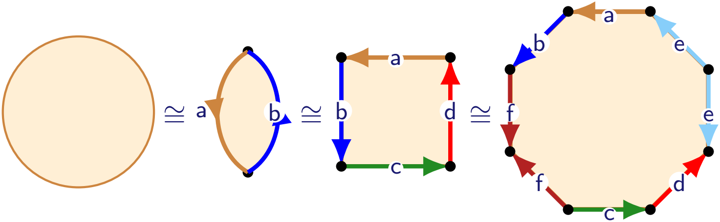
Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice
- The boundary of S is the union of the free edges
- A boundary circle is a cycle in the polygonal decomposition in which every edge is free

  We will show that boundary of S is a disjoint union of boundary circles

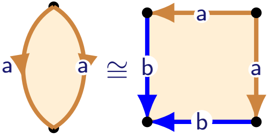
Example

$\mathbb{D}^2 =$

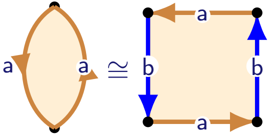


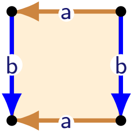
Example boundary circles...

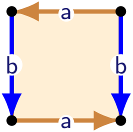
- Sphere

$S^2 \cong$ 

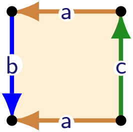
- Projective plane

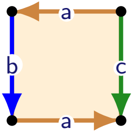
$\mathbb{P}^2 \cong$ 

- Torus $\mathbb{T} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

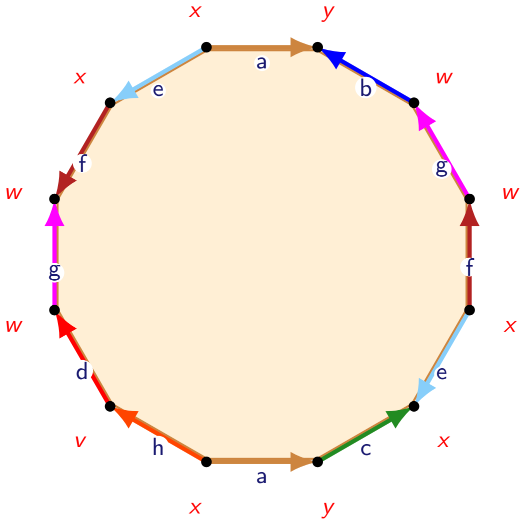
All edges paired $\Longrightarrow$ no boundary

- Annulus $\mathbb{A} \cong$ 

- Möbius $\mathbb{M} \cong$ 

Example boundary circles...

Exercise What is the boundary of the surface?



Free edges: b, c, d, h

Key observation

Paired edges imply that some vertices are equal

The Euler characteristic of a surface

Let $S = (V, E, F)$ be a surface with a polygonal decomposition

Definition

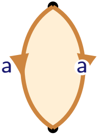
The Euler characteristic of S is $\chi(S) = |V| - |E| + |F|$

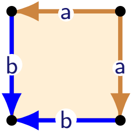
Remarks

- The Euler characteristic $\chi(S) = |V| - |E| + |F|$ of S is a higher dimensional generalization of the Euler characteristic of a graph $G = (V, E)$, which is $\chi(G) = |V| - |E|$
- The definition of $\chi(S)$ appears to depend on the choice of polygonal decomposition (V, E, F) of S. In fact, we will soon see that $\chi(S)$ is independent of this choice

Euler characteristic of basic surfaces.

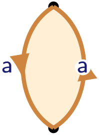
- Sphere

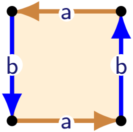
$S^2 \cong$ 

$\cong$ 

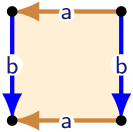
$, \chi = 2$

- Projective plane

$\mathbb{P}^2 \cong$ 

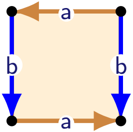
$\cong$ 

$, \chi = 1$

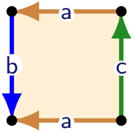
- Torus $\mathbb{T} \cong$ 

$, \chi = 0$

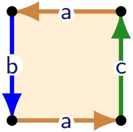
- Klein bottle

$\mathbb{K} \cong$ 

$, \chi = 0$

- Annulus $\mathbb{A} \cong$ 

$, \chi = 0$

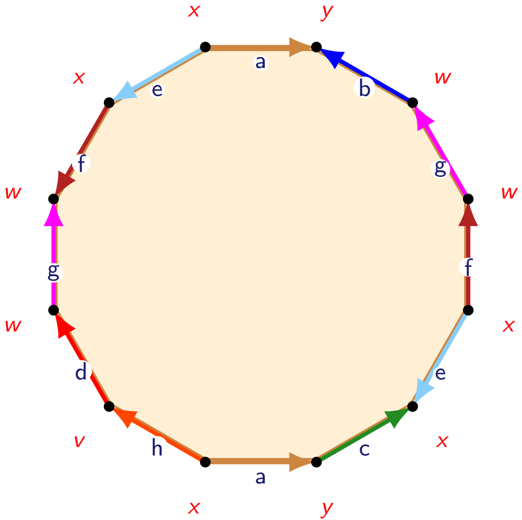
- Möbius $\mathbb{M} \cong$ 

$, \chi = 0$

Euler characteristic example

Example What is the Euler characteristic of the surface:

$S \quad =$



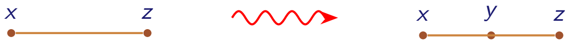
$\Longrightarrow \quad \chi(S) = -3$

Subdivision of a surface

Let S be a surface with a polygonal decomposition

A subdivision of S is any polygonal decomposition that is obtained from S by successively applying the following operations:

- Subdividing an edge by adding a new vertex



- Subdividing a face by adding a new edge

Remarks

- The subdivision of a subdivision of S is a subdivision of S
- If $\dot{S}$ has a polygonal decomposition that is a subdivision of a polygonal decomposition of S then $S \cong \dot{S}$

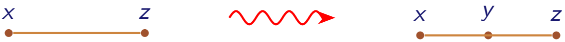
Subdividing and Euler characteristic

Proposition

Let $\dot{S}$ be a subdivision of S. Then $\chi(S) = \chi(\dot{S})$

Proof It is enough to check this for the two subdivision operations:

- Subdividing an edge:



- Subdividing a face:

Both operations preserve χ

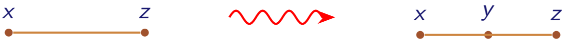
Subdividing and boundary circles

Proposition

Let $\dot{S}$ be a subdivision of S. Then S and $\dot{S}$ have the same number of boundary circles

Proof It is enough to check this for the two subdivision operations:

- Subdividing an edge:



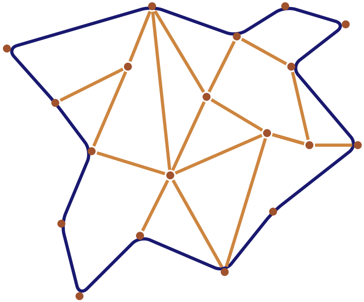
- Subdividing a face:

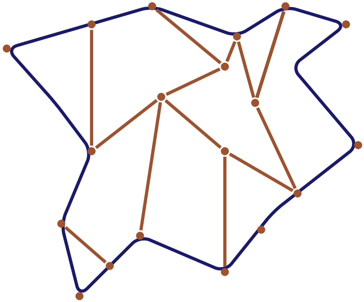
Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

Proof Merge the two subdivisions





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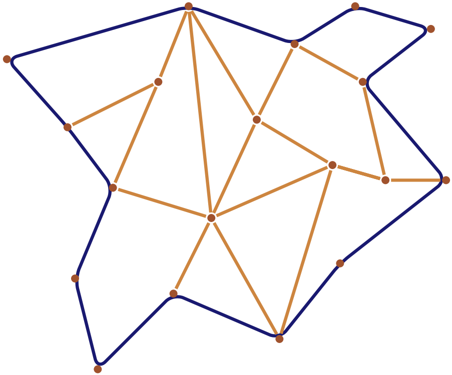

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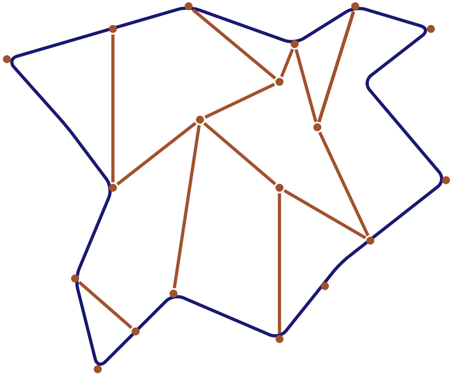

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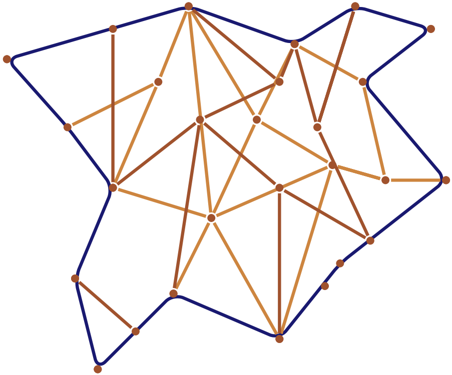

Common subdivisions

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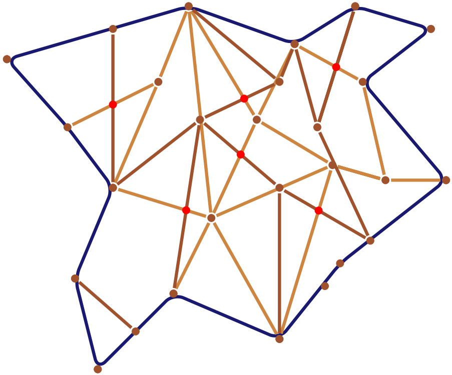
Proof Merge the two subdivisions — adding extra vertices as necessary



Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Observe that if P is a polygonal decomposition of S then $f(P)$ is a polygonal decomposition of T. Similarly, if Q is a polygonal subdivision of T then $g(T)$ is a polygonal decomposition of S

By the theorem we can assume that S and T have the same polygonal decomposition in the sense that $P = g(Q)$ and $Q = f(P)$

$$\implies \quad \chi(S) = \chi_P(S) = \chi_{f(P)} = \chi_Q(T) = \chi(T).$$

Similarly, S and T have the same number of boundary circles

Why are invariants useful?

Question

Let S and T be surfaces. Is $S \cong T$?

To show that S and T are homeomorphic is, in principle, easy: we find a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Showing that $S \ncong T$ is harder as we need to show that no such maps exist

Using invariants makes this easier because $S \cong T$ only if $\chi(S) = \chi(T)$ and if S and T have the same number of boundary circles

$\Longrightarrow$ if $\chi(S) \neq \chi(T)$, or if S and T have a different number of boundary circles, then $S \ncong T$

Exercise Using what we know so far, deduce that the surfaces

$$S^2, \mathbb{A}, \mathbb{D}^2, \mathbb{K}, \mathbb{M}, \mathbb{P}^2$$

are pairwise non-homeomorphic (see Tutorial 9)

Topology – week 9

Math3061

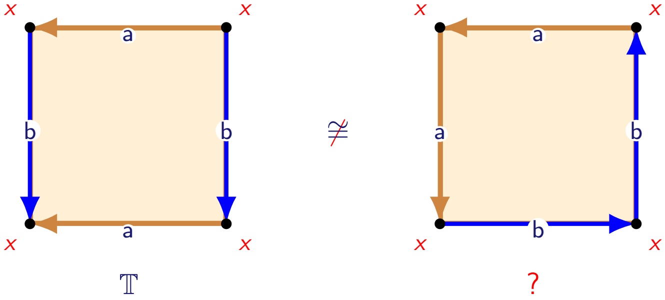
Daniel Tubbenhauer, University of Sydney

We have seen that homeomorphic surfaces must have:

- The same Euler characteristic
- The same number of boundary circles

These two invariants are both easy to compute but, by themselves, they are not enough to distinguish between all surfaces

Example



Orientability

Definition

A surface S is non-orientable if it contains a Möbius strip $\mathbb{M}$

If S does not contain a Möbius strip it is orientable

Remarks

- Even though this looks hard to apply we will see it isn't

Orientability

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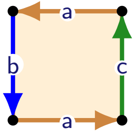
- Clearly, $\mathbb{M}$ is non-orientable, but there are no other "easy" examples

$\mathbb{M} =$ 

Orientability

Definition

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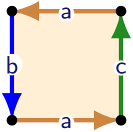
$\mathbb{M} =$ 

- Are S^2, $\mathbb{A}$, $\mathbb{D}^2$, $\mathbb{T}$, $\mathbb{P}^2$, $\mathbb{K}$, … orientable or non-orientable?

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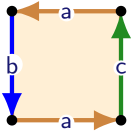
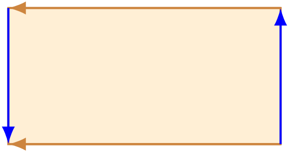
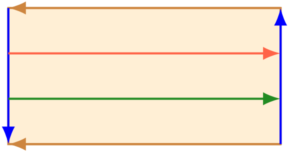
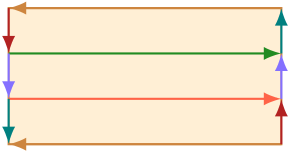
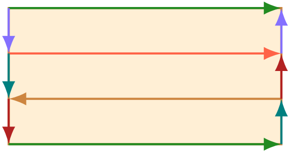
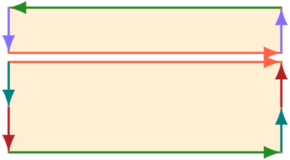
$\mathbb{M} =$ 

- Are S^2, $\mathbb{A}$, $\mathbb{D}^2$, $\mathbb{T}$, $\mathbb{P}^2$, $\mathbb{K}$, ... orientable or non-orientable?
- Can a surface be orientable and non-orientable for different polygonal decompositions? (That would be bad!)

$\mathbb{K}$ $=$  $=$ 

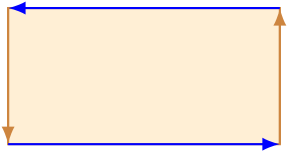
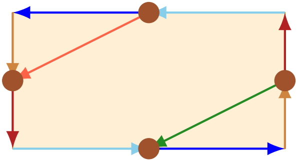
$=$  $=$ 

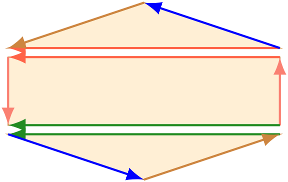
$=$  $=$ $\mathbb{M}$ $\mathbb{M}$

$\Longrightarrow$ The Klein bottle $\mathbb{K}$ is non-orientable!

... although it might be more accurate to say that the Klein bottle is a Möbius strip without boundary

The projective plane $\mathbb{P}^2$

$\mathbb{P}^2 \quad =$  $=$ 

$=$  $=$

$\mathbb{D}^2$
$\mathbb{M}$
$\mathbb{D}^2$

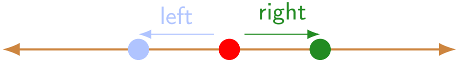
$\Longrightarrow$ The projective plane $\mathbb{P}^2$ is non-orientable

... or maybe $\mathbb{P}^2$ and not $\mathbb{K}$
is a Möbius strip without boundary?

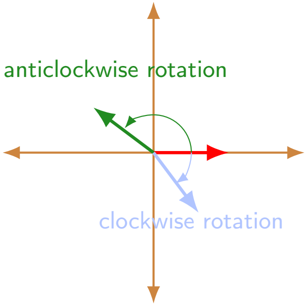
What does orientability mean?

Orientability is a generalisation of direction to higher dimensions

- One dimension $\mathbb{R}$



- Two dimensions $\mathbb{R}^2$



- Three dimensions $\mathbb{R}^3$???
- Higher dimensions $\mathbb{R}^n$, for $n \geq 3$???

Direction in higher dimensions

To generalise direction, choose an ordered basis $B = \{b_1, b_2, \dots, b_n\}$ of $\mathbb{R}^n$

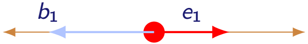
The order of the basis elements is the key to understanding direction

We can compare B to the standard basis $E = \{e_1, e_2, \dots, e_n\}$ of column vectors by computing the sign of the determinant

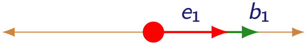
$$\det(B) = \det\begin{pmatrix} \vdots & \vdots & & \vdots \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & & \vdots \end{pmatrix} \quad \rightsquigarrow \quad \operatorname{sign}(B) = \pm 1$$

- One dimension $\mathbb{R}$

$\operatorname{sign}(B) = -1$



$\operatorname{sign}(B) = +1$



- Two dimensions $\mathbb{R}^2$

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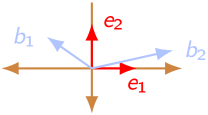

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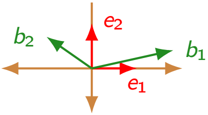
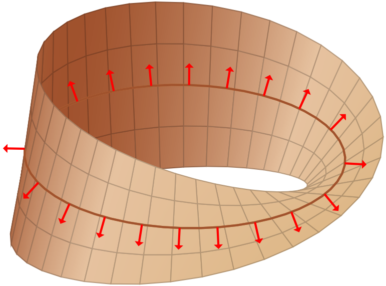


Direction on the Möbius strip

Pick a point $m \in \mathbb{M}$ on the Möbius strip and an ordered basis $B = \{b_1, b_2, b_3\}$ positioned at m with $b_3 = b_1 \times b_2$ pointing outwards

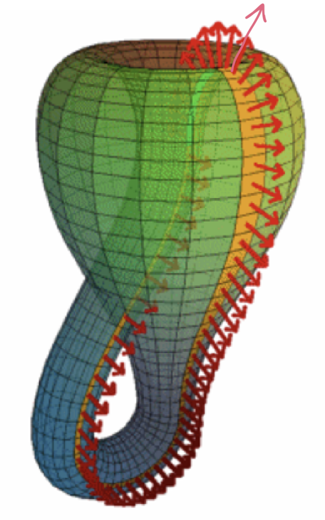
Now imagine m, and the coordinate axes moving, continuously around the Möbius strip so that the xyz-coordinate axes around $\mathbb{M}$



Initially, b_3 is pointing outwards but after one rotation it is pointing inwards

The vector b_3 is always normal to the surface of the Möbius strip. The direction of b_3 can change from pointing outside to inside because the Möbius strip is a surface with a boundary that only has one side

Direction on the Klein bottle $\mathbb{K}$



We can do the same experiment with the Klein bottle and we see the same phenomenon: the vector b_3 changes from pointing outside to pointing inside the surface

This time is slightly different because $\mathbb{K}$ is a surface without boundary

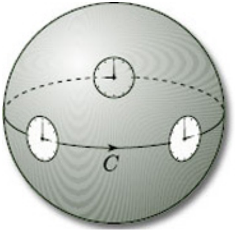
$\Longrightarrow$ The Klein bottle $\mathbb{K}$ does not have an inside and an outside !!

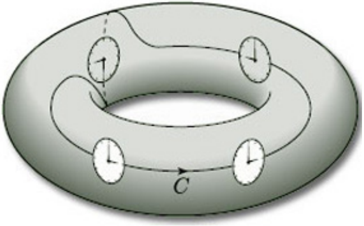
In contrast, orientable surfaces without boundary like S^2 and $\mathbb{T}$ do have an inside and an outside

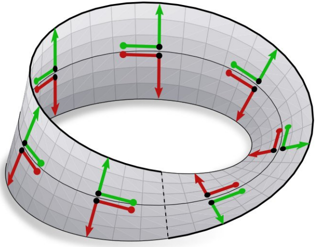
Warning: this is a drawing of $\mathbb{K}$ in $\mathbb{R}^3$ but it is not the actual Klein bottle! Similarly, the pictures of the sphere S^2 in $\mathbb{R}^3$ are not really the sphere!

Alternative description

Alternatively, think of an orientation as a consistent of a coordinate system for each point:





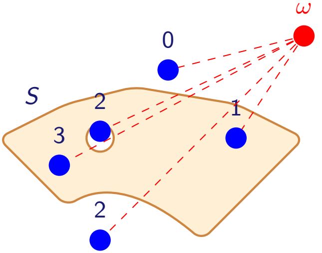


Theorem

Suppose that S is a connected surface without boundary that embeds in $\mathbb{R}^3$. Then S is orientable.

Proof Embed S in $\mathbb{R}^3$ and pick a point ω a "long" way from S

For each point $x \in \mathbb{R}^3$ draw a line from ω to x and define $s(x)$ to be the number of times this line crosses the boundary of S



Set $V_{\text{in}} = \{ x \in \mathbb{R}^3 \mid x \notin S \text{ and } s(x) \text{ is odd} \}$
$V_{\text{out}} = \{ x \in \mathbb{R}^3 \mid s(x) \text{ is even} \}$

$\implies \quad \mathbb{R}^3 = S \cup V_{\text{in}} \cup V_{\text{out}}$ (disjoint union)

Notice that since S is a closed surface it does not have boundary, so the "circle" in the picture, which contains a point x with $s(x) = 2$, should be interpreted as a tube through the surface

Orientable surfaces...

Now suppose that S is non-orientable, so that it contains a Möbius strip $\mathbb{M}$

Pick a point $m \in S$ that is on this Möbius strip and fix an ordered basis $\{b_1, b_2, b_3\}$ with b_1 and b_2 tangential to m and $b_3 = b_1 \times b_2$.

Replacing b_3 with $-b_3$, if necessary, we assume that b_3 points out of S

Now move m, and $B = \{b_1, b_2, b_3\}$, continuously around S

$\implies$ $\det(B)$ changes continuously as m moves around S

By moving m around the Möbius strip in M, we can move m to a point where b_3 now points inside S

$\implies$ By continuity, at some point b_3 must have been in the plane spanned by b_1 and b_2

$\implies$ $\det(B) = 0$ $\quad$ ↯↯↯ since B is linearly independent!

Corollary

Let S be a non-orientable closed surface. Then S does not embed in $\mathbb{R}^3$.

You can't fill a liquid into the Klein bottle

Strictly speaking the liquid is neither in- nor outside

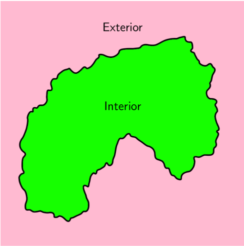
Jordan curve theorem

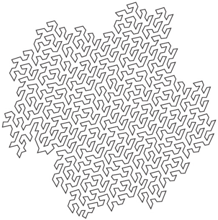
This argument used to prove theorem can be made rigorous for surfaces with finite polygonal decompositions but for "general surfaces" it is difficult to prove that $\mathbb{R}^3 = S \cup V_{\text{in}} \cup V_{\text{out}}$.

The corresponding result for curves in $\mathbb{R}^2$ is known as the Jordan Curve Theorem, which says that any closed curve C in $\mathbb{R}^2$ gives rise to a decomposition $\quad \mathbb{R}^2 = C \cup V_{\text{in}} \cup V_{\text{out}}$ (disjoint union)

This is really hard to prove!

To appreciate why this is a nontrivial result consider:

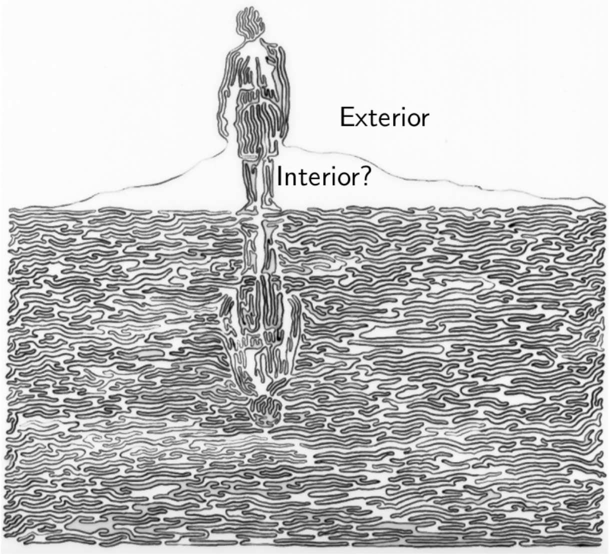




The left is easy, but can you tell for the right what is "in" or "out"?

Jordan curve theorem - 2

The main meat is that one needs to deal with “crazy” curves:



Embedding the projective plane in $\mathbb{R}^4$

The projective plane $\mathbb{P}^2$ is non-orientable, so it does not embed in $\mathbb{R}^3$

By definition, the projective plane is defined by identifying antipodal points on the sphere S^2:

$$\mathbb{P}^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\} / (x, y, z) \sim (-x, -y, -z)$$

We can embed $\mathbb{P}^2$ into $\mathbb{R}^4$ using the continuous map:

$$(x, y, z) \quad \mapsto \quad (xy, xz, yz, y^2 - z^2)$$

It is not hard to check that this is a well-defined injective function

$\implies$ $\mathbb{P}^2$ is homeomorphic to the image of this map in $\mathbb{R}^4$

Remark

We will soon see that every non-orientable surface can be constructed using projective planes, so this implies that every non-orientable surface embeds in $\mathbb{R}^4$

In contrast, every orientable surface embeds in $\mathbb{R}^3$

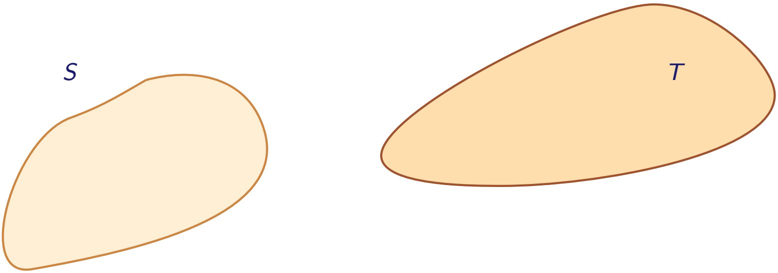
Connected sums

We need a way to build new surfaces from old surfaces

The boundary of a surface is the union of its boundary circles, or free edges. The interior of a surface is anything not on the boundary

Definition

The connected sum of surfaces S and T is the surface $S \# T$



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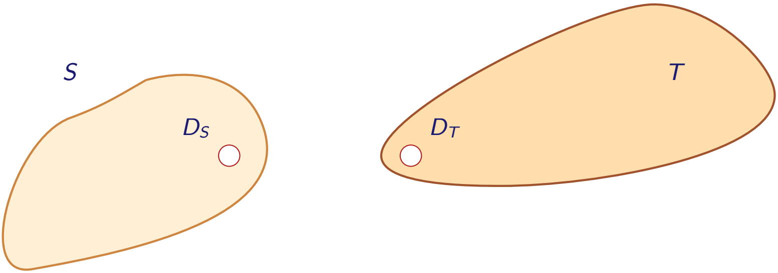

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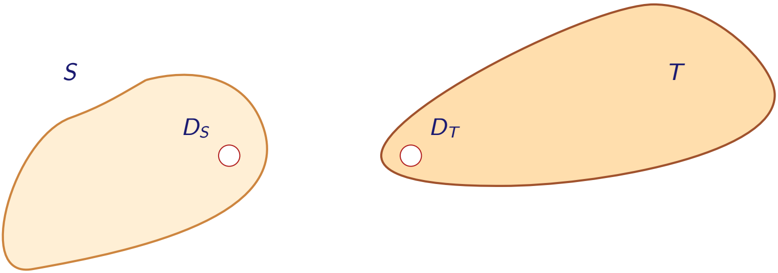
2. identifying the boundary circles of D_S and D_T



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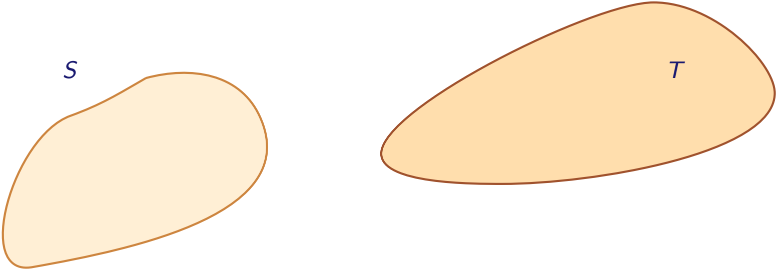

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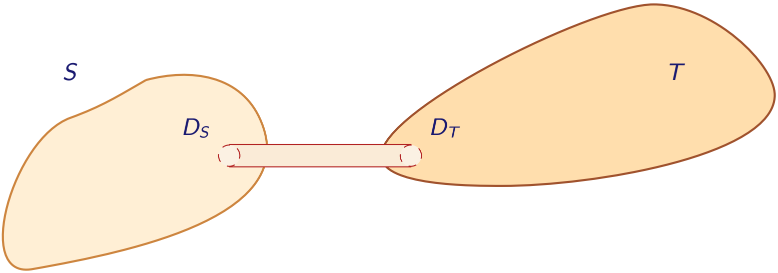
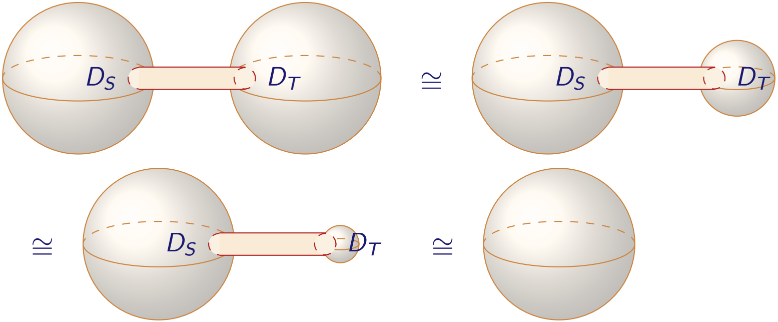
2. identifying the boundary circles of D_S and D_T



Identifying D_S and D_T is the same as connecting them with a cylinder

Connected sums with spheres

- What is $S^2 \# S^2$?



Hence, $S^2 \# S^2 \cong S^2$

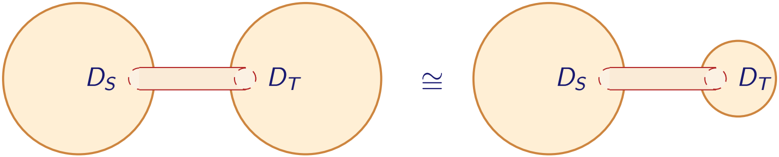
- If T is any surface then $T \# S^2 \cong T$

  This follows by exactly the same calculation!

So S^2 is the unit under the operation $\#$

Connected sums with disks

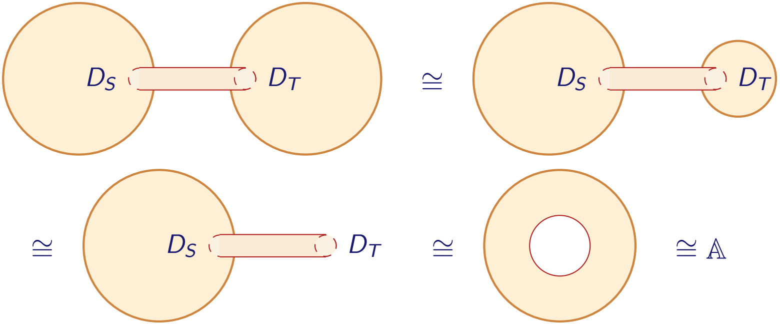
- What is $\mathbb{D}^2 \# \mathbb{D}^2$?



This is not the same as collapsing a sphere, which closes up the hole, because the disk has a boundary!

Connected sums with disks

- What is $\mathbb{D}^2 \# \mathbb{D}^2$?



  Hence, $\mathbb{D}^2 \# \mathbb{D}^2 \cong \mathbb{A}$, which is the annulus or cylinder

- If T is any surface then $T \# \mathbb{D}^2$ puts a puncture, or hole, in T

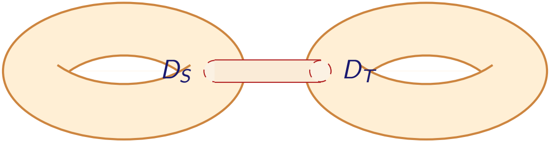
  This follows by exactly the same calculation!

  $$\implies \quad T \# \underbrace{\mathbb{D}^2 \# \ldots \# \mathbb{D}^2}_{d \text{ times}} = T \# \#^d \mathbb{D}^2 \text{ is equal to } T \text{ with } d$$

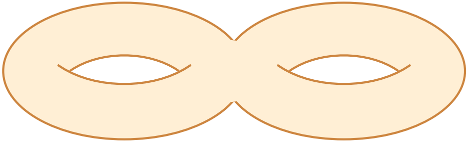
  punctures or, equivalently, T with d additional boundary circles

Connected sums with tori

- What is $\mathbb{T} \# \mathbb{T}$?

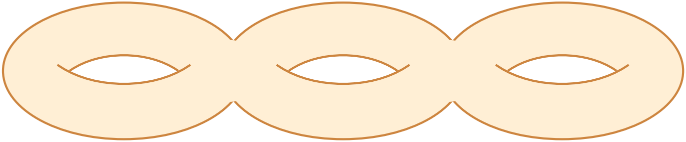


$\cong$



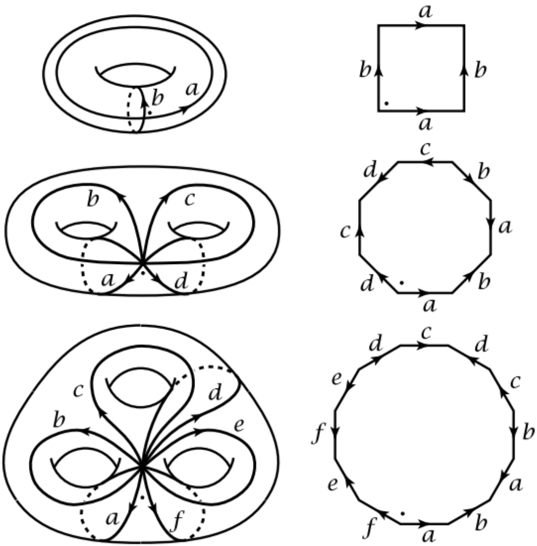
The double torus
$\mathbb{T} \# \mathbb{T} = \#^2\mathbb{T}$

Similarly, there are triple tori $\#^3\mathbb{T}$



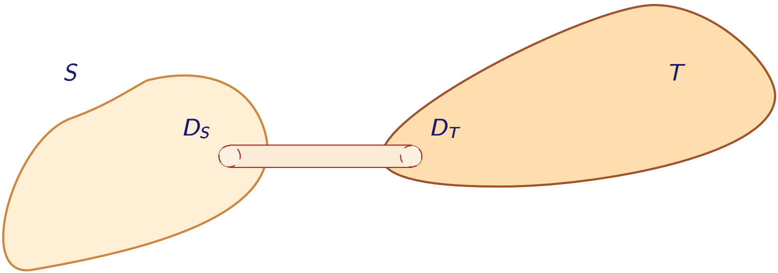
…and, more generally, t-tori $\#^t\mathbb{T}$

We already know t-tori



Properties of connected sums

- $S \# T$ is independent of the location of the disks D_S and D_T



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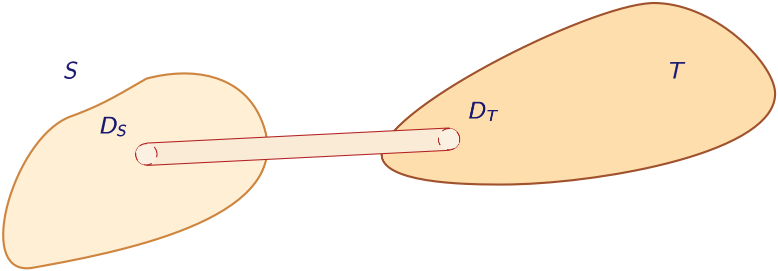

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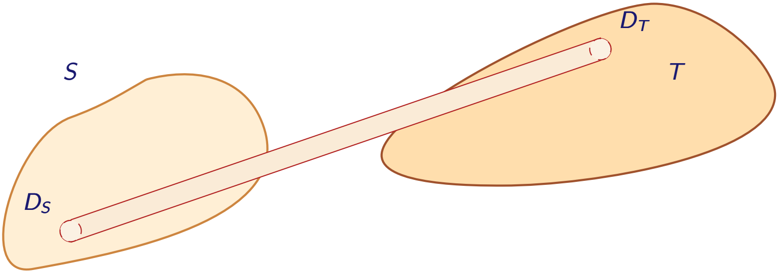

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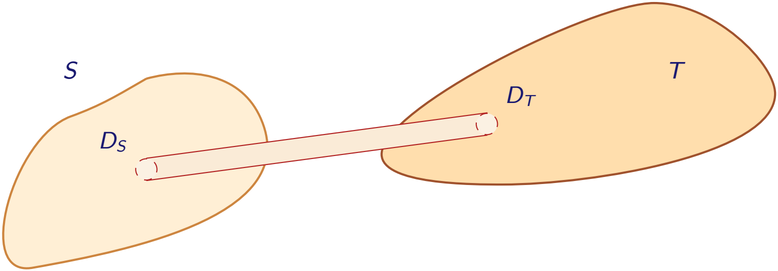
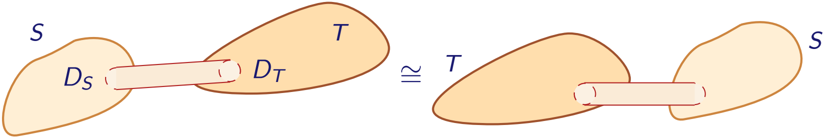


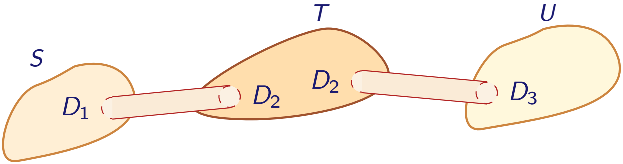
As long as D_S stays in the interior of S, and D_T in the interior of T, the surface $S \# T$ is unchanged up to homeomorphism

- $S \# T \cong T \# S$

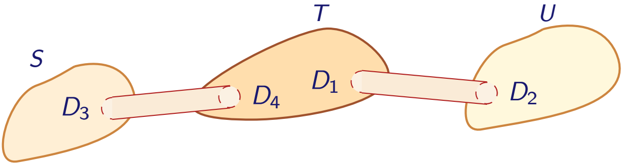


Associativity of connected sums...

- $S \# (T \# U) \cong (S \# T) \# U$



$\cong$



In these diagrams, D_1 and D_2 are cut first and then D_3 and D_4

$\Longrightarrow$ $\#$ is a "surface addition or multiplication"

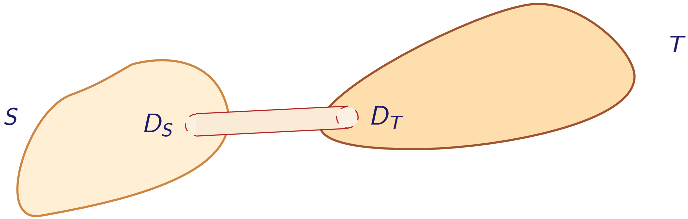
Connected sums of Euler characteristic

Theorem

Let S and T be surfaces with polygonal decompositions. Then

$$\chi(S \# T) = \chi(S) + \chi(T) - 2$$

Proof



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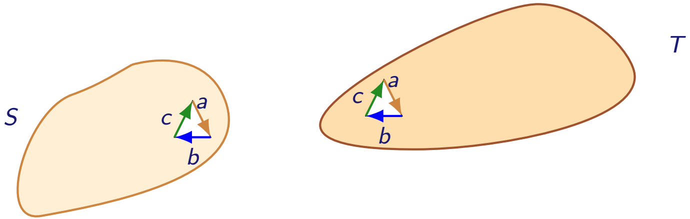


$$\implies \quad \chi(S \# T) = \big(\chi(S) - (3 - 3 + 1)\big) + \big(\chi(T) - (3 - 3 + 1)\big)$$

Moral The -2 comes from cutting out two disks

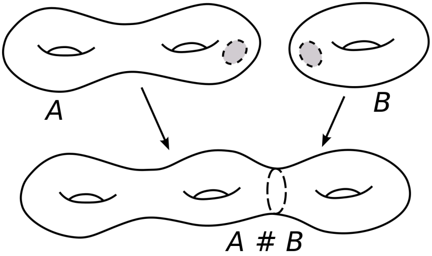
Examples

- If S is any surface then $S \cong S \# S^2$

$$\Longrightarrow \quad \chi(S) = \chi(S) + \underbrace{\chi(S^2)}_{=2} - 2 = \chi(S)$$

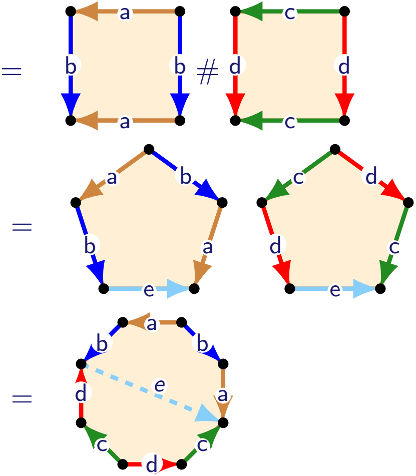
- $\mathbb{A} \cong \mathbb{D}^2 \# \mathbb{D}^2 \quad \Longrightarrow \quad \chi(\mathbb{A}) = \chi(\mathbb{D}^2) + \chi(\mathbb{D}^2) - 2 = 1 + 1 - 2 = 0$

- $\chi(\mathbb{T} \# \mathbb{T} \# \mathbb{T}) = (\chi(\mathbb{T}) + \chi(\mathbb{T}) - 2) + \chi(\mathbb{T}) - 2 = -4$



Connected sums and polygonal decompositions

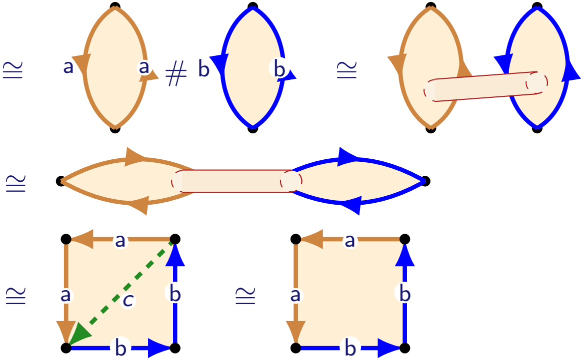
$\mathbb{T} \# \mathbb{T} =$



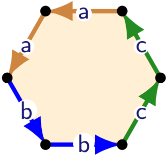
$\implies$ For surfaces without a boundary you can cut the disks anywhere!

- What is $\mathbb{P}^2 \# \mathbb{P}^2$?

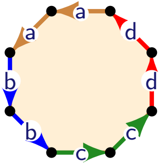
$\mathbb{P}^2 \# \mathbb{P}^2 \cong$



Similarly, $\#^3\mathbb{P}^2 \cong$



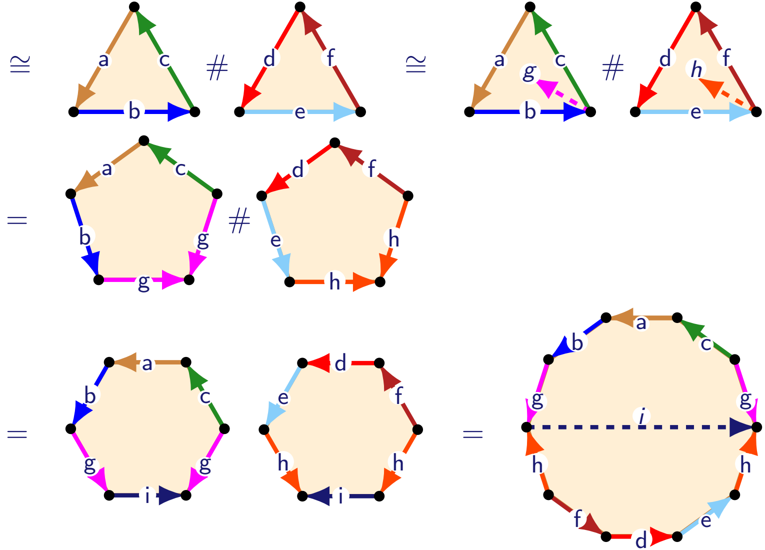
, $\#^4\mathbb{P}^2 \cong$



, ...

Connected sums and polygonal decompositions...

$\mathbb{D}^2 \# \mathbb{D}^2$



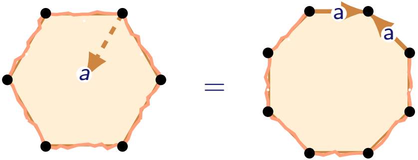
$\Longrightarrow$ For surfaces with a boundary, you can cut into the interior, if necessary, to form the connected sum

Surgery

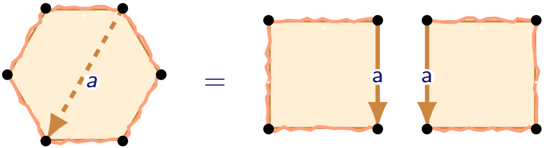
We have already seen that it is possible to change one polygonal decomposition into another using surgery

There are two basic operations:

- Adding and removing edges:



- Cutting and gluing



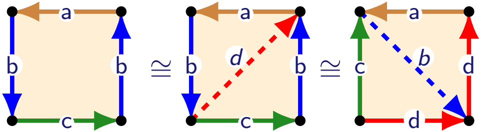
Perhaps surprisingly, these two operations and subdivision are all that we need

Lemma

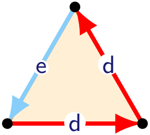
$\mathbb{M} \cong \mathbb{D}^2 \# \mathbb{P}^2$ (= *a punctured projective plane*)

Proof

$\mathbb{M} =$



$\cong$



where $e = a\bar{c}$

$\cong \mathbb{D}^2 \# \mathbb{P}^2$

$\Longrightarrow$ A Möbius strip is a punctured projective plane

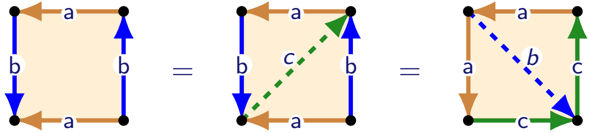
$\Longrightarrow$ Every non-orientable surface contains the projective plane

Lemma

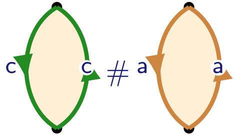
$$\mathbb{K} \cong \mathbb{P}^2 \# \mathbb{P}^2 \cong \#^2 \mathbb{P}^2$$

Proof

$\mathbb{K} \quad =$



$\cong$

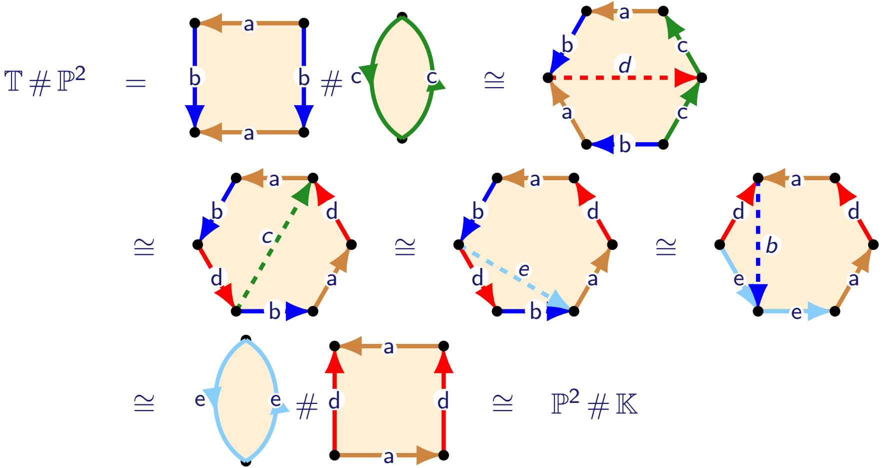


$$\cong \quad \mathbb{P}^2 \# \mathbb{P}^2$$

Theorem

$$\mathbb{T} \# \mathbb{P}^2 \cong \mathbb{K} \# \mathbb{P}^2$$

Proof



Projective planes dominate

On the last slide we saw that $\mathbb{T} \# \mathbb{P}^2 \cong \mathbb{K} \# \mathbb{P}^2$

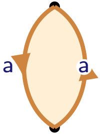
$\Longrightarrow$ $\mathbb{T} \# \mathbb{P}^2 \cong \#^3 \mathbb{P}^2$ since $\mathbb{K} \cong \#^2 \mathbb{P}^2$

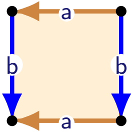
$\rightsquigarrow$ suggests that the connected sum of any surface with a projective plane is non-orientable

Warning Connected sums do not cancel since $\mathbb{T} \not\cong \mathbb{K}$

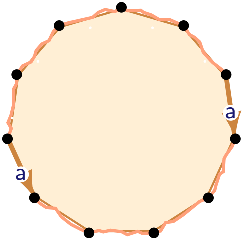
Why? $\mathbb{T}$ embeds in $\mathbb{R}^3$ but $\mathbb{K}$ does not!

Oriented and unoriented edges

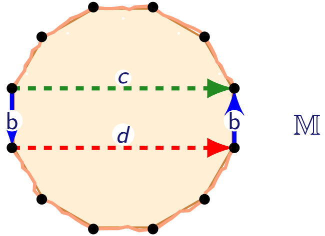
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

Paired edges on a polygon are oriented if they point in opposite directions and unoriented if they point in the same direction



Oriented



Unoriented

Oriented edges can be folded together without twisting whereas unoriented edges can only be brought together if the surface is twisted

Classification of connected surfaces

Theorem

Let S be a connected surface. Then there exist non-negative integers d, p and t such that

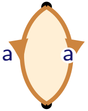
1. $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$
2. *the boundary of S is the disjoint union of d circles*
3. *S is orientable if and only if $p = 0$*

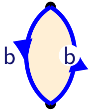
Moreover, we can assume that $pt = 0$, in which case S is uniquely determined up to homeomorphism by (d, p, t)

Remark If $d + p + t \neq 0$ we can omit the sphere S^2

Proof We argue by induction on the number of edges in a polygonal decomposition of S with one face to first prove ➊

Base case: If S has one edge then either

$S =$ 

$\cong S^2$ or $S =$ 

$\cong \mathbb{P}^2$

$\Longrightarrow$ The theorem is true in this case

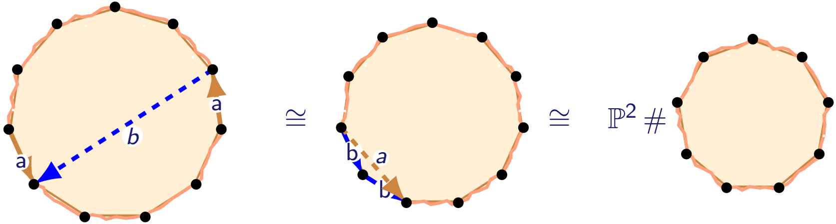
Proof of the classification theorem

Now suppose that S has at least two edges and that the theorem is true whenever all surfaces that have a polygonal decomposition with one face and fewer edges

If S has only free edges then $S \cong \mathbb{D}^2$ and the theorem holds

Hence, we can assume that S has at least one paired edge

Case I: S has an unoriented edge



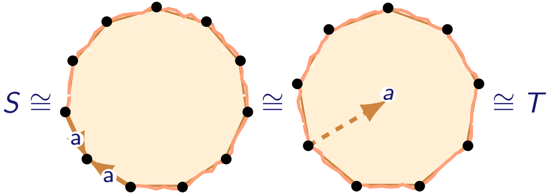
$$\implies \quad S \quad \cong \quad \mathbb{P}^2 \# T$$

By induction, $T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ since T has fewer edges

$$\implies \quad S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^{p+1} \mathbb{P}^2 \# \#^t \mathbb{T}$$ as required

Case II: All paired edges in S are oriented

If S has adjacent oriented edges then



$\Longrightarrow \quad S \cong T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ by induction

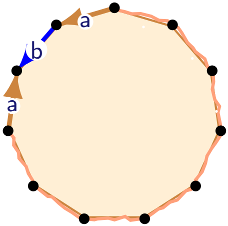
Hence, we can assume that the paired edges are not adjacent

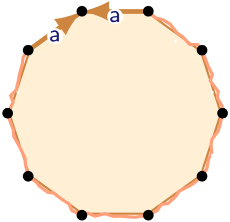
Similarly, we can assume that S does not have any adjacent free edges as such edges can be replaced with a single free edge

Fix an (oriented) paired edge a such that the number of edges between the two copies of a is minimal

Proof of the classification theorem...

Case IIa: All edges on one side of a are free

$\Longrightarrow \quad S \quad \cong$ 

$\cong \quad \mathbb{D}^2 \#$ 

$\cong \quad \mathbb{D}^2 \# T$

By induction, $T \cong S^2 \# \#^d\mathbb{D}^2 \# \#^p\mathbb{P}^2 \# \#^t\mathbb{T}$

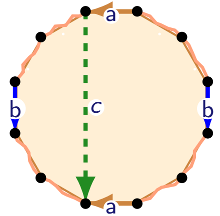
$$\Longrightarrow \quad S \cong \mathbb{D}^2 \# T \cong S^2 \# \#^{d+1}\mathbb{D}^2 \# \#^p\mathbb{P}^2 \# \#^t\mathbb{T}$$

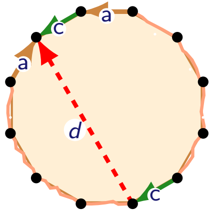
Hence, we can assume that there are paired edges on both sides of a

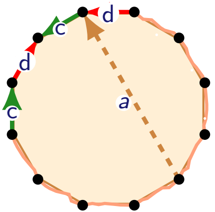
Proof of the classification theorem...

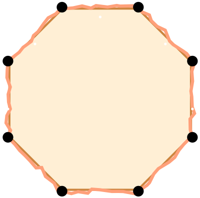
Case IIb: There are paired edges on both sides of a

The number of edges between the ends of a is minimal, so

$\Longrightarrow \quad S \quad \cong$ 

$\cong$ 

$\cong$ 

$\cong \quad \mathbb{T}\,\#$  $\cong \mathbb{T}\,\#\,U$

By induction, $U \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$

$\Longrightarrow \quad S \cong \mathbb{D}^2 \# U \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^{t+1} \mathbb{T}$

Proof of the classification theorem...

We have now proved that every surface can be written in the form

$$S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$$

for non-negative integers d, p and t

The proof so far shows that d is the number of boundary circles

Next, note that if $p > 0$ then $\mathbb{P}^2$ is contained in S

$\implies$ S is non-orientable if $p \neq 0$

On the other hand, $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T} \hookrightarrow \mathbb{R}^3$ is orientable if $p = 0$

$\implies$ S is orientable if and only if $p = 0$

We have now proved ❶, ❷ and ❸ from the theorem!

Next, observe that if $p \neq 0$ and $t \neq 0$ then S contains $\mathbb{P}^2 \# \mathbb{T} \cong \#^3 \mathbb{P}^2$

$\implies$ $\#^t \mathbb{T} \# \mathbb{P}^2 \cong \#^{t-1} \mathbb{T} \# \#^3 \mathbb{P}^2 \cong \ldots \cong \#^{2t+1} \mathbb{P}^2$

$\implies$ Hence, we can assume $t = 0$ if $p \neq 0$

That is, we can assume $pt = 0$ — equivalently, $p = 0$ or $t = 0$

Proof of the classification theorem...

It remains to prove if $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ with $tp = 0$ then S is uniquely determined up to homeomorphism by (d, p, t)

Let $T = S^2 \# \#^e \mathbb{D}^2 \# \#^q \mathbb{P}^2 \# \#^s \mathbb{T}$, with $sq \neq 0$

$\Longrightarrow$ We need to show that $S \cong T$ if and only if $(d, p, t) = (e, q, s)$

If $(d, p, t) = (e, q, s)$ there is nothing to prove, so suppose $S \cong T$

- $d = e$ as homeomorphism preserve boundary circles
- $p \neq 0 \Leftrightarrow q \neq 0$ as homeomorphisms preserve orientability
- Homeomorphisms preserve Euler characteristic. By tutorial 9,
  - $\chi(S^2 \# \#^a \mathbb{D}^2 \# \#^b \mathbb{P}^2) = 2 - a - b$
  - $\chi(S^2 \# \#^a \mathbb{D}^2 \# \#^c \mathbb{T}) = 2 - a - 2c$

$\Longrightarrow$ $(d, p, t) = (e, q, s)$ since $\chi(S) = \chi(T)$

All parts of the classification theorem are now proved!!

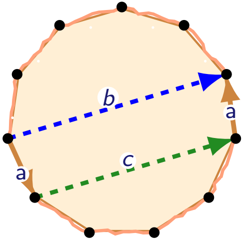
Hence, we now know all surfaces up to homeomorphism!

Orientability

Corollary

A surface S is non-orientable if and only if its polygonal decomposition contains an unoriented edge

Proof Any unoriented edge gives a Möbius band inside S:



Conversely, $S = S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T}$ embeds in $\mathbb{R}^3$, so it is orientable.
Hence, a polygonal decomposition of S can only contain oriented edges

It is now not hard to find an explicit polygonal decomposition of

$$S = S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T}$$

and check that surgery cannot create unoriented edges in S

Standard forms

Theorem

Let S be a connected surface. Then there exist non-negative integers d, p and t with $pt = 0$ such that

1. $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$
2. *the boundary of S is the disjoint union of d circles*
3. *S is orientable if and only if $p = 0$*

The surface S is in standard form when written as

$$S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$$

with $pt = 0$ — that is, $p = 0$ or $t = 0$

- The standard form uniquely identifies S
- S is orientable if and only if $p = 0$
- S has d boundary circles
- S has Euler characteristic $\chi(S) = 2 - d - p - 2t$ (tutorials!)

The standard form of a surface that is not connected has each component in standard form

Corollary of classification

Corollary

A connected surface is uniquely determined, up to homeomorphism by

1. *the number of boundary circles*
2. *its orientability*
3. *its Euler characteristic*

Proof Write $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ in standard form with $tp = 0$

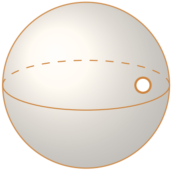
$$\implies \quad \chi(S) = 2 - d - p - 2t$$

Hence, the standard form uniquely determines the number of boundary circles, orientability and Euler characteristic of S

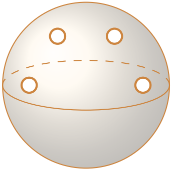
Conversely, these three characteristics of S determine (d, p, t)

- $S^2 \# \#^d \mathbb{D}^2$ is a sphere with d punctures

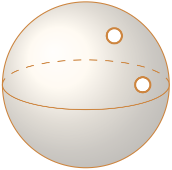
$S^2 \# \mathbb{D}^2 \quad =$



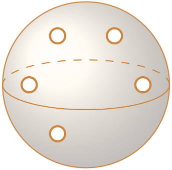
$S^2 \# \#^4 \mathbb{D}^2 \quad =$



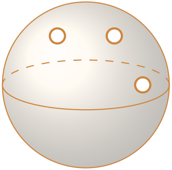
$S^2 \# \#^2 \mathbb{D}^2 \quad =$



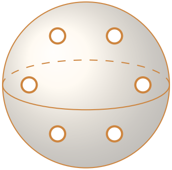
$S^2 \# \#^5 \mathbb{D}^2 \quad =$



$S^2 \# \#^3 \mathbb{D}^2 \quad =$



$S^2 \# \#^6 \mathbb{D}^2 \quad =$

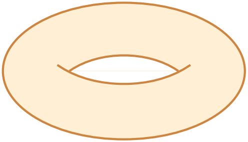
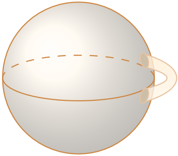


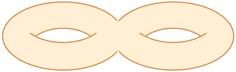
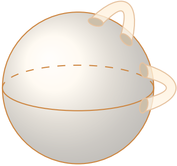
More generally, $S \# \#^d \mathbb{D}^2$ is S with d punctures

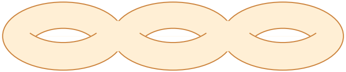
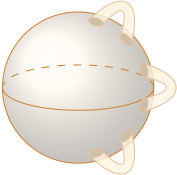
A spheres with zero and one puncture

Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

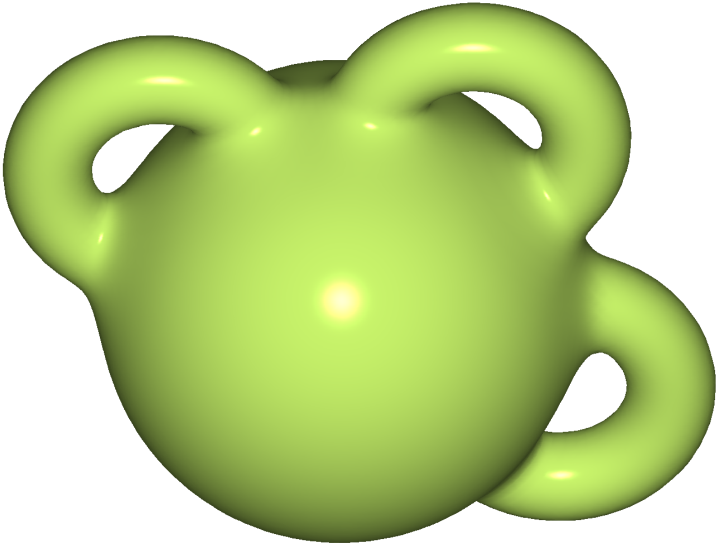
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^3 \mathbb{T} \quad \cong$  $\cong$ 

Continuing like this constructs a sphere with t-handles $\#^t \mathbb{T}$

Handle decomposition

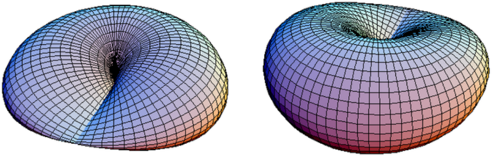


Sphere with cross-caps

- $S^2 \# \#^p \mathbb{P}^2$ is a sphere with p cross-caps

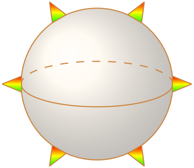
A cross-cap is what you get when you sew a Möbius strip onto the sphere

This shape lives in $\mathbb{R}^4$, so difficult to visualize but Wikipedia draws it as:



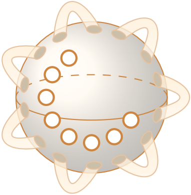
In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^6 \mathbb{P}^2 \quad \cong$$

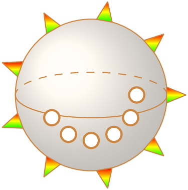


We can combine the pictures above to draw all of the standard surfaces:

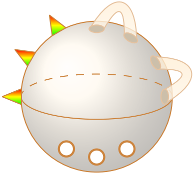
$$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \quad \cong$$



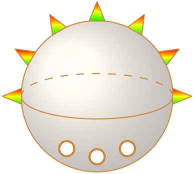
$$\#^6\mathbb{D}^2 \# \#^9\mathbb{P}^2 \quad \cong$$



$$\#^3\mathbb{D}^2 \# \#^2\mathbb{T} \# \#^3\mathbb{P}^2 \quad \cong$$



$$\cong$$



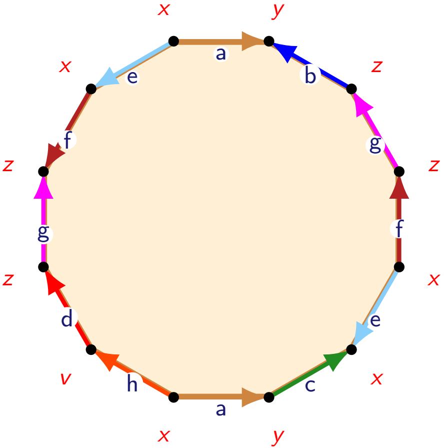
Putting a surface in standard form

Given a polygonal decomposition for a surface we can put it in standard form by:

- Find all of the vertices (identified edges implicitly identify vertices)
- Count the number d of boundary circles
- S is orientable $(p = 0)$ if all edges are oriented otherwise it is non-orientable $(t = 0)$
- Compute $\chi(S) = 2 - d - p - 2t$ to determine the missing variable, which is t if S is orientable and or p if non-orientable

Example 1

What is the surface with the below polygonal decomposition?



$a\,c\,\overline{e}\,f\,g\,b\,\overline{a}\,e\,f\,\overline{g}\,\overline{dh}$ (overline=opposite direction)

$\Longrightarrow$ This is $\#^1\mathbb{D}^2 \# \#^0\mathbb{T} \# \#^4\mathbb{P}^2$

Topology – week 10

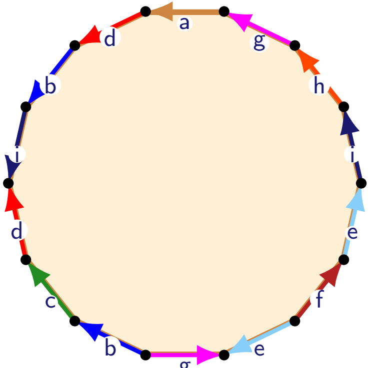
Math3061

Daniel Tubbenhauer, University of Sydney

Words for surfaces

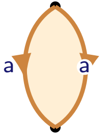
A polygonal decomposition for a surface that has one face can be encoded in a word

$$a\ d\ b\ i\ \overline{d}\ \overline{c}\ \overline{b}\ g\ \overline{e}\ f\ e\ i\ h\ g$$

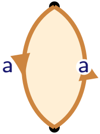


- write x for an edge pointing anticlockwise
- write $\overline{x}$ for an edge pointing clockwise
- We always read the word in anticlockwise order

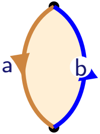
Words for basic surfaces

- $S^2 =$ 

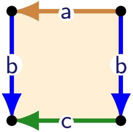
$= a\ \overline{a}$

- $\mathbb{P}^2 =$ 

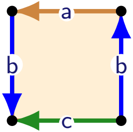
$= a\ a$

- $\mathbb{D}^2 =$ 

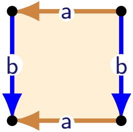
$= a\ b$

- $\mathbb{A} =$ 

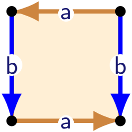
$= a\ b\ \overline{c}\ \overline{b}$

- $\mathbb{M} =$ 

$= a\ b\ \overline{c}\ b$

- $\mathbb{T} =$ 

$= a\ b\ \overline{a}\ \overline{b}$

- $\mathbb{K} =$ 

$= a\ b\ a\ \overline{b}$

Properties of words

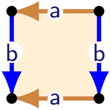
- Words encode orientability
  - Orientable: $\dots a \dots \overline{a} \dots$ or $\dots \overline{a} \dots a \dots$
  - Non-orientable: $\dots a \dots a \dots$ or $\dots \overline{a} \dots \overline{a} \dots$
- Words give a compact and easily readable way of describing surfaces
- Words can be read in clockwise or anticlockwise order (we always read in anticlockwise order)
- The word of a surface is well-defined only up to cyclic permutation and reversing the direction of any edge

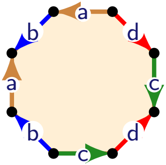
Example The following words are all words for the torus $\mathbb{T}$:

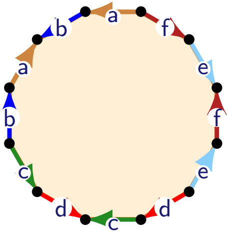
$$\begin{array}{cccc} a\,b\,\overline{a}\,\overline{b} & b\,\overline{a}\,\overline{b}\,a & \overline{a}\,\overline{b}\,a\,b & \overline{b}\,a\,b\,\overline{a} \\ a\,\overline{b}\,\overline{a}\,b & \overline{b}\,\overline{a}\,b\,a & \overline{a}\,b\,a\,\overline{b} & b\,a\,\overline{b}\,\overline{a} \end{array}$$

- The word of a surface can be used to give generators and relations for the first homotopy group of the surface — this generalises independent cycles and are beyond the scope of this unit

Standard words for closed orientable surfaces

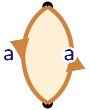
- Connected sums of tori: $\#^t\mathbb{T}$
  - $\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}$

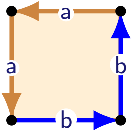
  - $\#^2\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d}$

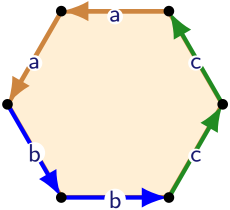
  - $\#^3\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d}\ e\ f\ \overline{e}\ \overline{f}$

  - … $\#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \ldots\ a_t\ b_t \overline{a_t}\ \overline{b_t}$

Words for closed non-orientable surfaces

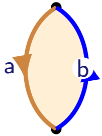
- Connected sums of projective plans $\#^p\mathbb{P}^2$
  - $\mathbb{P}^2 =$ 

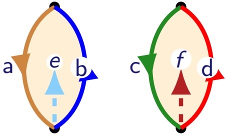
$= a\ a$
  - $\#^2\mathbb{P}^2 =$ 

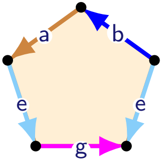
$= a\ a\ b\ b$
  - $\#^3\mathbb{P}^2 =$ 

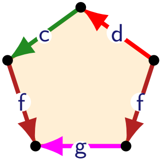
$= a\ a\ b\ b\ c\ c$
  - … $\#^p\mathbb{P}^2 = a_1\ a_1\ a_2\ a_2\ \dots\ a_p\ a_p$

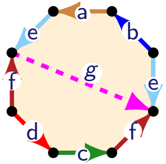
Standard words for surfaces with boundary

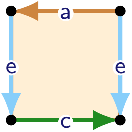
- $\#^d\mathbb{D}^2$
  - $\mathbb{D}^2 =$ 

$= a\ b$
  - $\#^2\mathbb{D}^2 =$ 

$\cong$ 

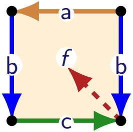
$\#$ 

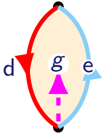
$\cong$ 

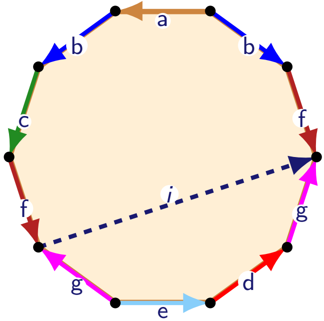
$\cong$ 

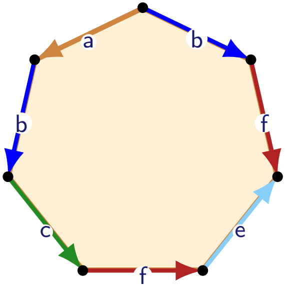
$= a\ e\ c\ \overline{e}$

Standard words for surfaces with boundary

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

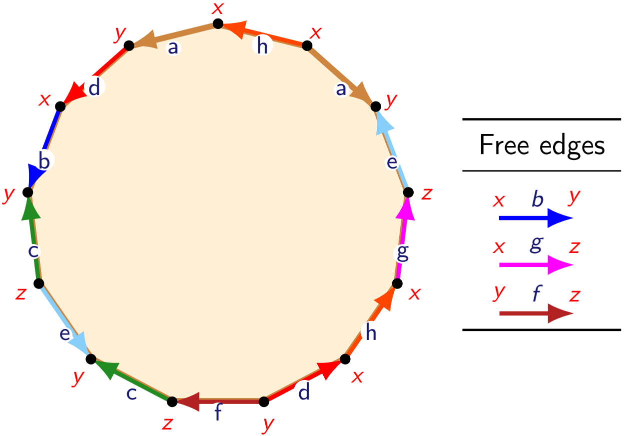
  $\cong$ 

$\cong$ 

  $= a\ b\ c\ f\ e\ \overline{f}\ \overline{b}$

- $\#^d\mathbb{D}^2 = a_1\ b_1\ a_2\ b_2\ \ldots\ b_{d-1} a_d\ \overline{b}_{d-1}\ \ldots \overline{b}_2\ \overline{b}_1$

What standard surface is given by the word $a\ d\ b\ \overline{c}\ e\ \overline{c}\ \overline{f}\ d\ h\ g\ e\ \overline{a}\ h$?



$\Longrightarrow \quad d = 1$ and $\chi(S) = 3 - 8 + 1 = -4$

$\Longrightarrow \quad S \cong \mathbb{D}^2 \# \#^5 \mathbb{P}^2$

$\Longrightarrow \quad S = a\ b\ b\ c\ c\ d\ d\ e\ e\ f\ f$

The vertex-degree equation revisited

When we looked at graphs we proved the vertex-degree equation:

$$\sum_{v \in V} \deg(v) = 2|E| \qquad \text{for } G = (V, E) \text{ a graph}$$

The best way to understand this formula is to note that each edge $\{x, y\} \in E$ contributes 2 to both sides of this equation

- $+1$ to each of $\deg(x)$ and $\deg(y)$ on the left-hand side
- $+2 = 2 \cdot 1$ to the right-hand side for the edge $\{x, w\}$

We want similar formulas for a surface $S = (V, E, F)$ with a polygonal decomposition

Question What is the correct definition of degree in S ?

The problem

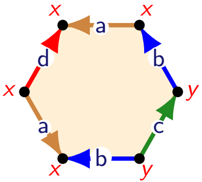
We are identifying edges in S and hence implicitly identifying vertices

- Do we identify edges and vertices when computing $\deg(v)$ and $|E|$?

Answer Yes and no!

The degree of a vertex

Consider the surface with polygonal decomposition



Using identified vertices and edges + count with multiplicities

$\Longrightarrow$ $\deg(x) = 5$, $\deg(y) = 3$, so $\deg(x) + \deg(y) = 8 = 2|E|$

Not using identified edges or vertices (i.e. as a graph, ignoring the face)

$\Longrightarrow$ six vertices of degree 2 and six edges, so $12 = 2 \cdot 6$

The vertex-degree equation holds using either identified or non-identified edges and vertices because in both cases the degree of a vertex is defined to be the number of incident edges to the vertex

The surface degree-vertex equation

Proposition

Let $S = (V, E, F)$ be a surface with polygonal decomposition. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof The proof is the same as before: the edge $\{x, y\}$ contributes $+2$ to both sides of this equation because edge contributes $+1$ to $\deg(x)$ and $+1$ to $\deg(y)$.

Therefore, we have two degree-vertex equations:

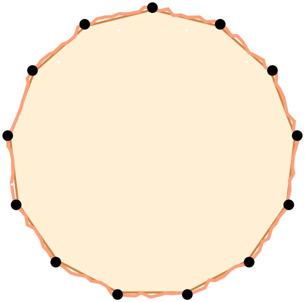
- The graph degree-vertex equation where we do not identify edges and vertices in S
- The surface degree-vertex equation where we do identify edges and vertices in S

The degree of a face

Let $S = (V, E, F)$ be a surface with polygonal decomposition

Let $f \in F$ be a face of S. The degree of f is
$$\deg(f) = \text{number of edges (count with multiplicities) incident with } f$$

Examples Suppose that $f \in F$ is an n-gon



$\Longrightarrow \quad \deg(f) = n$

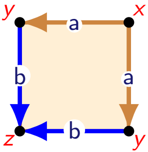
Notice that faces are never identified in the polygonal decomposition

Question How are $\sum \deg(f)$ and $2|E|$ related?

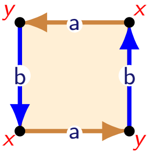
Face degrees of basic surfaces

In all cases $\deg(\text{face}) = 4$ as there are 4 non-identified edges

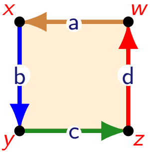
- Sphere

$S^2 \cong$ 

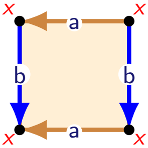
- Projective plane

$\mathbb{P}^2 \cong$ 

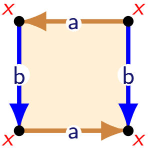
- Disk

$\mathbb{D}^2 \cong$ 

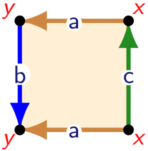
- Torus

$\mathbb{T} \cong$ 

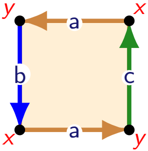
- Klein bottle

$\mathbb{K} \cong$ 

- Annulus

$\mathbb{A} \cong$ 

- Möbius band

$\mathbb{M} \cong$ 

The face-degree equation

Recall that for any graph $G = (V, E)$ we proved that $\sum_{v \in V} \deg(v) = 2|E|$

Let (V, E, F) be a polygonal decomposition

The degree of a face $f \in F$ is $\deg(f) = n$ if P is an n-gon

Proposition (The surface face-degree equation)

Let $S = (V, E, F)$ be a closed surface (no boundary). Then

$$\sum_{f \in F} \deg(f) = 2|E|,$$

Proof By definition, $\deg(f) = n$ if f is an n-gon

Since S is a closed surface, every edge meets two faces (potentially the same face), so it contributes $+2$ to both sides of this equation

$$\implies \quad \sum_{f \in F} \deg(f) = 2|E|$$

Remark To use this formula we need to know the number of identified edges in the polygonal decomposition

Dual surfaces

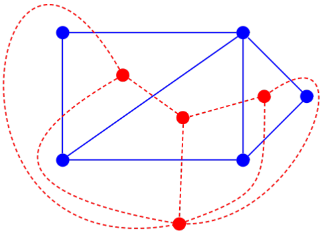
Let $S = (V, E, F)$ be a closed surface with a polygonal decomposition such that the vertices around each polygon are distinct

The dual surface S^* has polygonal decomposition (V^*, E^*, F^*), where

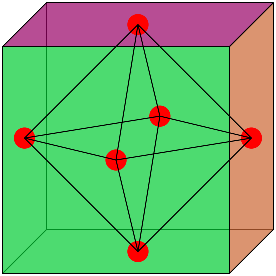
- the vertex set of S^* is $V^* = F$, the set of faces of S
- there is an edge between two vertices f and f' of S^* if the faces f and f' in S are separated by an edge

$\Longrightarrow$ the faces of S^* are the vertices of S

Examples



The dual of the cube



$\Longrightarrow$ the dual surface to the cube is the octahedron

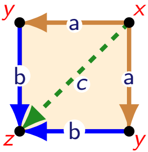
Dual surfaces and the degree equations

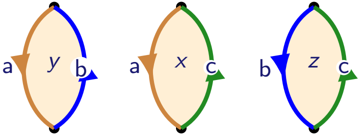
Taking the dual of a surface swaps the vertices and faces

$\implies$ if $v \in V$ then $v \in F^*$ and $\deg_S(v) = \deg_{S^*}(v)$

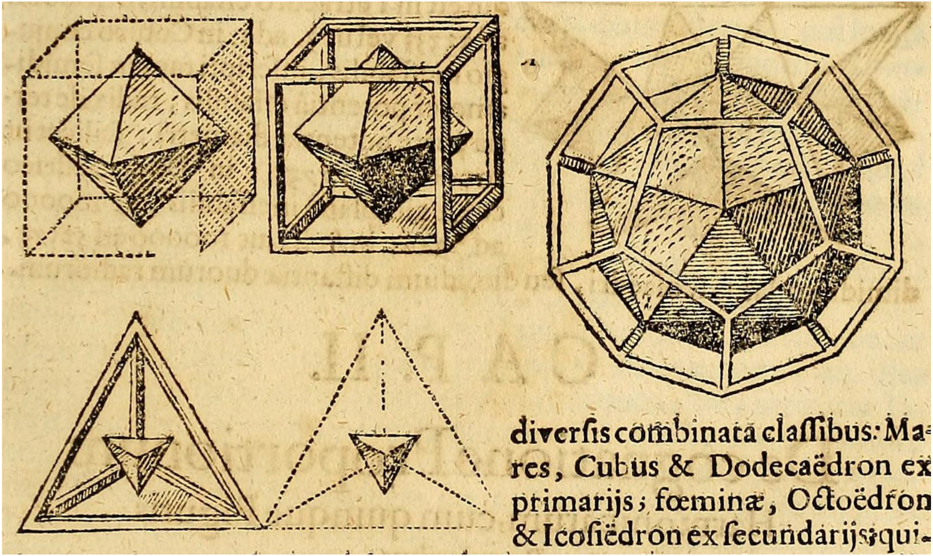
$\implies$ the vertex-degree equation for S is the same as the face-degree equation for S^*

Example

$S^2 =$ 

$(S^2)^* =$ 

We will see better examples when we look at Platonic solids


diverſis combinata claſſibus: Ma-
res, Cubus & Dodecaëdron ex
primarijs; fœminæ, Octoëdron
& Icoſiëdron ex ſecundarijs; qui-

Graphs on surfaces

Recall that a graph G is planar if it can be drawn in $\mathbb{R}^2$ without edge crossings

Let S be a surface and $x, y \in S$. A path from x to y on S is a continuous map $p\colon[0,1]\longrightarrow S$ such that $p(0) = x$ and $p(1) = y$

Let $\mathscr{P}(S)$ be the set of all paths on S

If S is any surface and $G = (V, E)$ is a graph then an embedding of G in S is a pair of maps

$$f\colon V \longrightarrow S \quad \text{and} \quad \mathrm{p}\colon E \longrightarrow \mathscr{P}(S)$$

such that:

- The map f is injective
- If $e = \{v, w\} \in E$ then $\mathrm{p}(e) \in \mathscr{P}(S)$ is an injective path from $f(v)$ to $f(w)$
- If $e, e' \in E$ then the paths $F(e)$ and $F(e')$ can intersect only at the images of their endpoints

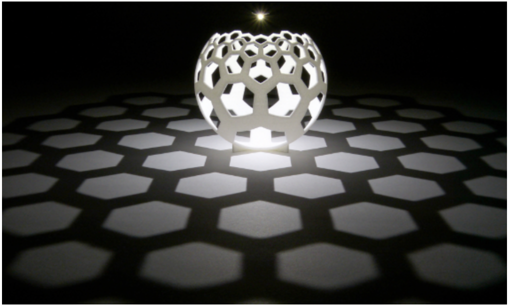
Planar graphs

Theorem

Let G be a (finite) graph. Then the following are equivalent.

1. *There is an embedding of G in $\mathbb{R}^2$ (= the graph is planar)*
2. *There is an embedding of G in $\mathbb{D}^2$*
3. *There is an embedding of G in S^2*

Proof Stereographic projection! (Move G away from ∞.)



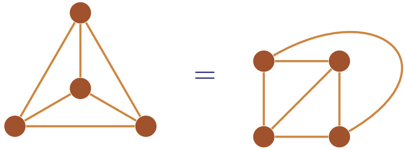
Faces of embedded graphs

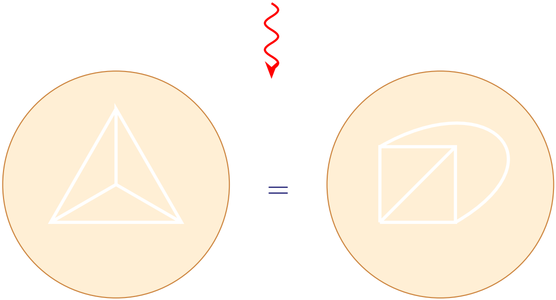
Suppose that G has an embedding on a surface S

Identify G with its image in S

The faces of G are the connected components of $S \setminus G$

Example Taking $S = \mathbb{D}^2$ and $G = K_4$ gives four faces:

$K_4 =$ 



Planar graphs and polygonal decompositions

Theorem

*Let G be a connected planar graph without leaves.
Then G gives a polygonal decomposition of S^2 where the polygons correspond to the non-trivial cycles in G*

Proof Since G is connected, and S^2 does not have a boundary, $S^2 \setminus G$ is a disjoint union of a finite number of regions each of which is bounded by a non-trivial cycle in G.

Every vertex v in G has degree at least 2 and, by assumption, every edge is included in a non-trivial cycle in G

$\Longrightarrow$ there are two faces adjacent to every edge in G

$\Longrightarrow$ the embedding of G in S^2 induces a polygonal decomposition on S^2

Remark The argument cheats slightly because we are implicitly assuming that the edges are "nice" curves. This allows us to side-step issues connected with the Jordan curve theorem

Planar graphs and Euler characteristic

Theorem

Let $G = (V, E)$ be a connected planar graph with face set F. Then $2 = |V| - |E| + |F|$

Proof Use the previous theorem or argue by induction on $|E|$

Case 1 G is a tree

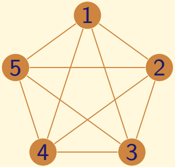
Combine $|V| - |E| = 1$ (previous lectures) and that there is only one face

Case 2 G is not a tree

By $\chi(S^2) = 2$ and the previous theorem

Planarity of K_5

Proposition

The complete graph $K_5 =$ 

is not planar

Proof Assume that K_5 is planar with $|F|$ faces

We have $|V| = 5$ and $|E| = 10$, so $2 = |V| - |E| + |F| \implies |F| = 7$

Let's count the number of faces in this polygonal decomposition differently

- The faces correspond to cycles in K_5
- Every face has at least 3 edges, so by the degree-face equation

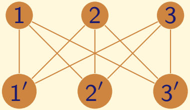
$$\implies \quad 2|E| = \sum_{f \in F} \deg(f) \geq 3|F|$$

$$\implies \quad 2|E| = 20 \geq 21 = 3|F| \qquad \lightning\lightning\lightning$$

Hence, the complete graph K_5 is not planar

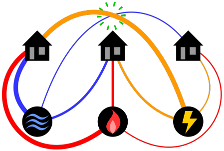
Planarity of bipartite graphs

Proposition

The bipartite graph $K_{3,3} =$ 

is not planar

Proof Tutorials



Theorem (Kuratowski)

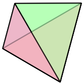
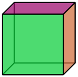
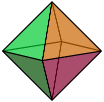
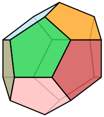
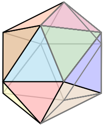
Let G be a graph. Then G if planar if and only if it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$

The proof is out of the scope of this unit!

Platonic solids

A Platonic solid is a surface that has a polygonal decomposition that is constructed using regular n-gons of the same shape and size such that the same number of polygons meet at every vertex

Examples

	Tetrahedron	Cube	Octahedron	Dodecahedron	Isosahedron
					
n	3	4	3	5	3
|V|	4	8	6	20	12
|E|	6	12	12	30	30
|F|	4	6	8	12	20

Questions

- Are there any others?
- Can we understand them as polygonal decompositions of the sphere?

Vertices, edges and faces of Platonic solids

Let P be a polygonal decomposition of S^2 obtained by gluing together (regular) n-gons so that p polygon meet at each vertex

Suppose there are $|V|$ vertices, $|E|$ edges and $|F|$ faces

$\Longrightarrow$ each vertex has degree p and each face degree n

$\Longrightarrow$ $p|V| = 2|E|$ by the vertex-degree equation

$\Longrightarrow$ $2|E| = n|F|$ by the face-degree equation

$\Longrightarrow$ $2 = \chi(S^2) = |V| - |E| + |F| = \frac{2|E|}{p} - |E| + \frac{2|E|}{n}$

$\Longrightarrow$ $\frac{1}{2} + \frac{1}{|E|} = \frac{1}{p} + \frac{1}{n}$

$\Longrightarrow$ $\frac{1}{p} + \frac{1}{n} = \frac{1}{2} + \frac{1}{|E|} > \frac{1}{2}$

We require $p \geq 3$, $n \geq 3$ and $|E| \geq 2$

The equations above give:

$$|E| = \left(\frac{1}{p} + \frac{1}{n} - \frac{1}{2}\right)^{-1}, \ |V| = \frac{2|E|}{p} \text{ and } |F| = \frac{2|E|}{n}$$

Classification of Platonic solids

Theorem

The complete list of Platonic solids is:

p	n	$\frac{1}{p}+\frac{1}{n}$	$e=\left(\frac{1}{p}+\frac{1}{n}-\frac{1}{2}\right)^{-1}$	$v=\frac{2e}{p}$	$f=\frac{2e}{n}$	*Platonic solid*
3	3	$\frac{2}{3}$	6	4	4	*Tetrahedron*
3	4	$\frac{7}{12}$	12	8	6	*Cube*
3	5	$\frac{8}{15}$	30	20	12	*Dodecahedron*
4	3	$\frac{7}{12}$	12	6	8	*Octahedron*
5	3	$\frac{8}{15}$	30	12	20	*Isosahedron*

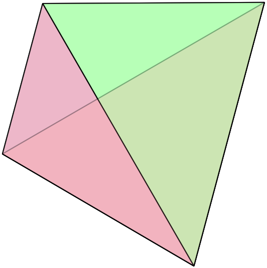
Proof Since $\frac{1}{p}+\frac{1}{n}>\frac{1}{2}$ and $p, n \geq 3$ we get $n<6$ since $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$
Case-by-case we then get the above values for p, n as the only possible values for Platonic solids.

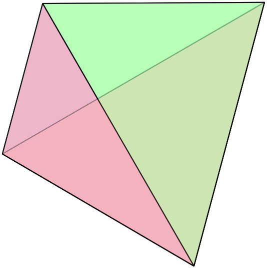
To prove existence we need to actually construct them

Proof Continued Their construction is well-known:

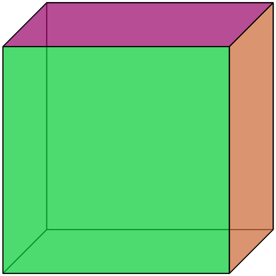
Dual tetrahedron = tetrahedron

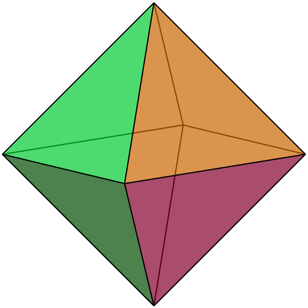
There is a symmetry in the Platonic solids given by $(p, n) \leftrightarrow (n, p)$. This corresponds to taking the dual surface



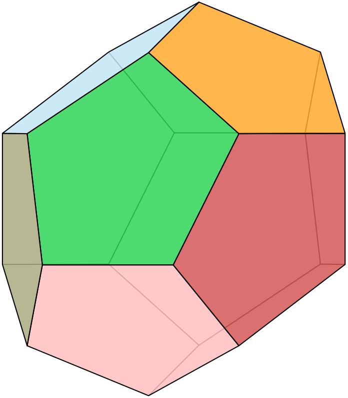


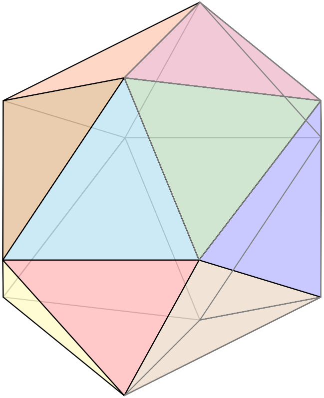
Cube and octahedron





Dodecahedron and icosahedron





Platonic soccer balls

Here are two dodecahedral decompositions of S^2

Soccer ball

Example A ball is made by gluing together triangles and octagons so that each octagon is connected to four non-touching triangles. Determine the number of octagons and triangles used

Let there be $|V|$ vertices, $|E|$ edges and $|F|$ faces

Write $|F| = o + t$, where $o = \#$octagons and $t = \#$triangles

$$\implies \quad 2 = |V| - |E| + o + t$$

We have:

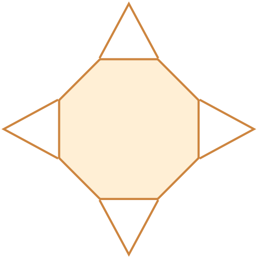
- vertex-degree equation: $3|V| = 2|E|$
- face-degree equation: $2|E| = 3t + 8o$
- Every octagon meets 4 triangles,

$\implies \quad 3t = 4o \quad \implies \quad 2|E| = 12o$

$\implies \quad 2 = o\left(4 - 6 + 1 + \frac{4}{3}\right) = \frac{o}{3}$

$\implies \quad o = 6$ and $t = 8$

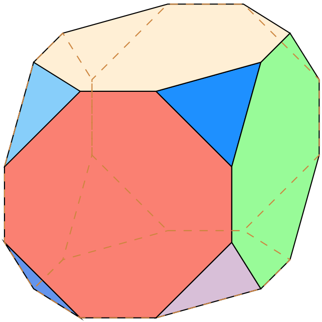
$\implies \quad |E| = 36$ and $|V| = 24$



The octacube

As with the Platonic solids, we have only shown that if such a surfaces exists then there are 6 octagons, 8 triangles, 24 vertices and 36 edges but we have not shown that such a surface exists!

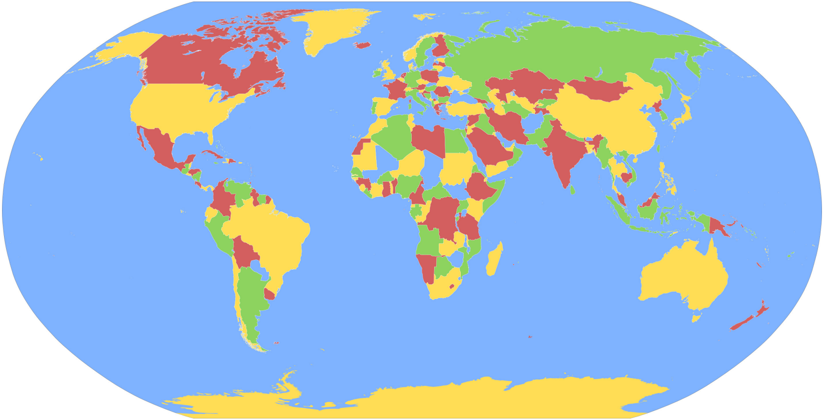
In fact, this surface does exist and it can be constructed by cutting triangular corners off a cube



Coloring maps

Question

How many different colors do you need to color a map so that adjacent countries have different colors?



A map is a polygonal decomposition. The answer to this question involves the same ideas we used to understand Platonic solids

Let $P = (V, E, F)$ be a polygonal decomposition of a surface S

Polygons in P are adjacent if they are separated by an edge

Let $C_P(S)$ be the minimum number of colours needed to colour the polygons in P such that adjacent polygons have different colors

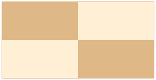
Definition

The chromatic number of S is $C(S) = \max\{\, C_P(S) \mid P \text{ is a “map” on } S \,\}$

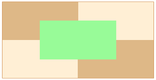
We still need to say what a map in in terms of polygonal decompositions

That is, $C(S)$ is the smallest number of colors that we need to be able to color any polygonal decomposition, or “map”, on S

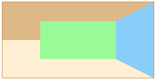
Examples



$C_P(\mathbb{D}^2) = 2$



$C_P(\mathbb{D}^2) = 3$



$C_P(\mathbb{D}^2) = 4$

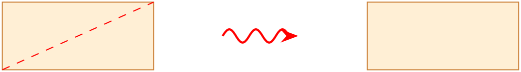
$\Longrightarrow \quad C(\mathbb{D}^2) \geq 4$

For maps of the world we are most interested in $C(\mathbb{D}^2) = C(S^2)$

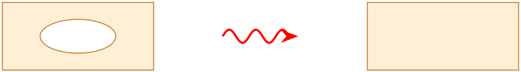
Map colouring assumptions

A map on a surface S is a polygonal decomposition such that:

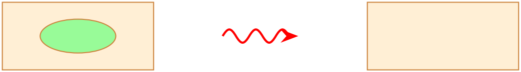
- All vertices have degree at least 3
- No region (i.e. face or polygon) has a border with itself



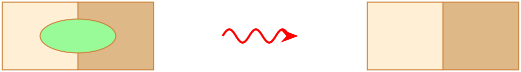
- No region contains a hole



- No region is completely surrounded by another



- No internal region has only two borders (i.e. edges)



These assumptions are purely for convenience because, in each case, we can colour these maps using the same number of colours

Understanding map colourings

The basic idea is to use the Euler characteristic and the degree-vertex and degree-face equations to understand colourings

Let $M = (V, E, F)$ be a map on a surface S. Set

- $\partial_V = \frac{2|E|}{|V|}$, the average vertex-degree
- $\partial_F = \frac{2|E|}{|F|}$, the average face-degree

By definition, $\partial_V |V| = 2|E| = \partial_F |F|$

Moreover,

- $\partial_V \geq 3$ since vertices have degree at least 3
- $\partial_F \leq |F| - 1$ as no region borders itself

Remark For a Platonic solid that is made from n-gons with p polygons meeting at each vertex we have $\partial_V = p$ and $\partial_F = n$

Bounding the face degree

Lemma

Suppose that M is a map on a closed surface S. Then
$$\partial_F = \left(1 - \frac{\chi(S)}{|F|}\right) \Big/ \left(\frac{1}{2} - \frac{1}{\partial_V}\right)$$

Proof This is a simple calculation with the Euler characteristic:

$$\chi(S) = |V| - |E| + |F| = \frac{|F|\partial_F}{\partial_V} - \frac{|F|\partial_F}{2} + |F|$$
$$\implies \quad \frac{\chi(S)}{|F|} = \frac{\partial_F}{\partial_V} - \frac{\partial_F}{2} + 1$$
$$\implies \quad \partial_F = \left(1 - \frac{\chi(S)}{|F|}\right) \Big/ \left(\frac{1}{2} - \frac{1}{\partial_V}\right)$$

Corollary

Let M be a map on a closed surface S. Then $\partial_F \le 6\left(1 - \frac{\chi(S)}{|F|}\right)$

Proof By assumption, $\partial_V \ge 3 \quad \implies \quad \frac{1}{2} - \frac{1}{\partial_V} \ge \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\implies \quad \partial_F \le 6\left(1 - \frac{\chi(S)}{|F|}\right)$ as required

Maps on sphere and projective planes

Corollary

Let M be a map on S^2 or $\mathbb{P}^2$. Then $\partial_F < 6$

Proof By the last corollary, $\partial_F \leq 6\left(1 - \frac{\chi(S)}{|F|}\right)$

Hence the result follows since $\chi(S^2) = 2$ and $\chi(\mathbb{P}^2) = 1$

Remarks

1. A Platonic solid constructed out of n-gons is a special type of map on S^2. As $\partial_F = n$ this reproves the fact that Platonic solids only exist when $3 \leq n \leq 5$
2. If the average face degree $\partial_F < 6$ then there must be at least one face f with $\deg(f) \leq 5$
   This observation will be important when we prove the Five color theorem (not quite the four color theorem)

Topology – week 11

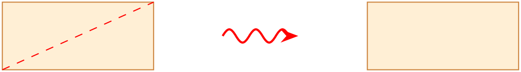
Math3061

Daniel Tubbenhauer, University of Sydney

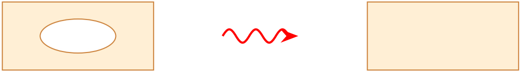
Map coloring assumptions

A map on a surface S is a polygonal subdivision such that:

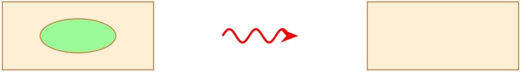
- All vertices have degree at least 3
- No region (i.e. face or polygon) has a border with itself



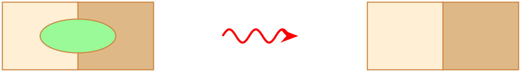
- No region contains a hole



- No region is completely surrounded by another



- No internal region has only two borders (i.e. edges)



The last three assumptions are purely for convenience because, in each case, we can color these maps using the same number of colors

Recall: Notation for map colorings

The basic idea is to use the Euler characteristic and the degree-vertex and degree-face equations to understand colorings

Let $M = (V, E, F)$ be a map on a surface S.

Set

- $\partial_V = \frac{2|E|}{|V|}$, the average vertex-degree
- $\partial_F = \frac{2|E|}{|F|}$, the average face-degree

By definition, $\partial_V|V| = 2|E| = \partial_F|F|$

Moreover,

- $\partial_V \geq 3$ since vertices have degree at least 3
- $\partial_F \leq |F| - 1$ because no region borders itself
- If M is a map on a closed surface S, then we proved that
$$\partial_F \leq 6\Big(1 - \frac{\chi(S)}{|F|}\Big)$$

Maps on surfaces with $\chi(S) \le 0$

Lemma

Let M be a map on a closed surface S with $\chi(S) \le 0$. Then

$$\partial_F \le \tfrac{1}{2}\left(5 + \sqrt{49 - 24\chi(S)}\right)$$

Proof Recall that $\partial_F < |F|$ since no region bounds itself

$$\partial_F < |F| \quad\Longrightarrow\quad |F| \ge \partial_F + 1$$

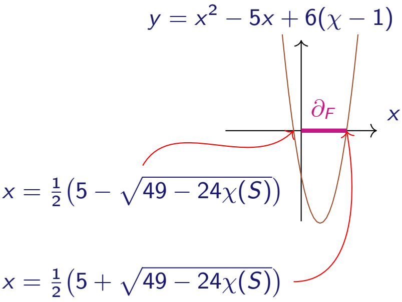
Using the corollary from last lecture, and the fact that $\chi(S) \le 0$,

$$\partial_F \le 6\left(1 - \frac{\chi(S)}{|F|}\right) \le 6\left(1 - \frac{\chi(S)}{1+\partial_F}\right)$$

$$\Longleftrightarrow \quad \partial_F^2 - 5\partial_F + 6\big(\chi(S) - 1\big) \le 0$$

$$\Longrightarrow \quad \partial_F \le \tfrac{1}{2}\big(5 + \sqrt{49 - 24\chi(S)}\big)$$

as required

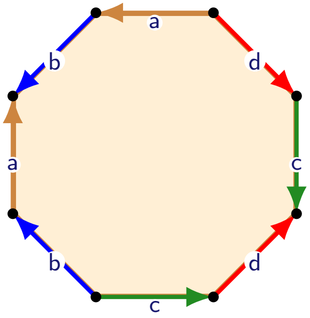


Average face degree for the double torus

Example Let $S = \#^2\mathbb{T}$.

$$\implies \qquad \partial_F \leq \tfrac{1}{2}\big(5 + \sqrt{49 - \chi(S)}\big) = \tfrac{1}{2}\big(5 + \sqrt{49 - 24(-2)}\big) \approx 7.4$$

The standard polygonal decomposition for $S = \#^2\mathbb{T}$ is



This has $\partial_F = 8$!?

This is not a contradiction because we are assuming that no region has a border with itself, which is never true for a polygonal decomposition that has only one face

Heawood's theorem

Theorem

Suppose that S is a closed surface. Then

$$C(S) \leq \begin{cases} 6, & \text{if } S = S^2 \text{ or } S = \mathbb{P}^2, \\ \frac{7+\sqrt{49-24\chi(S)}}{2}, & \text{otherwise} \end{cases}$$

Proof Let c be the integer part of the right-hand side. Then:

- If $S = S^2$ or $S = \mathbb{P}^2$ then $\partial_F < 6 = c$ by last week's discussion
- Otherwise, $\partial_F \leq \frac{1}{2}\left(5 + \sqrt{49 - 24\chi(S)}\right) = c - 1 < c$ by the last lemma

$\Longrightarrow \quad \partial_F < c$ for all S

Claim If M is a map on S then $C_M(S) \leq c$

We argue by induction on $|F|$

- If $|F| \leq 6$ then M has at most 6 faces, so $C_M(S) \leq 6 \leq c$
- Assume now that $|F| > 6$ and that the claim holds for smaller $|F|$

Since $\partial_F < c$ there is at least one face f with $\deg(f) < c$

Proof of Heawood's theorem...

We are now assuming that $|F| > 6$ and f is a face with $\deg(f) < c$

We construct a new map N by shrinking f to a point x:

This gives a new map N on S with $|F| - 1$ faces

$\implies \quad C_N(S) \le c$ by induction

Since $\deg(f) < c$ we need at most $c - 1$ colors around x:

As we used at most $c - 1$ colors around x, we can color the map M with c colors $\quad \implies \quad C_M(S) \le c \quad \implies \quad C(S) \le c$

Chromatic numbers

Heawood's theorem gives an upper bound for the chromatic number $C(S)$

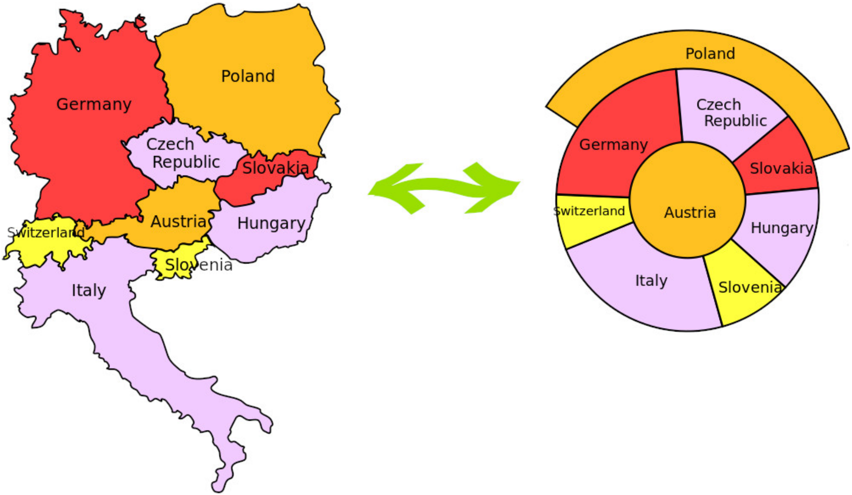
This estimate is exactly right except when $S = S^2$ or $S = \mathbb{K}$

Surface	Heawood's bound	real $C(S)$
S^2	6	4
$\mathbb{K}$	7	6
$S \neq S^2, \mathbb{K}$	$c = \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor$	c

Remarks

1. To prove this for $S \neq S^2, \mathbb{K}$ it is necessary to construct maps that require this many colors and show no more colors are ever needed
2. It is easy to see that $C(S^2) \geq 4$ but it is really hard to show that $C(S^2) = 4$: the first proofs of the Four color theorem used complicated reductions and then exceedingly long brute force computer calculations
3. If $S = S^2$ then $\chi(S^2) = 2$ so $\frac{7+\sqrt{49-24\chi(S)}}{2} = 4$!?

Why is $C(S^2) \geq 4$ easy to see? Well:



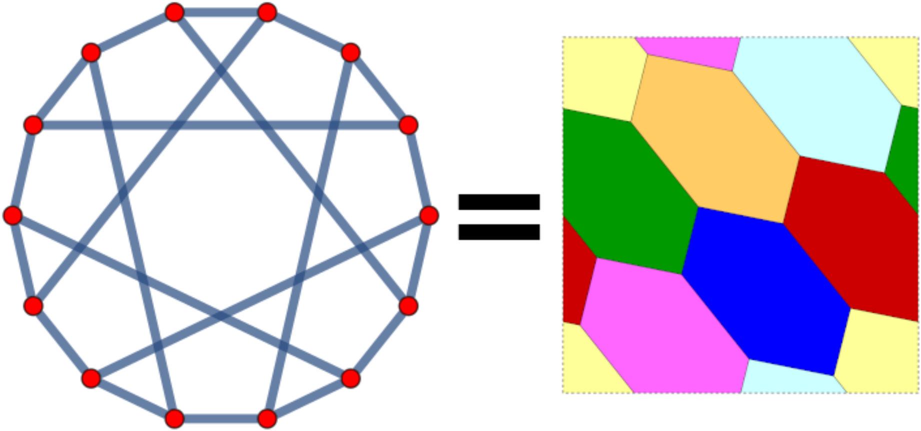
Coloring the torus

Heawood's estimate for the torus is $C(\mathbb{T}) \leq \frac{7+\sqrt{49-24\chi(\mathbb{T})}}{2} \leq 7$

Here is a map on the torus that requires 7 colors

Hence, $C(\mathbb{T}) = 7$ (see the tutorials)

Hexagons on the torus

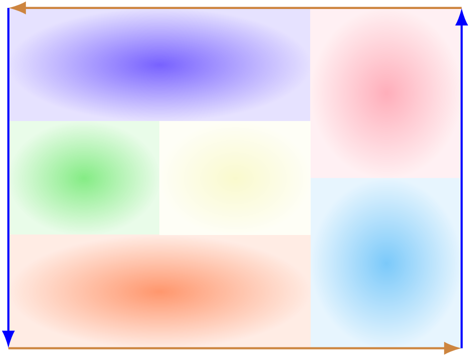


Coloring the projective plane

Heawood's estimate for the projective plane $\mathbb{P}^2$ is

$$C(\mathbb{P}^2) \leq \frac{7+\sqrt{49-24\chi(\mathbb{P}^2)}}{2} \leq 6$$

Here is a map on $\mathbb{P}^2$ that requires 6 colors:

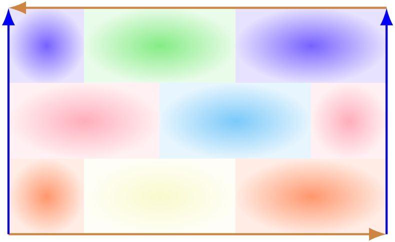


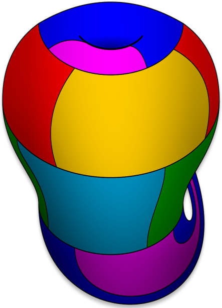
Hence, $C(\mathbb{P}^2) = 6$

Heawood's estimate for the Klein bottle is

$$C(\mathbb{K}) \leq \frac{7+\sqrt{49-24\chi(\mathbb{K})}}{2} \leq 7$$

In fact, Franklin (1930) proved that $C(\mathbb{K}) = 6$





Using these maps you can show that $C(\mathbb{K}) \geq 6$

The four color theorem

Theorem

Every map on $\mathbb{D}^2$ can be colored using four colors.
That is, $C(\mathbb{D}^2) = C(\mathbb{R}^2) = C(S^2) = 4$

Remark All known proofs have a computational component

There were several incorrect proofs published before Appel and Haken proved this result. One of the incorrect proofs was due to Kempe and 11 years later Heawood found a counterexample to their proof. In doing this, Heawood gave their upper bound for the chromatic number $C(S)$ of any closed surface and he gave a conjecture for coloring surfaces and graphs, which was finally proved in 1968 by Ringel and Young.

At the same time, Heawood proved the Five color theorem

Theorem

Every map on $\mathbb{D}^2$ can be colored with five colors

By stereographic projection, it is enough to show that $C(S^2) \leq 5$

Proof of Heawood's Five color Theorem

Let $M = (V, E, F)$ be a map on S^2. We argue by induction on $|F|$

If $|F| \leq 5$ then we can color M with $|F|$ colors, starting the induction

Suppose then that $|F| > 5$. Recall that we have proved $\partial_F < 6$

$\implies$ M has a face f with $\deg(f) \leq 5$

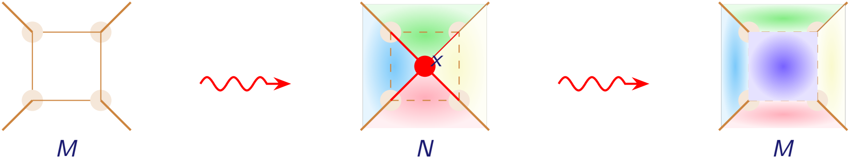
As we did in the proof of Heawood's theorem, construct a new map N by shrinking f to a point:

By induction the new map N is 5-colorable

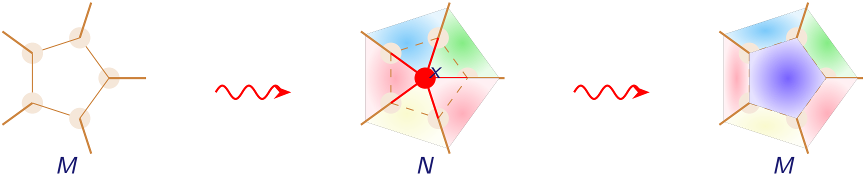
As in the proof of Heawood's theorem, the idea is now to modify the 5-coloring on N to give a 5-coloring on M. This time the proof is more complicated and there are several cases to consider

Case 1: $\deg(f) < 5$

If $\deg(f) < 5$ then the 5-coloring of N has at most 4 colors for the faces in N around x $\implies$ M is 5-colorable:

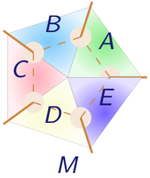


Case 2: $deg(f) = 5$ and the colors around x are not distinct



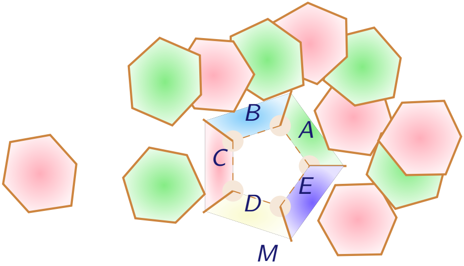
As we have used at most 4 colors in N around x, it follows that M is 5-colorable

Case 3: $\deg(f) = 5$ and all of the colors in N around x are different

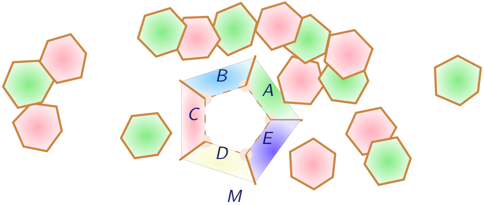


Label the regions A–E as shown.

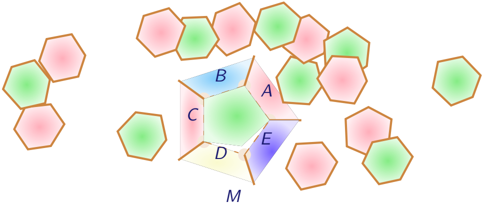
Consider the polygonal decomposition P contained in N that has these five faces together with all of the regions in N that have the same colors as the faces A and C



Case 3a: The regions A and C are not connected in P



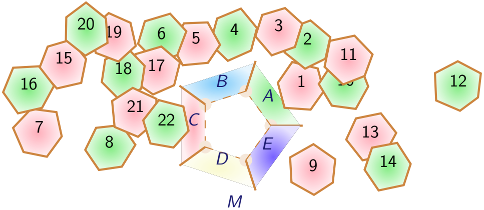
$\Longrightarrow$ Swapping the colors A and C in the connected component of P that contains A gives a new map N' with a valid coloring



$\Longrightarrow$ A and C now have the same color and we are back in Case 2

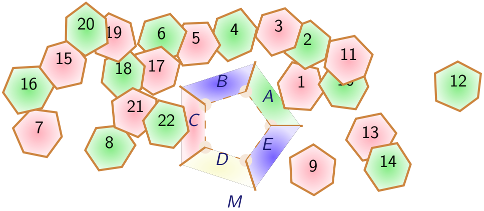
$\Longrightarrow$ The map M is 5-colorable

Case 3b: The regions A and C are connected in P



$\Longrightarrow$ As A and C are connected, B and E cannot be connected!

Swap colors B and E in the "color connected component" containing D



$\Longrightarrow$ We are back in Case 2, so M is 5-colorable

This completes the proof of the Five color Theorem

Knots

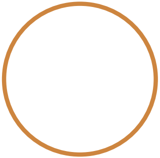
Intuitive definition A knot is a piece of string with the ends tied together

Definition

A knot is the image of an injective continuous map from S^1 into $\mathbb{R}^3$, where $S^1 = \{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \,\}$ is the unit circle in $\mathbb{R}^2$

Equivalently, a knot is a closed path in $\mathbb{R}^3$ that has no self-intersections

Examples



Unknot

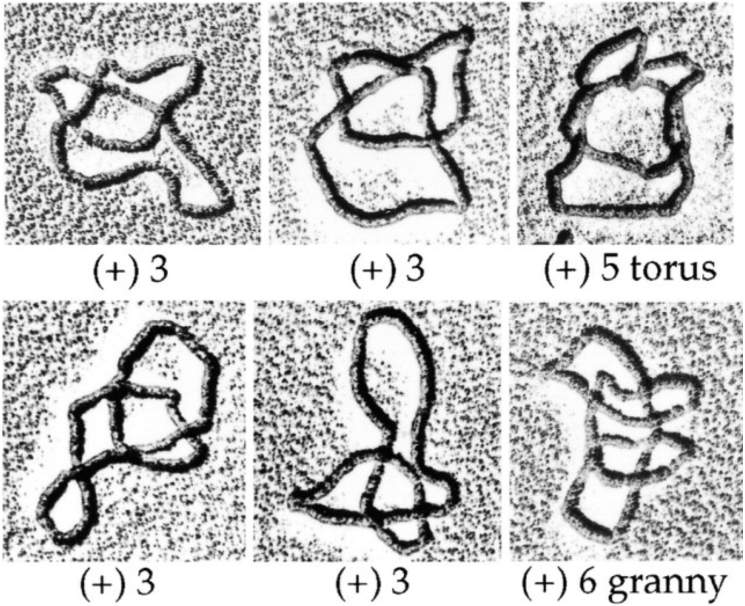
Trefoil

Reverse trefoil

Heart knot

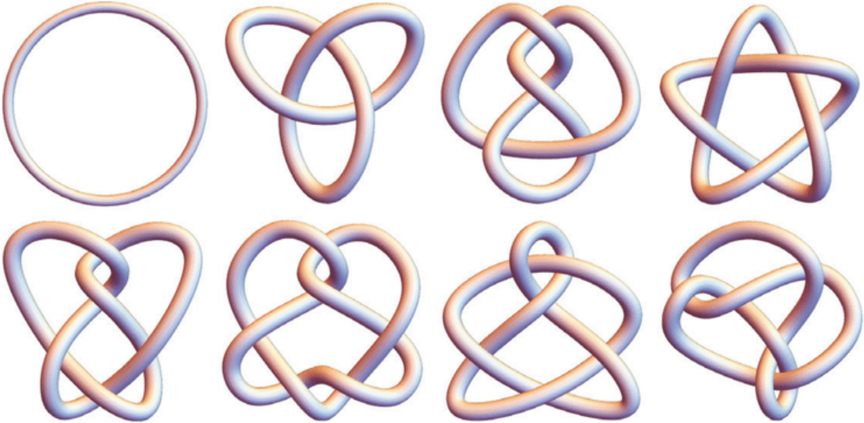
Knot theory is a beautiful mathematical subject with applications in mathematics, computer science, computer chip design, biology, . . .

A picture of life



Another picture of life

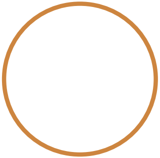
More knots



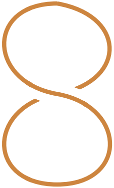
Basic question in knot theory

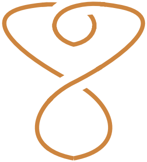
Question

When is a knot the unknot?



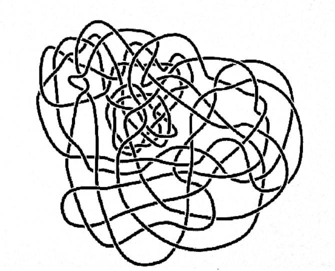
Unknot





Another unknot

It is difficult to tell if a knot is the unknot



When are two knots the same?

- Can we tell when two knots are equal?
- What does it even mean for two knots to be equal?

Question Is being homeomorphic enough?

No! Every knot is homeomorphic to S^1

$\Longrightarrow$ Homeomorphism is not the right equivalence relation for knots!

Definition

Two knots K and L are equivalent, and we write $K \cong L$, if there exists a continuous map, or ambient isotopy, $f : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$ such that

1. for each $t \in [0,1]$ the map $\mathbb{R}^3 \to \mathbb{R}^3; x \mapsto f(x,t)$ is a homeomorphism
2. if $x \in K$ then $f(x,0) = x$, and
3. there is a homeomorphism $K \to L$ given by $x \mapsto f(x,1)$

Intuitively, f continuously deforms $K = f(K,0)$ into the knot $L = f(K,1)$

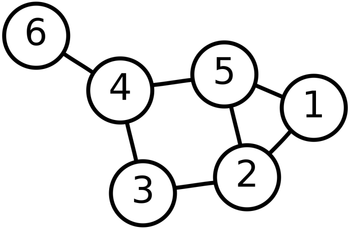
In practice, we will never use this definition but you should see it

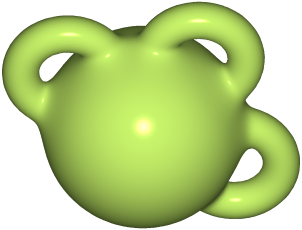
A knot K is trivial if it is equivalent to the unknot otherwise it is non-trivial

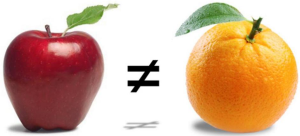
Different notions of "equal"

Objects	Graphs	Surfaces	Knots
Equivalence	Isomorphism of graphs	Homeomorphism	Equivalence of knots

In other words, graphs, surfaces and knots should never be directly compared – they are different beasts





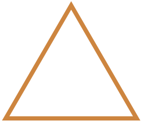


Polygonal knots

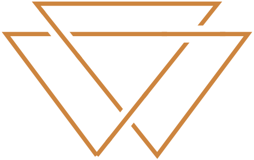
A polygonal knot is a finite union of (straight) line segments in $\mathbb{R}^3$ that is homeomorphic to S^1

$\rightsquigarrow$ just like the polygonal decompositions of surfaces, polygonal knots reduce the study of knots to combinatorics

Examples



Unknot



Trefoil

Figure eight

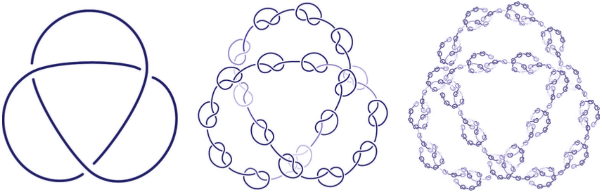
Remark Two polygonal knots K and L are equivalent if they have a common subdivision

Only polygonal knots

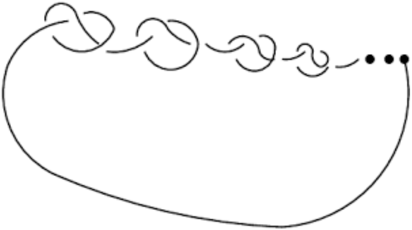
From now on all knots are polygonal knots and we drop the adjective polygonal

This is not a huge restriction: anything you can draw is polygonal. Any "finite thing" is a polygonal knot, but "limits" are not so we ignore them

Good (but the limit is not):

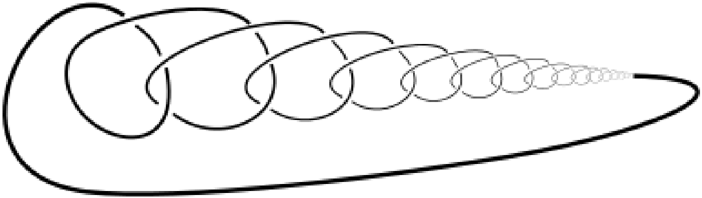


Not good:



Polygonal knots avoid pathologies

These are not polygonal knots:



Knot projections

Question What do our drawings of knots actually mean?

A knot projection is a drawing of a knot in $\mathbb{R}^2$ such that:

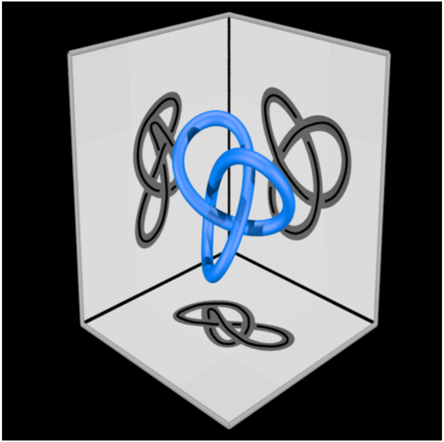
- crossings only involve two string segments, or connected components
- over and under crossings indicate relative string placement

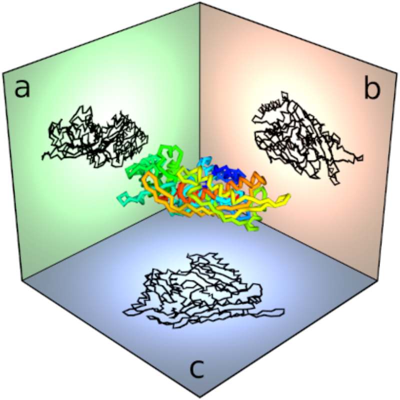
Warning!

Knot projections are a convenient way of drawing knots but they involve a choice of projection

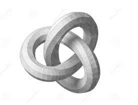
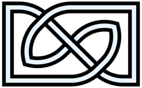
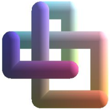
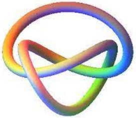
$\Longrightarrow$ Knot projections can be misleading so we have to check that our constructions are independent of the choice of knot projection

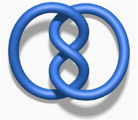
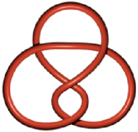
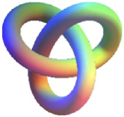
Projections = shadows





The trefoil knot times nine

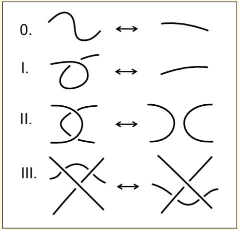






Reidemeister's theorem

Theorem

Two knot diagrams represent the same knot if and only if they are related by a (finite) sequence of moves of the following three types

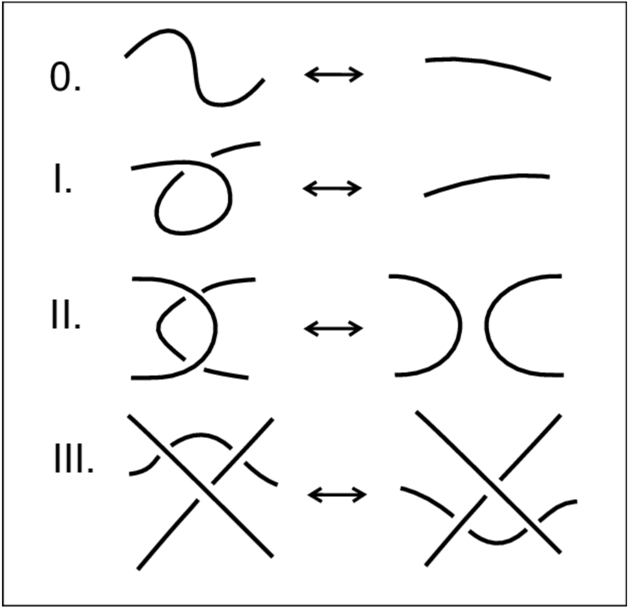


Here the 0th move is usually used silently

We won't prove Reidemeister's theorem in this lecture - the proof is a bit technical and uses the definition of equivalence of knots

The point: Reidemeister's theorem reduces topology to combinatorics of diagrams

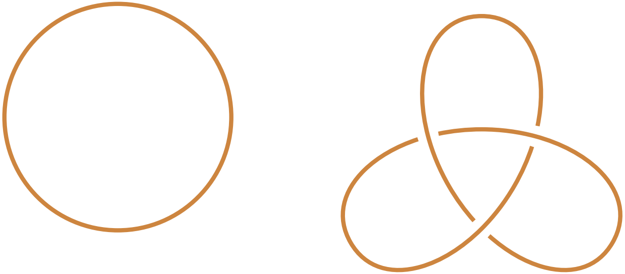
The Reidemeister moves on one slide



The knotty trefoil

Question

Is the trefoil knot equivalent to the unknot?



It seems clear that these two knots are different but, so far, we have not seen an easy way to distinguish between them

Knot colorings

Definition

A coloring of a knot (projection) is the assignment of colors to the different segments,or connected components, so that at each crossing all segments have either the same color or they all have different colors and at least two colors are used

or

$\Longrightarrow$ If a knot (projection) is 3-colorable then it has a coloring that uses exactly 3 colors

Let $C_3(K)$ be the number of different colorings of K using 3 colors

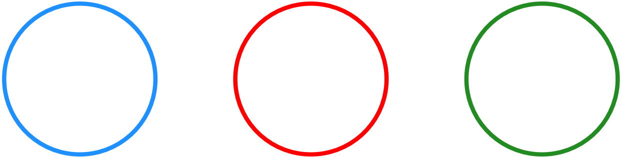
Remark

- A knot can always be colored using a single color, so $C_3(K) \geq 3$ for all knots K
- As soon as more than one color is used we must use all three colors, so K is 3 colorable if and only if $C_3(K) > 3$

Three colorings

As the unknot has no crossings, it has only one segment that must always be colored using the same color

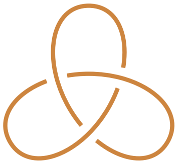
$\Longrightarrow \quad C_3(\text{Unknot}) = 3$ and the Unknot is not 3-colorable



Which of the following are knots are 3-colorable?

coloring the trefoil knot

Question What is $C_3(T)$ if T is the trefoil knot?

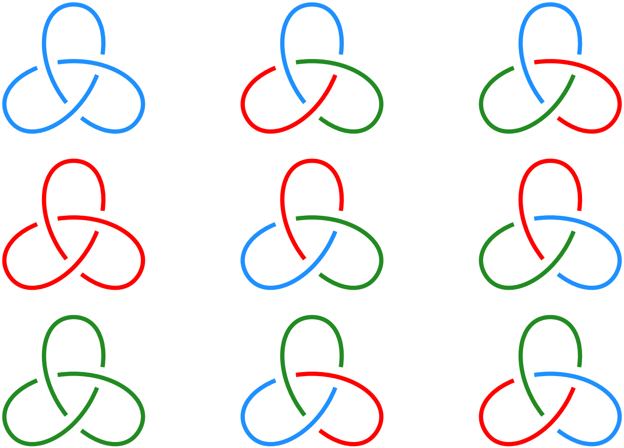


Claim $C_3(T) = 9$ since the components of T can be colored independently

coloring the trefoil knot

Question What is $C_3(T)$ if T is the trefoil knot?

Claim $C_3(T) = 9$ since the components of T can be colored independently



Three colorability

Theorem

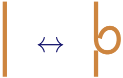
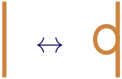
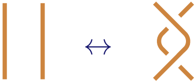
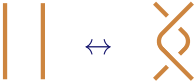
The integer $C_3(K)$ is a knot invariant
That is, $C_3(K)$ depends only on K, up to ambient isotopy, and it is independent of the choice of knot projection

Corollary

Being 3-colorable is a knot invariant

The corollary follows because K is 3-colorable if and only if $C_3(K) > 3$

To prove the theorem it suffices to check that $C_3(K)$ is invariant under the three Reidemeister moves

- Twisting  and 
- Looping  and 

- Braiding

 $\leftrightarrow$  and  $\leftrightarrow$ 

 $\leftrightarrow$  and  $\leftrightarrow$ 

 $\leftrightarrow$  and  $\leftrightarrow$ 

 $\leftrightarrow$  and  $\leftrightarrow$ 

Key point For each Reidemeister move there is a unique way to complete any coloring given the existing colors of the segments going in and out

Topology – week 12

Math3061

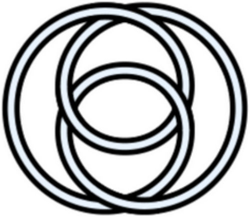
Daniel Tubbenhauer, University of Sydney

Reidemeister moves are powerful but might be tricky

This is the unknot: $K =$

These two knots are equivalent: $K =$

, $K' =$



How to show that? Use Reidemeister moves (this is a strongly recommended exercise). But that might be tricky in general, so invariants is what we want.

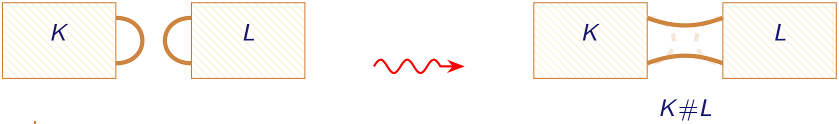
Connected sums of knots

We used connected sums to construct and classify surfaces

We want an analog of connected sums for knots

Definition

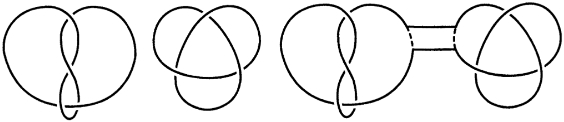
Given two knots K and L their connected sum is the knot $K\#L$ that is obtained by cutting both knots and splicing them together



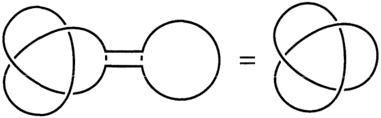
Remarks

- $\#$ does not depend on the choice of knot projections or where you cut either knot, and it is an "addition" or "multiplication":
  - $K\#\bigcirc \cong K$
  - $K\#L \cong L\#K$
  - $(K\#L)\#M \cong K\#(L\#M)$

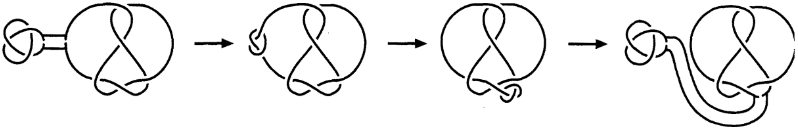
Examples of



J K $J\#K$



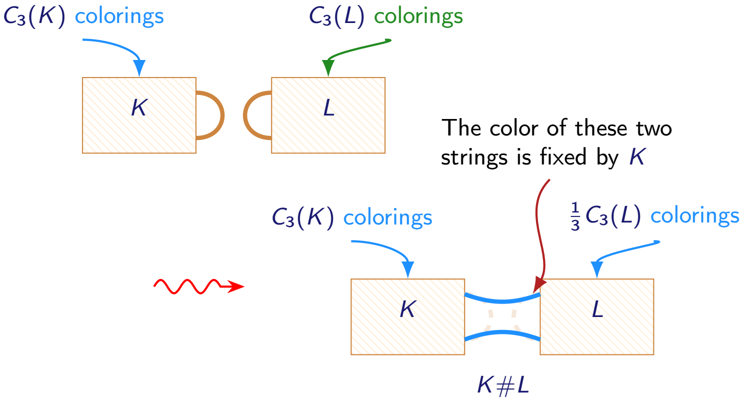
K unknot K



Proposition

Let K and L be knots. Then $C_3(K\#L) = \frac{1}{3}C_3(K)\cdot C_3(L)$

Proof We need to count the possible colorings of $K\#L$



Since the colors of the connecting strands are fixed, there are only $\frac{1}{3}C_3(L)$ ways to 3-color the strands of L inside $K\#L$

Corollary

There are infinitely many inequivalent knots

Proof Since $C_3(K)$ is a knot invariant, it is enough to find an infinite family of knots that have a different number of 3-colorings

Let T be the trefoil knot

$$\implies \quad C_3(T) = 9 = 3^2 > 3$$

$\implies$ if $n \geq 1$ then

$$\begin{aligned} C_3(\#^k T) &= \tfrac{1}{3} C_3(T) \cdot C_3(\#^{k-1} T) = \tfrac{1}{3} \cdot 9 \cdot C_3(\#^{k-1} T) \\ &= 3 C_3(\#^{k-1} T) \\ &= 3^2 C_3(\#^{k-2} T) \cdots = 3^{k-1} C_3(T) = 3^{k+1} \end{aligned}$$

Therefore, the knots T, $\#^2 T$, $\#^3 T$, ... are all inequivalent because they all have a different number of 3-colorings

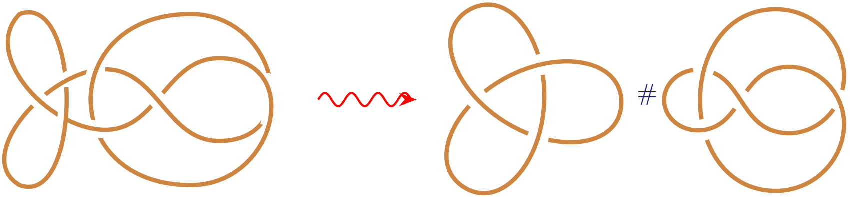
More generally, the same argument shows that if K is 3-colorable then the knots K, $\#^2 K$, $\#^3 K$,... are all inequivalent

Prime knots

Definition

The knot K is a composite knot if it has a factorisation $K = L\#M$, where L and M are not the unknot
A knot K is prime if it is not composite

Example



Remark The definition of prime knots is hard to apply because it is difficult to tell when a knot is not the unknot!

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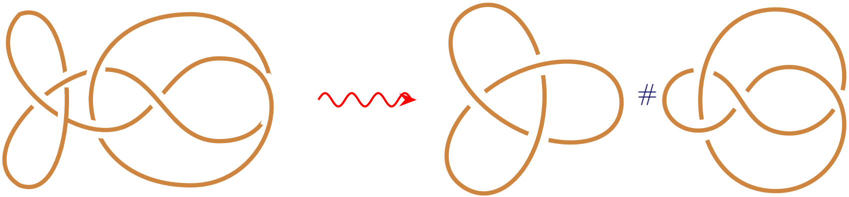


Remark The definition of prime knots is hard to apply because it is difficult to tell when a knot is not the unknot!

In fact, we don't yet know that the figure eight knot is not the unknot!!

The crossing number of a knot

Definition

The crossing number of a projection is the number of crossings you see. The crossing number $\mathrm{cross}(K)$ of a knot K is the smallest number of crossings in any knot projection

This is obviously a knot invariant but not obvious how to compute it !!!

Examples

- $\mathrm{cross}($$) = 0$. In fact, $\mathrm{cross}(K) = 0$ if and only if K is the unknot
- $\mathrm{cross}($$) = 3$

Lemma

Let K and L be knots. Then $\mathrm{cross}(K\#L) \leq \mathrm{cross}(K) + \mathrm{cross}(L)$

Remark It is a big open question if $\mathrm{cross}(K\#L) = \mathrm{cross}(K) + \mathrm{cross}(L)$

This is only known to be true for certain types of knots such as alternating knots, which we will meet soon

The crossing number and prime knots

Lemma

Let K and L be knots. Then $\mathrm{cross}(K\#L) \leq \mathrm{cross}(K) + \mathrm{cross}(L)$

Proof Note that $K\#L$ has a projection with $\mathrm{cross}(K) + \mathrm{cross}(L)$ crossings

Corollary

Let K be a knot. Then $K = P_1\# \dots \# P_n$, for prime knots $P_1, \dots, P_n$

Proof Immediate by induction on $\mathrm{cross}(K)$, the minimal number of crossings in K

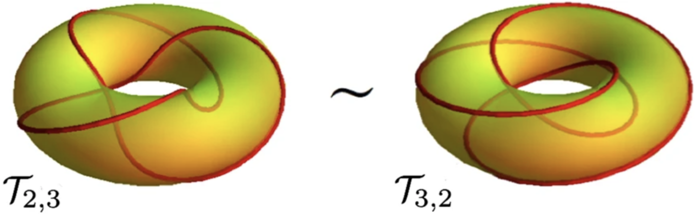
Conversely, we can ask how many prime knots there are

Torus knots

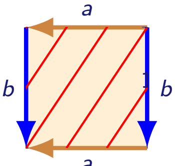
For $a, b \in \mathbb{R}$ write $a \equiv b$ if $a - b \in \mathbb{Z}$ $\iff$ same fractional part

Definition

Then the (p, q)-torus knot $\mathcal{T}_{p,q}$ is the closed path $\{\, (x, y) \in T \mid py \equiv qx \,\}$ on the standard polygonal decomposition of the torus on the unit square, where $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$

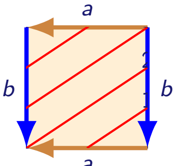


$\mathbb{T} \cong$

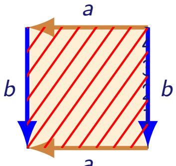


Torus knot $(2, 3)$

$\cong$



Torus knot $(3, 2)$



Torus knot $(5, 7)$

Torus knots are prime knots

Theorem

Suppose that $\gcd(p,q)=1$. *Then the* (p,q)*-torus knot is prime*

This is intuitively clear because whenever we try to write a torus knot as the connected sum of two smaller knots, each of the smaller knots is the unknot; we sketch the proof momentarily

Corollary

There are an infinite number of prime knots

Proof If $p<q$ then $\mathrm{cross}(\mathcal{T}_{p,q})=(p-1)q$ — true but won't prove

$\Longrightarrow$ the torus knots $\mathcal{T}_{2,q}$ with $q>2$ odd are all inequivalent

The number of prime knots with n-crossings

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0	1	1	2	3	7	21	49	165	552	2176	9988	46972

As is common, knots and their mirror images are only counted once

Torus knots are prime - proof sketch

Proof

For $p, g \geq 2$ let the (p, q)-torus knot K lie on an unknotted torus $T \subset S^3$ and let the 2-sphere S define a decomposition of K. We assume that S and T are in general position, that is, $S \cap T$ consists of finitely many disjoint simple closed curves.

Such a curve either meets K, is parallel to it or it bounds a disk D on T with $D \cap K = \emptyset$. Choose γ with $D \cap S = \partial D = \gamma$. Then γ divides S into two disks D', D'' such that $D \cup D'$ and $D \cup D''$ are spheres, $(\cup D') \cap (\cup D'') = D$; hence, D' or D'' can be deformed into D by an isotopy of S^3 which leaves K fixed. By a further small deformation we get rid of one intersection of S with T.

Torus knots are prime - proof sketch

Proof Continued

Consider the curves of $S \cap T$ which intersect K. There are one or two curves of this kind since K intersects S in two points only. If there is one curve it has intersection numbers $+1$ and -1 with K and this implies that it is either isotopic to K or nullhomotopic on T. In the first case K would be the trivial knot. In the second case it bounds a disk D_0 on T and $D_0 \cap T$, plus an arc on S, represents one of the factor knots of K; this factor would be trivial, contradicting the hypothesis.

Torus knots are prime - proof sketch

Proof Continued

The case remains where $S \cap T$ consists of two simple closed curves intersecting K exactly once. These curves are parallel and bound disks in one of the solid tori bounded by T. But this contradicts $p, q \geq 2$

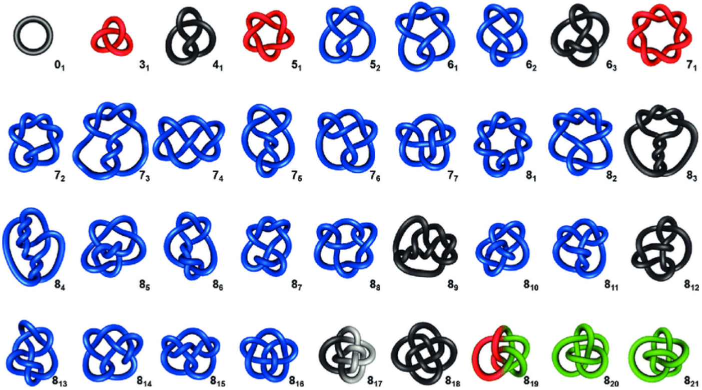
Prime factorisation of knots

Theorem

Suppose that K is not the unknot. Then $K = P_1 \# P_2 \# \ldots \# P_n$, for prime knots $P_1, \ldots, P_n$. Moreover, the multiset of prime knots is a knot invariant

This can be proved using Seifert surfaces (that we meet later)

Here is a table of the unknot and the first 36 prime knots:



Colorable knots

Question

Is the figure eight knot the unknot?

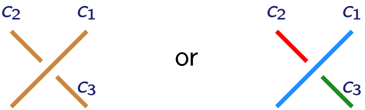
$\Longrightarrow$ We need another knot invariant to show that the figure eight knot is not the unknot

To do this we first need to better understanding 3-colorings

Rather than colors, lets color the segments with 0, 1 and 2

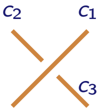
Question

What can we say about $c_1 + c_2 + c_3$ for a 3-coloring?

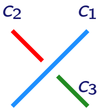


Possible colorings and the values of $c_1 + c_2 + c_3$

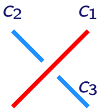
Allowed colorings



or



Disallowed colorings



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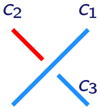

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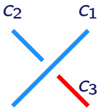
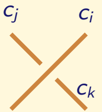


Knot colorings with p-colors

Definition

Let $p \in \mathbb{N}$. A p-coloring of a knot K is a coloring of the segments of K that using colors from $\{0, 1, \ldots, p-1\}$ such that



$$\Longrightarrow \quad 2c_i \equiv c_j + c_k \pmod{p}$$

Let $C_p(K)$ be the number of p-colorings of K.
A knot is p-colorable if it has a p-coloring that uses at least two colors

- $a \equiv b \pmod{p} = a - b$ is divisible by p. When $p = 3$ this agrees with the previous definition of 3-coloring
- As with 3-coloring this depends on the choice of knot projection
- For any p the constant coloring is a p-coloring

$\Longrightarrow \quad C_p(K) \geq p$

$\Longrightarrow \quad K$ is p-colorable if and only if $C_p(K) > p$

Colorability is a knot invariant

Theorem

Suppose that $p \geq 3$. Then $C_p(K)$ and p-colorability are both knot invariants

Proof Repeat the argument used for 3-colorings to show that $C_p(K)$ is unchanged by the Reidemeister moves and hence is a knot invariant

$\Longrightarrow$ p-colorability is a knot invariant since K is p-colorable if and only if $C_p(K) > p$

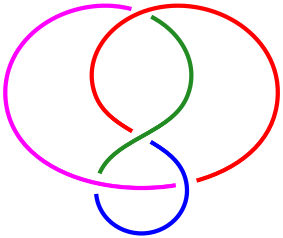
Similarly, $C_p(K \# L) = \frac{1}{p} C_p(K) C_p(L)$, for knots K and L

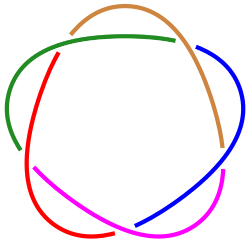
Question

Is there an easy way to tell if a knot is p-colorable?

Examples of p-colorings

Are the following knots 4-colorable, 5-colorable, ... ?





We need a better way to determine if a knot is p-colorable!

$\rightsquigarrow$ Use linear algebra!

The trefoil knot is knotted!

Corollary

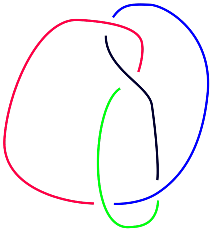
The trefoil knot is not the unknot

Proof The trefoil is 3-colorable and the unknot is not

Corollary

The trefoil knot is not equivalent to the figure eight knot

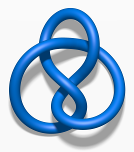
Proof The trefoil is 3-colorable and the figure eight knot is not



The trefoil knot in comparison

$\neq$

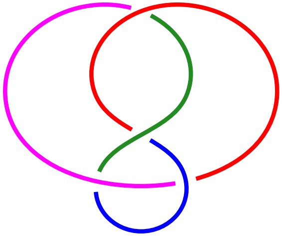
or



Colorful linear algebra

Consider the figure eight knot.

Label the segments c_1, c_2, c_3, c_4 in traveling order around the knot



$\Longrightarrow$ We require:

$$\begin{array}{ccccc} 2c_1 & & -c_3 & -c_4 & \equiv 0 \\ -c_2 & 2c_2 & & -c_4 & \equiv 0 \\ -c_1 & -c_2 & 2c_3 & & \equiv 0 \\ & -c_2 & -c_3 & 2c_4 & \equiv 0 \end{array}$$

In matrix form this becomes $M_K \underline{C} \equiv \underline{0} \pmod{p}$, where

$$M_K = \begin{bmatrix} 2 & 0 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix} \quad \text{and } \underline{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

That is, $\underline{C}$ is a p-coloring $\Longleftrightarrow$ $M_K \underline{C} \equiv 0 \pmod{p}$

We have reduced finding $c_1, \ldots, c_4$ to linear algebra!

Let K be knot projection with n crossings.

$\Longrightarrow$ Each segment starts and ends at a crossing, and each crossing has two under-crossings, so the knot projection has n segments.

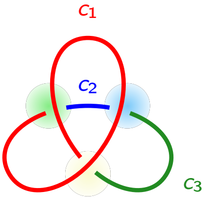
Traveling around the knot in an anti-clockwise direction let the colors of the segments be $c_1, \dots, c_n$ and let the crossings be $x_1, \dots, x_n$

The knot matrix of K is the matrix $M_K = \big(m_{ij}\big)$, where m_{ij} is the sum of the contributions of the jth segment of color c_j to the ith crossing x_i with

$$\begin{cases} +2, & \text{for over-crossings} \\ -1, & \text{for under-crossings} \end{cases}$$

$\Longrightarrow$ crossings label rows and segments label columns

An atypical example

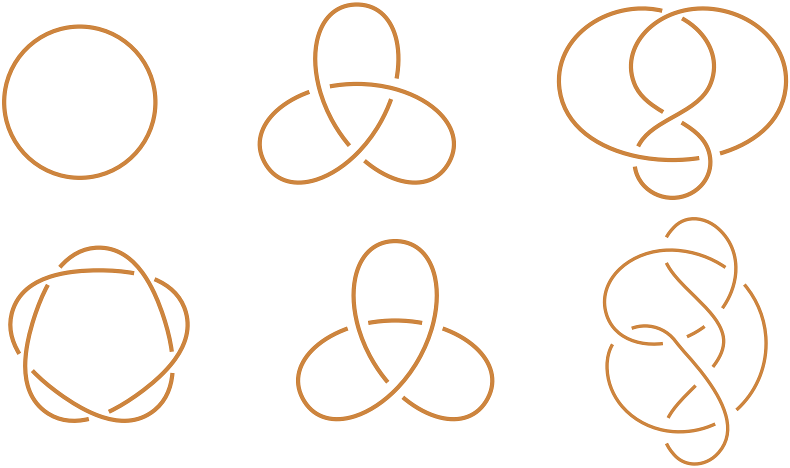


$$M_K = \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 \end{matrix} \\ \begin{bmatrix} 2-1 & -1 & 0 \\ 2 & -1 & -1 \\ 2-1 & 0 & -1 \end{bmatrix} \end{array}$$

Alternating knots

We mainly consider colorings of alternating knots

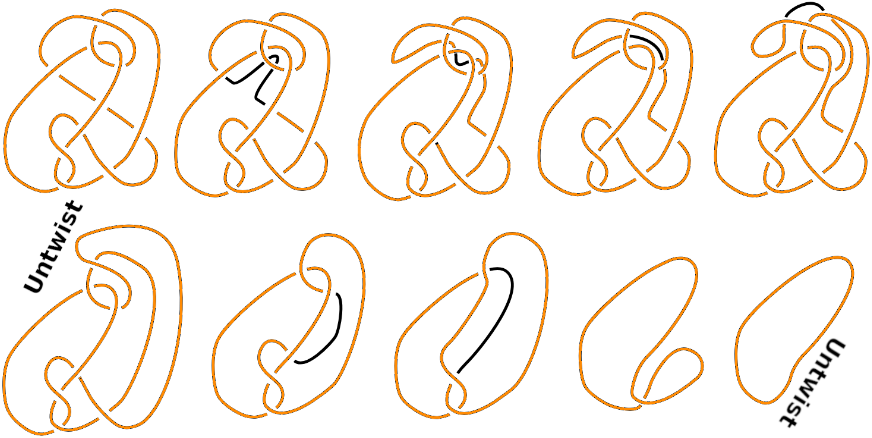
A knot projection is alternating if the crossings alternate between over and under crossings as you travel around the knot in an anti-clockwise direction



$\Longrightarrow$ Being alternating is not a knot invariant

Alternating knots – careful with projections

The unknot is alternating, but it can have non-alternating projections:



Similarly, for other knots

Knot matrices for alternating knots

If K is an alternating knot then:

$\Longrightarrow$ every segment starts as an under-string, becomes an over-string and finishes as an under-string

$\Longrightarrow$ when read in traveling order the segments and crossings alternate as $c_1, x_2, c_2, x_2, \dots, c_n, x_n$

$\Longrightarrow$ if K is alternating and no segment meets itself then each row of M_K will contain one 2 and two -1's

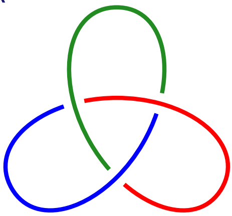
$\Longrightarrow$ if K is alternating the row and column sums of M_K are all 0

We will mainly consider colorings of alternating knots

Knot matrix examples

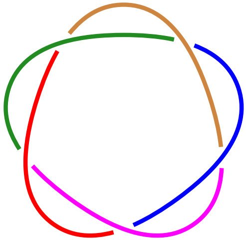
$$M_K = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$K =$



$$M_L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L =$



Properties of the knot matrix

Lemma

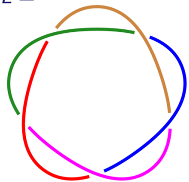
Let K be an alternating knot.

1. *The row and column sums of M_K are all 0*
2. $M_K \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \underline{0}$
3. $\det M_K = 0$

Proof

(1) Since the knot is alternating every colored strand contributes 2 once and -1 twice (see below) and dually from crossings

$$M_L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L =$ 

Properties of the knot matrix

Proof Continued

(2) By (1), the respective vector is an eigenvector with eigenvalue zero

(3) By (2) there is an zero eigenvector, so the kernel is nontrivial

Minors of a matrix

The (r, c)-minor of an $n \times n$ matrix M is the $(n-1) \times (n-1)$-matrix M_{rc} obtained by deleting row r and column c from M)

$$M = \begin{bmatrix} a_{11} & \cdots & a_{1c} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rc} & \cdots & a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nc} & \cdots & a_{nn} \end{bmatrix}$$

$$\rightsquigarrow \qquad M_{rc} = \begin{bmatrix} a_{11} & \dots & a_{1c} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r1} & \dots & a_{rc} & \dots & a_{rn} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nc} & \dots & a_{nn} \end{bmatrix}$$

The knot determinant

Definition

Let K be a knot. The knot determinant of K is $det(K) = \left|\det(M_K)_{11}\right|$

Lemma

Let $M = (m_{rc})$ be an $n \times n$ matrix with zero row and column sums. Then $\det M_{rc} = \pm \det M_{11}$, for $1 \leq r, c \leq n$

Proof   Let $\mathbb{I}$ be the $n \times n$-matrix with every entry equal to 1

Then $\det(M + \mathbb{I}) = \det \begin{bmatrix} m_{11}+1 & m_{12}+1 & \cdots & m_{1n}+1 \\ m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1}+1 & m_{n2}+1 & \cdots & m_{nn}+1 \end{bmatrix}$

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$$= \det \begin{bmatrix} n+\sum_i m_{i1} & n+\sum_i m_{i2} & \cdots & n+\sum_i m_{in} \\ m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1}+1 & m_{n2}+1 & & m_{nn}+1 \end{bmatrix}$$

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Proof Let $\mathbb{I}$ be the $n \times n$-matrix with every entry equal to 1

Then $\det(M + \mathbb{I})$

$$= \det \begin{bmatrix} n+\sum_i m_{i1} & n+\sum_i m_{i2} & \cdots & n+\sum_i m_{in} \\ m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1}+1 & m_{n2}+1 & & m_{nn}+1 \end{bmatrix}$$

$$= \det \begin{bmatrix} n & n & \cdots & n \\ m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1}+1 & m_{n2}+1 & & m_{nn}+1 \end{bmatrix}$$

The knot determinant

Definition

Let K be a knot. The knot determinant of K is $det(K) = \left|\det(M_K)_{11}\right|$

Lemma

Let $M = (m_{rc})$ be an $n \times n$ matrix with zero row and column sums.
Then $\det M_{rc} = \pm \det M_{11}$, for $1 \leq r, c \leq n$

Proof Let $\mathbb{I}$ be the $n \times n$-matrix with every entry equal to 1

Then $\det(M + \mathbb{I})$ $= \det \begin{array}{c} \begin{array}{cccc} n & n & \cdots & n \end{array} \\ \begin{bmatrix} m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1}+1 & m_{n2}+1 & & m_{nn}+1 \end{bmatrix} \end{array}$

$$= \det \begin{array}{c} \begin{array}{cccc} n^2 & n & \cdots & n \end{array} \\ \begin{bmatrix} n+\sum_i m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ n+\sum_i m_{n1}+1 & m_{n2}+1 & & m_{nn}+1 \end{bmatrix} \end{array}$$

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$$= \det \begin{bmatrix} n^2 & n & \cdots & n \\ n & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ n & m_{n2}+1 & & m_{nn}+1 \end{bmatrix}$$

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$$\text{Then } \det(M + \mathbb{I}) \quad = \det \begin{bmatrix} n^2 & n & \cdots & n \\ n & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ n & m_{n2}+1 & & m_{nn}+1 \end{bmatrix}$$

$$= n^2 \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & m_{n2}+1 & & m_{nn}+1 \end{bmatrix}$$

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$$= n^2 \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & m_{22} & \cdots & m_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & m_{n2} & & m_{nn} \end{bmatrix}$$

The knot determinant

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Lemma

Let $M = (m_{rc})$ be an $n \times n$ matrix with zero row and column sums. Then $\det M_{rc} = \pm \det M_{11}$, for $1 \leq r, c \leq n$

Proof Let $\mathbb{I}$ be the $n \times n$-matrix with every entry equal to 1

$$\text{Then } \det(M + \mathbb{I}) = n^2 \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & m_{n2}+1 & & m_{nn}+1 \end{bmatrix}$$

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$$= n^2 \det M_{11}$$

The knot determinant

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$$= n^2 \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & m_{22} & \cdots & m_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & m_{n2} & & m_{nn} \end{bmatrix}$$

By the same argument, if $1 \le r, c \le n$ then

$$\det(M + \mathbb{I}) = (-1)^{r+c} n^2 \det M_{rc}$$

The knot determinant

Definition

Let K be an alternating knot. The knot determinant of a knot K is

$$det(K) = \left|\det(M_K)_{11}\right| \qquad \text{— can take any minor of } M_K$$

Theorem

Let K be an alternating knot and $p \geq 3$ be a prime. Then K is p-colorable if and only if p divides the knot determinant $\det(K)$

Proof

By definition, K is p-colorable if and only if there exist $c_1, \ldots, c_n$ such that $M_K \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \pmod{p}$.

Now $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is a 0-eigenvector of M_K, so if $d \in \mathbb{Z}$ then

$$M_K \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = M_K \begin{bmatrix} c_1+1 \\ \vdots \\ c_n+1 \end{bmatrix} = M_K \begin{bmatrix} c_1+2 \\ \vdots \\ c_n+2 \end{bmatrix} = \cdots = M_K \begin{bmatrix} c_1+d \\ \vdots \\ c_n+d \end{bmatrix}$$

Proof Continued

$\implies$ We can assume that $c_1 = 0$ by taking $d = -c_1$

Hence, K is p-colorable if and only if and only if there exist $c_2, \ldots, c_n$ such that

$$M_K \begin{bmatrix} 0 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \equiv 0 \pmod p \iff (M_K)_{11} \begin{bmatrix} c_2 \\ \vdots \\ c_n \end{bmatrix} \equiv 0 \pmod p$$

$$\iff \det(K) \neq 0 \pmod p$$

Remarks

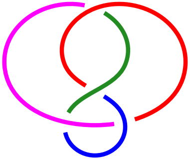
1. The Reidemeister moves show that the knot matrix M_K is not a knot invariant but $\det(K) = |\det(M_K)_{11}|$ is a knot invariant
2. If K and L are knots then $\det(K\#L) = \det(K)\det(L)$
   $\implies$ if $\det(K\#L) = p$ is prime, then either $\det(K) = p$ or $\det(L) = p$
3. If K is not alternating then the row sums of M_K are still 0. Therefore, the argument used to prove the theorem shows that K is p-colorable if and only if p divides $(M_K)_{rc}$, for some r, c.

Colorability of the figure eight knot

Summary of how to determine p-colorability

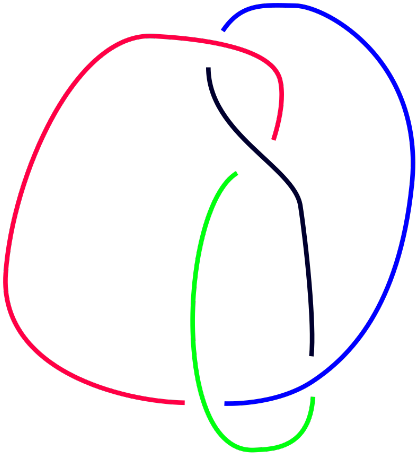
1. Label the segments in traveling order
2. Compute the entries of the knot matrix M_K
3. Compute the knot determinant $\det(K) = |\det(M_K)_{11}|$
4. Check if p divides $\det(K)$

$$M_K = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \end{pmatrix}$$

$K =$ 

The determinant is five, so the figure eight knot is five-colorable (and only five colorable)

Colorability of the figure eight knot – part 2



Thus, the figure eight knot is not trivial (it has strictly more than five 5-colorings) and also not the trefoil knot

Seifert surfaces

Definition

A Seifert surface for a knot K is an orientable surface that has K as its boundary

Theorem

Every knot has a Seifert surface

Remark In general, a Seifert surface is not unique

We will prove this result by giving an algorithm for constructing a Seifert surface for any knot

Constructing Seifert surfaces

Proof Real world version

Take a knot, build out of wire, and put it into soap

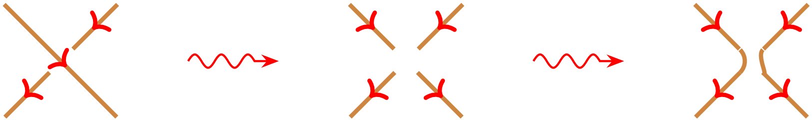
The minimal surface you get is a Seifert surface

Constructing Seifert surfaces

Proof Math version

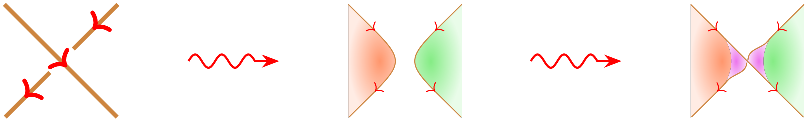
Step 1 Pick an orientation of the knot
That is, fix a direction to travel around the knot

Step 2 At each crossing cut the over-string and join the incoming and outgoing strings; the knot is then a disjoint union of Seifert circles

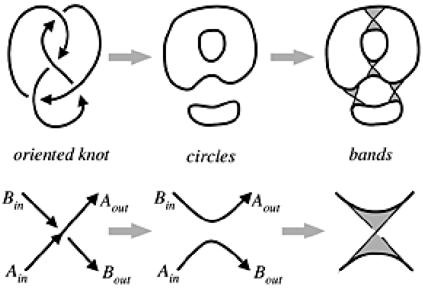


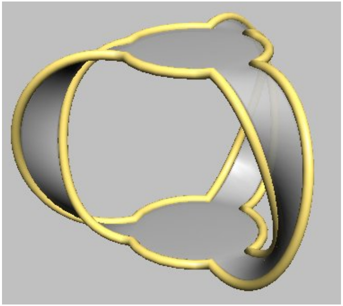
Step 3 Imagine the Seifert circles as being at different heights and glue a disk onto each one of the Seifert circles

Step 4 Now each crossing in K, glue on a twisted strip that has the crossing as a boundary

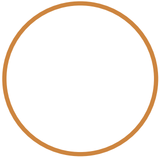
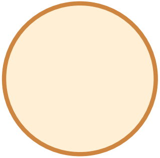


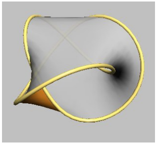
The platform construction

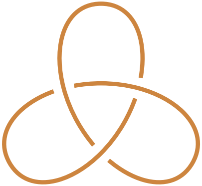


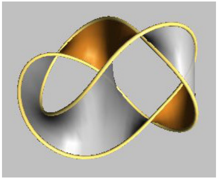


Examples of Seifert surfaces

- Unknot:  



- Trefoil 



- Figure eight 

More examples of Seifert surfaces

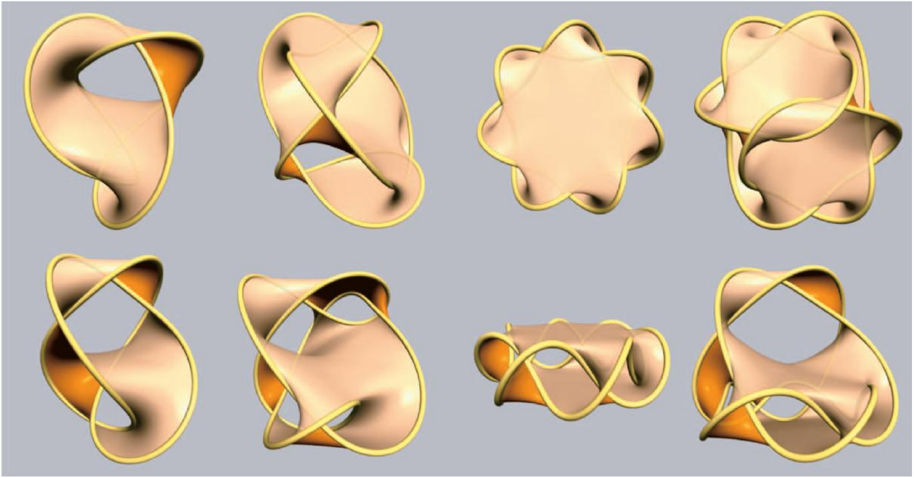


Figure 8= 4_1 6_1 7_1 8_5

The genus of a knot

Let S be a Seifert surface of a knot K

$\implies$ S is orientable + has one boundary circle since it embeds in $\mathbb{R}^3$

$\implies$ $S \cong \mathbb{D}^2 \# \#^t \mathbb{T}$, where $t = \frac{1-\chi(S)}{2} \geq 0$

Definition

The genus of K is $g(K) = \min\left\{ \frac{1-\chi(S)}{2} \,\middle|\, S \text{ a Seifert surface of } K \right\}$

Remark Used to prove uniqueness of factorization of prime knots

Example (with proof!)

- $K =$  $\implies$ $g(K) = 0$ as $S \cong \mathbb{D}^2$ and g cannot be smaller, so just checking this one diagram  is sufficient

Fact $g(K) = 0 \iff K =$ 

Problem K is the trefoil:  ... not very clear how to calculate $g(K)$!

Calculating the knot genus

Proposition

Let S be the Seifert surface with s Seifert circles that is constructed from a knot projection for a knot K with c crossings.
Then $\chi(S) = s - c$ and $g(K) \leq \frac{1+c-s}{2}$

Proof Recall from tutorials that $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$

Write $S = A \cup B$, where A the union of the Seifert circles and B the union of the twists in S

$\implies$ $A \cap B$ is a union of c pairs

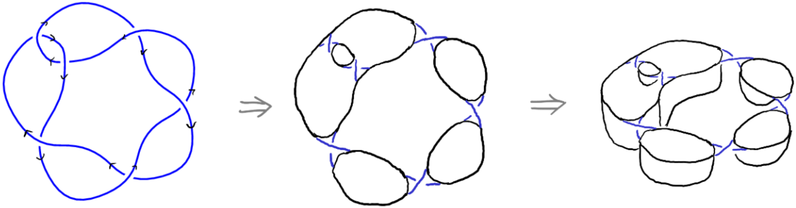
$\implies$ $\chi(S) = \chi(A) + \chi(B) - \chi(A \cap B) = s + c - 2c = s - c$

Hence, $g(K) \leq \frac{1-\chi(S)}{2} = \frac{1+c-s}{2}$

Genus of trefoil and figure eight knots

If K has c crossings and s Seifert circles then $g(K) \leq \frac{1+c-s}{2}$

So $g(K) \leq \frac{1+4-3}{2} = 1$



genus=1

Genus of alternating knots

Bad news: It can happen that $g(K) < \frac{1-\chi(S)}{2}$!!

The good news is that there is no bad news for alternating knots

Theorem

Let S be the Seifert surface constructed from an alternating knot projection of K. Then $g(K) = \frac{1-\chi(S)}{2}$

Proof Nontrivial and omitted!

Knot genus is additive

Theorem

Let K and L be knots. Then $g(K\#L) = g(K) + g(L)$

Start of proof It is not hard to see that $S_{K\#L} \cong S_K \#_{\text{strip}} S_L$ (connected sum along a strip connecting the surfaces and boundary cycles). This implies that $g(K\#L) \leq g(K) + g(L)$. The reverse implication is much harder!

The theorem gives another proof that the trefoil and figure eight knots are non-trivial because both knots have genus 1

Corollary

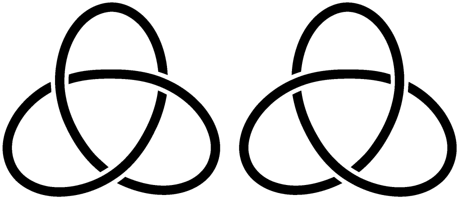
Let K and L be knots, which are not the unknot. Then $K \not\cong (K\#L)\#M$ *for any knot* M

Proof If such a knot M existed then

$$g(K) = g\big((K\#L)\#M\big) = g(K) + g(L) + g(M)$$
$$\implies \quad g(M) = -g(L) < 0 \qquad ↯↯↯$$

Left = right-handed trefoil? No idea...

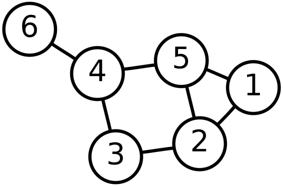
No method we have seen distinguishes these two fellows:

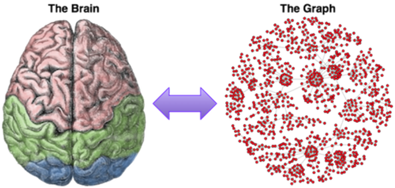


But that has to wait for another time...

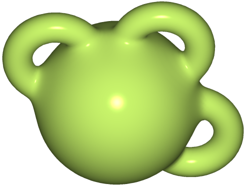
A few take away pictures

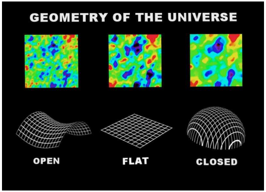
Topic 1: graphs!



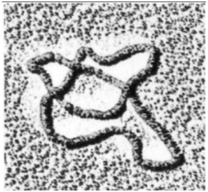


Topic 2: surfaces!





Topic 3: knots!



This was my last slide!

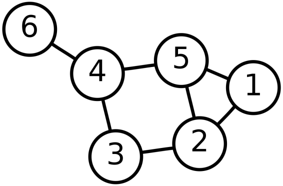
Topology – recollection

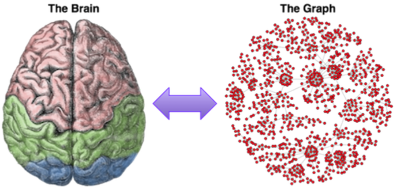
Math3061

Daniel Tubbenhauer, University of Sydney

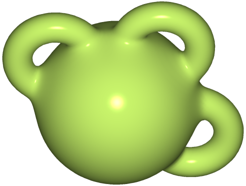
The three main topics

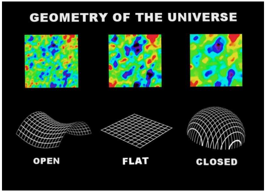
Topic 1: graphs!



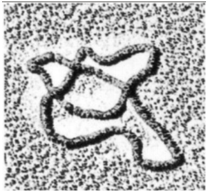


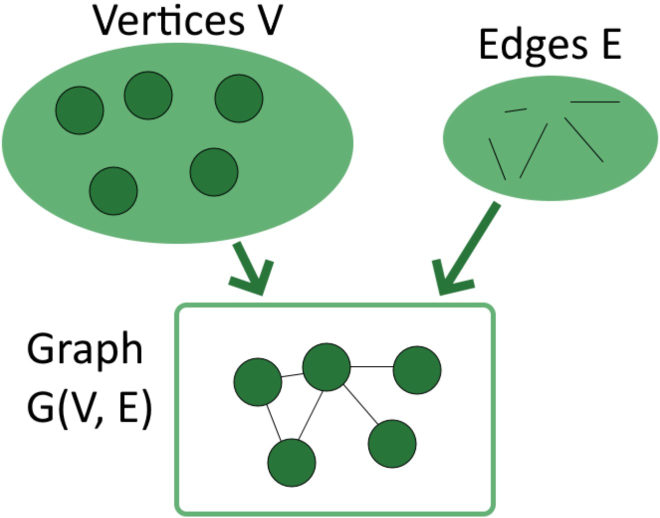
Topic 2: surfaces!





Topic 3: knots!




Vertices V
Edges E
Graph
G(V, E)

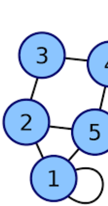
Questions we ask about a graph G

1. Have we seen G before? Is it one of the standard ones (lines, cycles, complete graphs, complete bipartite graphs)?
2. How many vertices and edges does G have?
3. What is its Euler characteristic?
4. Is G connected? How many connected components does G have?
5. Is G a tree? If not, then can we find a spanning tree?
6. What are its paths (start and endpoint might be different)? What are its circuits?
7. Does G have an Eulerian circuit? Does G have an Eulerian path?
8. Is G planar, i.e. does it embed into the plane = the disc = S^2?
9. Does G embed into other surfaces?
10. How many colors do we need to color maps defined by G?

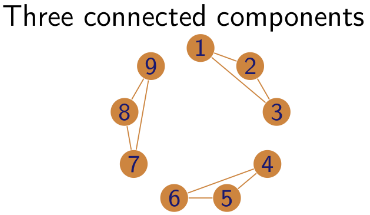
Let us answer 1-10 for the Pappus graph

But before, let us recall what the above are!

A connected graph with $|V| = 6$, $|E| = 8$, $\chi = -2$ and one loop:

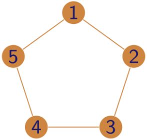


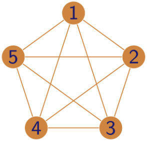
A non-connected graph with $|V| = 9$, $|E| = 9$, $\chi = 0$:

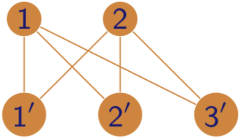


Standard graphs

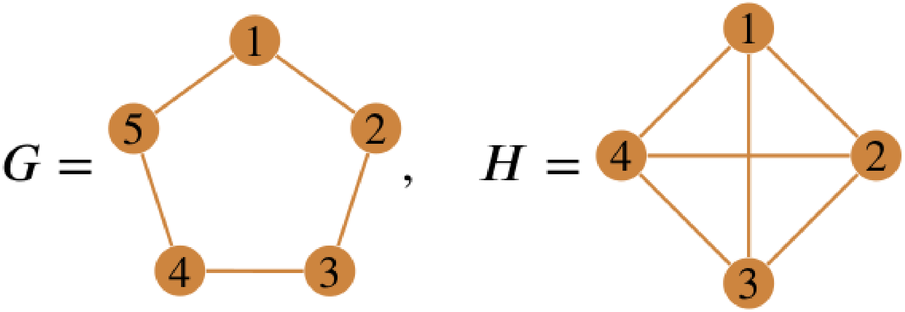
Line: 

Cycle: 

Complete: 

Complete bipartite: 

Exercise Check whether you understand how the various standard graphs are related and what properties they have. For example, which ones are subgraphs, which ones are planar etc.



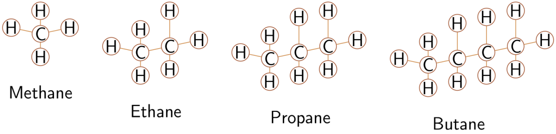
Trees

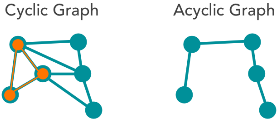
Trees are acyclic, so only the right graph below is a tree:

A tree is a connected graph that has no non-trivial circuits

Examples

- Saturated hydrocarbons





Trees satisfy many properties and are always amenable for induction, e.g. prove the following as an exercise:

Corollary

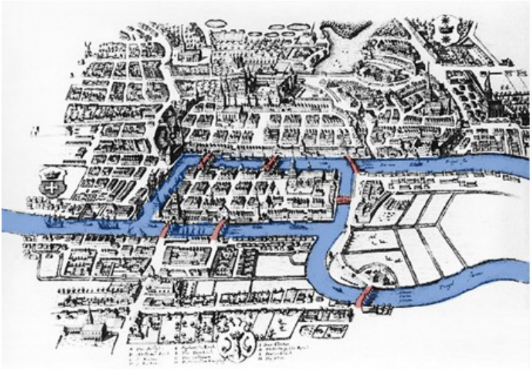
Suppose that $T = (V, E)$ *is a tree. Then* $|V| = |E| + 1$.

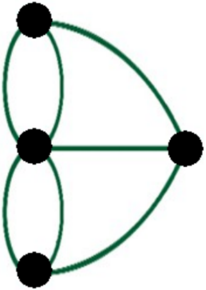
Euler and cycles

Euler's famous criterion:

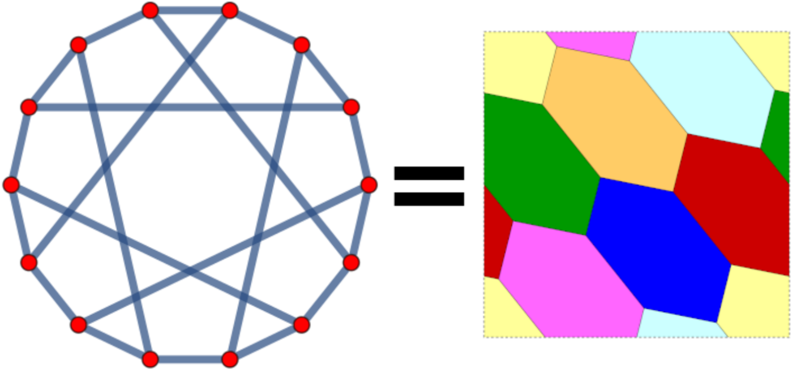
Theorem

Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree





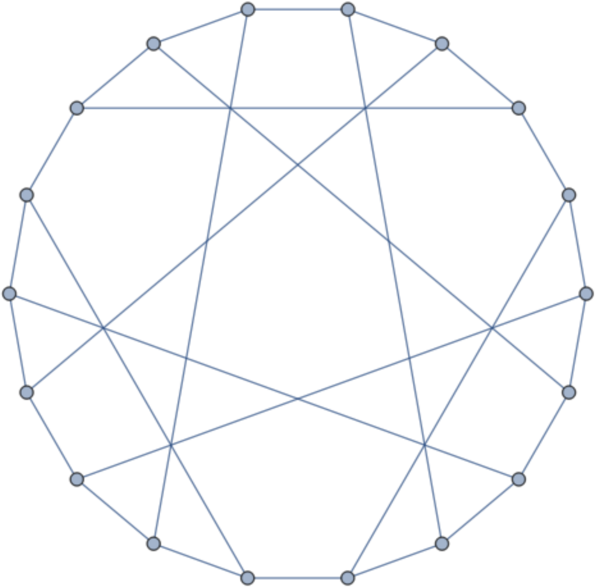
Embeddings on surfaces



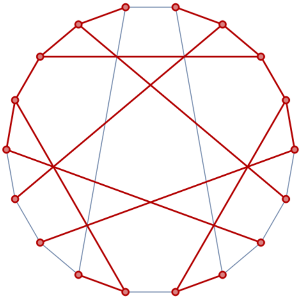
Heawood's coloring formula:

$$c = \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor$$

The Pappus graph G



The Pappus graph G – answering 1–10, part 1

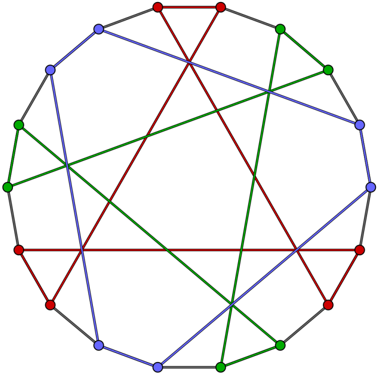


The Pappus graph is a not a standard graph – it is neither a line nor a cycle nor complete nor complete bipartite

We clearly have $|V| = 18$ and $|E| = 27$, so that $\chi(G) = |V| - |E| = -9$, and G is connected

The Pappus graph is not a tree and a spanning tree is illustrated above (there are many more spanning trees)

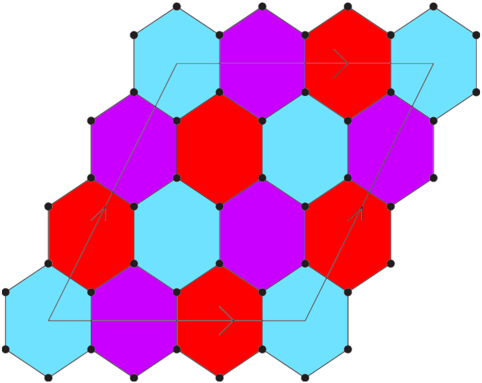
The Pappus graph G – answering 1–10, part 2



The Pappus graph has many cycles that are hexagons, as illustrated above. In fact, one checks that the length of the smallest cycle is 6

Every vertex in the Pappus graph is of degree 3, so there are neither Eulerian circuits nor paths

The Pappus graph G – answering 1–10, part 3

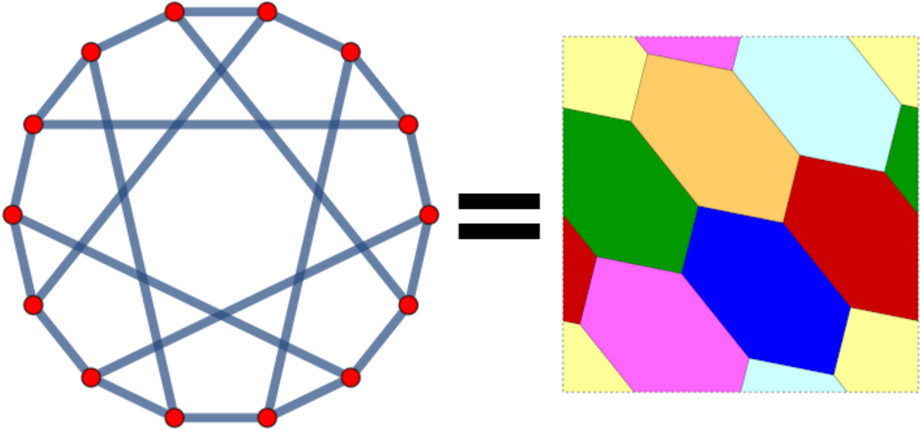


The Pappus graph does not embed into S^2

G embeds onto the torus and then needs 3 colors to color it, see above

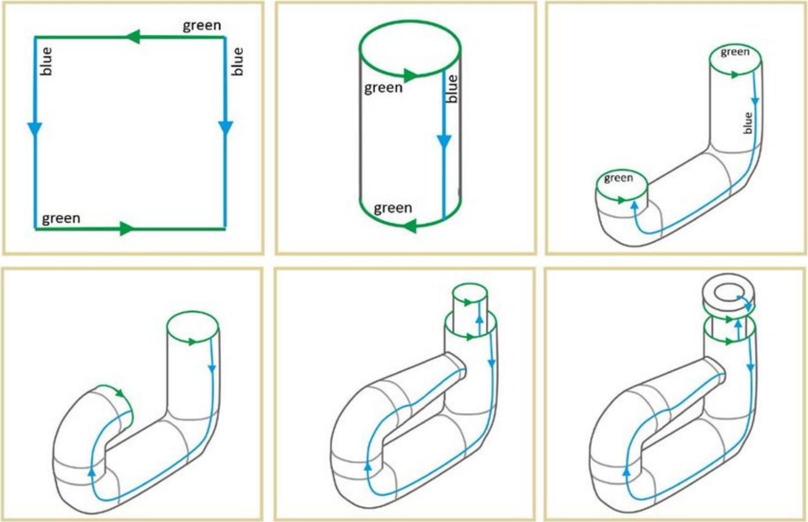
Heawood's theorem for the torus would give $\lfloor\frac{7+\sqrt{49-24\cdot 0}}{2}\rfloor = 7$ as the number of colorings needed in the worst case, so G does better

The Heawood graph – answer 1–10 as an exercise



The above graph is called the Heawood graph – **try yourself!**

Topic 2: surfaces!

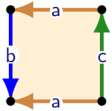


Questions we ask about a surface S

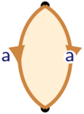
1. Have we seen S before? Is it one of the standard ones (sphere, torus, Klein bottle, projective plane etc.)?
2. How many boundary cycles = punctures does S have?
3. What is its Euler characteristic?
4. Is S connected? How many connected components does S have?
5. Is S orientable?
6. Can we find a polygonal form of S?
7. What is its standard form?
8. How many cross-caps are there in standard form?
9. How many handles are there in standard form?
10. If $d = 0$, then what is the chromatic number of S?

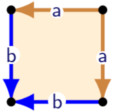
Let us answer 1-10 for a randomly generated polygonal form

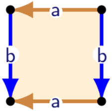
But before, let us recall what the above are!

- Annulus $\mathbb{A} \cong$ 

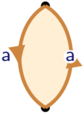
- Sphere

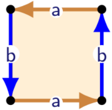
$S^2 \cong$ 

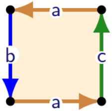
$\cong$ 

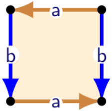
- Torus $\mathbb{T} \cong$ 

- Projective plane

$\mathbb{P}^2 \cong$ 

$\cong$ 

- Möbius strip $\mathbb{M} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

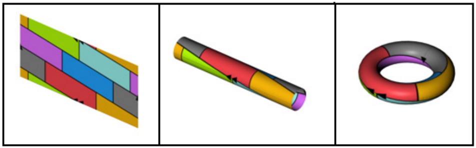
These are 2 dimensional objects, e.g. the torus is hollow:

$\mathbb{T} =$ 

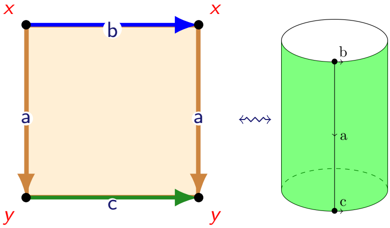
From polygons to surfaces

Recall that one goes from a polygon to a surface by identifying paired edges

For the torus that means e.g.



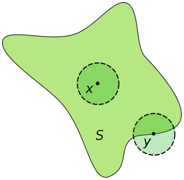
For an annulus one gets

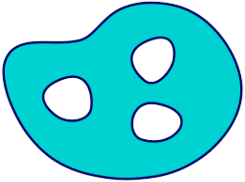


One can build of most these, e.g. a Möbius, strip out of paper

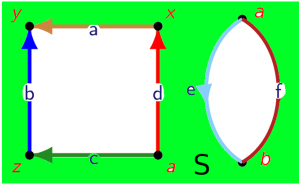
The boundary

Boundary points have neighborhoods that are half-discs; all other point have disc neighborhoods





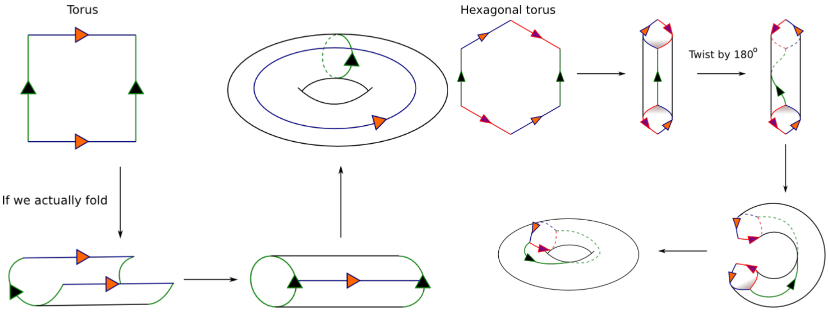
In a polygonal form, the free edges wrap around boundary components:



Note that the surface S is on the outside in these pictures

Euler characteristic

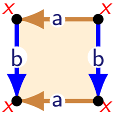
Every surface S has infinitely many polygonal forms and they might look wildly different, e.g.:

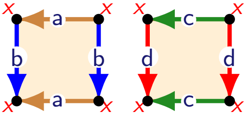


The Euler characteristic $\chi(S) = |V| - |E| + |F|$ is the same for any polygonal form

$$\text{left: } \chi(\mathbb{T}) = 1 - 2 + 1 = 0, \quad \text{right: } \chi(\mathbb{T}) = 2 - 3 + 1 = 0$$

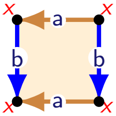
We have $\chi(S) \neq \chi(T) \Rightarrow S \ncong T$ but the converse is not true:

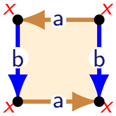
$\mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T}) = 0,$ $\mathbb{T} \cup \mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T} \cup \mathbb{T}) = 0$

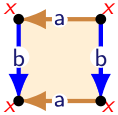
Fix: check connectivity

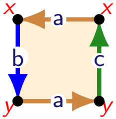
$\mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T}) = 0,$ $\mathbb{K} \leftrightsquigarrow$ 

$, \chi(\mathbb{K}) = 0$

Fix: check orientability

$\mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T}) = 0,$ $\mathbb{A} \leftrightsquigarrow$ 

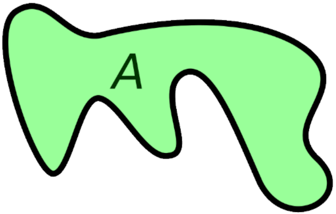
$, \chi(\mathbb{A}) = 0$

Fix: check boundary

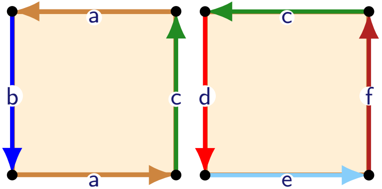
Connectivity – we can eyeball it

Connected = we can go from every point of S to any other point of S

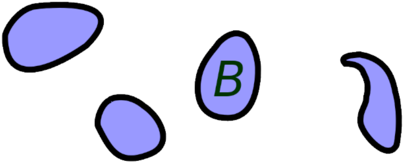
Connected:



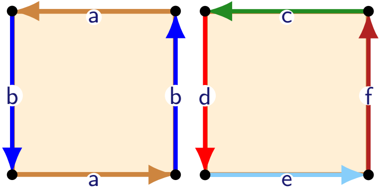
or



Not connected:



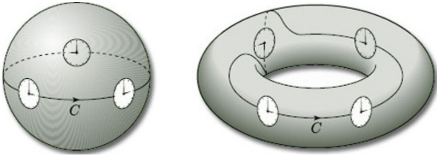
or

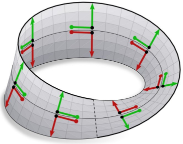


Orientability – we can tell on the words

Orientable = consistent choice of a coordinate system

Top: orientable, bottom: not orientable



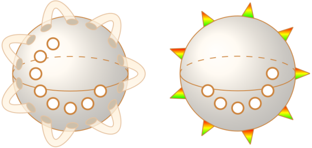


This is hard to check on the surface itself but:

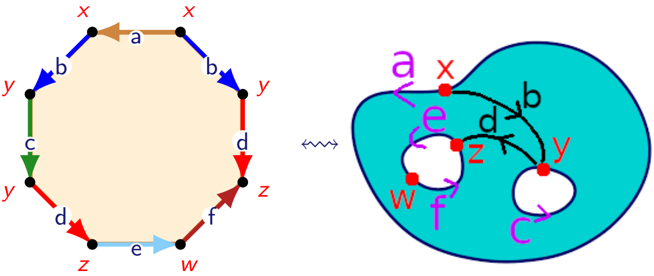
- Words encode orientability
  - Orientable: $\dots a \dots \overline{a} \dots$ or $\dots \overline{a} \dots a \dots$
  - Non-orientable: $\dots a \dots a \dots$ or $\dots \overline{a} \dots \overline{a} \dots$

Boundary = punctures = holes

Eight and six boundary components, respectively:



On the polygon this is the free-edge game: identify free edges, and check what cycles they form, e.g.:



The classification theorem

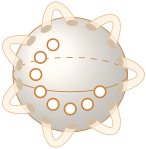
Theorem

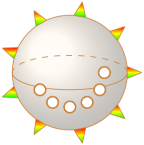
Let S be a connected surface. Then there exist non-negative integers d, p and t such that

1. $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$
2. *the boundary of S is the disjoint union of d circles*
3. *S is orientable if and only if $p = 0$*

Moreover, we can assume that $pt = 0$, in which case S is uniquely determined up to homeomorphism by (d, p, t)

Thus, every surfaces is of either of the following two forms, called standard:

$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \cong$ 

$\#^6\mathbb{D}^2 \# \#^9\mathbb{P}^2 \cong$ 

$d \leftrightsquigarrow$ punctures=boundary=holes

$p \leftrightsquigarrow$ projective planes=cross caps

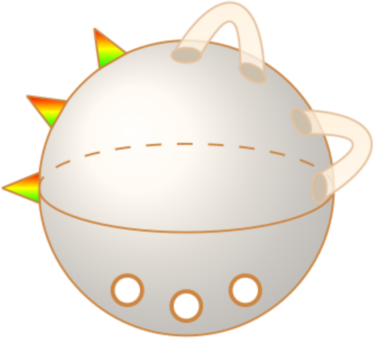
$t \leftrightsquigarrow$ handles=tori

Handles and cross-caps do not want to go along

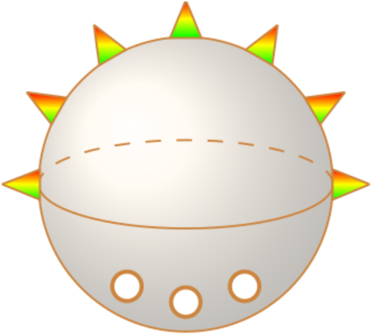
$$\mathbb{T} \# \mathbb{P}^2 \cong \mathbb{K} \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \leftrightsquigarrow \text{“}t = 2p\text{”}$$

Not true: $\mathbb{T} \cong \mathbb{K}$

We can use this to always get rid of all tori in the presence of $\mathbb{P}^2$, e.g.:



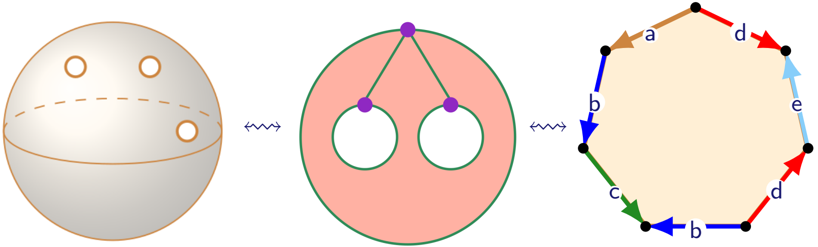
$\cong$



The left-hand surface is not in standard form

From a surface to a polygon

Here is an example how to find a word for the 3-times punctured sphere:



In general, using the classification theorem, we had standard words that we can paste together:

$$\#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \dots\ a_t\ b_t\overline{a_t}\ \overline{b_t}$$

$$\#^p\mathbb{P}^2 = a_1\ a_1\ a_2\ a_2\ \dots\ a_p\ a_p$$

$$\#^d\mathbb{D}^2 = a_1\ b_1\ a_2\ b_2\ \dots\ b_{d-1}a_d\ \overline{b}_{d-1}\ \dots\overline{b}_2\ \overline{b}_1$$

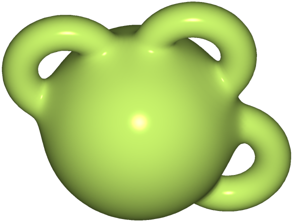
Heawood's exciting theorem

For a connected closed surface $S \ncong \mathbb{K}$ we have that the chromatic number $C(S)$ is

$$C(S) = \left\lfloor \tfrac{1}{2}(7 + \sqrt{49 - 24\chi(S)}) \right\rfloor$$

Additionally $C(\mathbb{K}) = 6$

Example

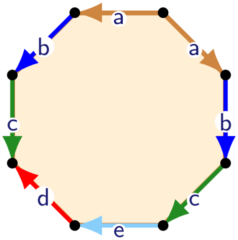
$S =$  $, \chi(S) = -4, C(S) = \lfloor 9.5208 \rfloor = 9$

recall the formula:

$$\chi(S) = 2 - d - p - 2t$$

for $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$

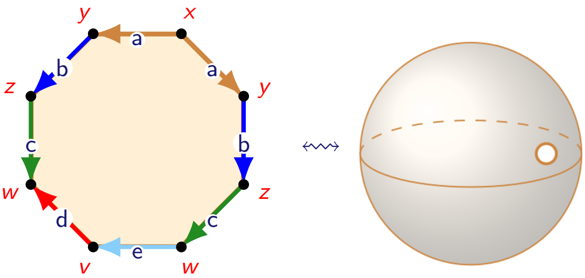
A random example



To find (d, p, t) for S we go through a list of steps:

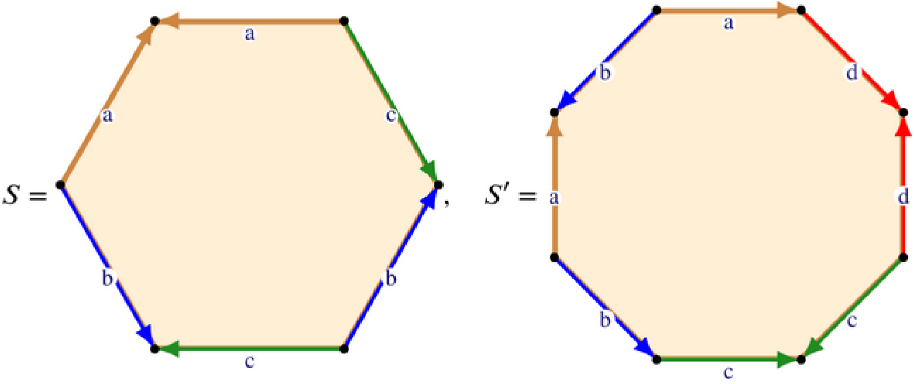
1. Identify vertices and count them $\Rightarrow |V|$
2. Count edges and faces $\Rightarrow |E|$ and $|F|$
3. Compute $\chi(S) = |V| - |E| + |F|$
4. Check how free edges arrange themselves in cycles $\Rightarrow d$
5. Check for $a...a$ and $\bar{a}...\bar{a}$; if we find them, then $t = 0$ otherwise $p = 0$ $\Rightarrow$ we get either p or t
6. Use $\chi(S) = 2 - d - p - 2t$ to determine the remaining entry t or p

A random example – part 2



Lets do it!

1. From the above we get $|V| = 5$
2. Counting edges and faces gives $|E| = 5$ and $|F| = 1$
3. We get $\chi(S) = |V| - |E| + |F| = 1$
4. The only free edges $d\colon v \to w$ and $e\colon w \to v$ form one cycle, so $d = 1$
5. No pairs $a...a$ or $\bar{a}...\bar{a}$, so $p = 0$
6. $1 = \chi(S) = 2 - 1 - 0 - 2t$ gives $t = 0$



These two surfaces are well-known and want to be identified – **try yourself!**

Exercise Write down some word representing a polygonal form and identify its corresponding standard form, meaning (d, p, t)

Topic 3: knots!

Questions we ask about a knot K

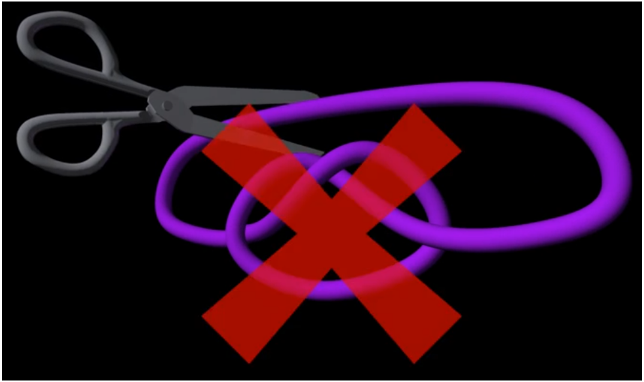
1. Have we seen K before? Is it one of the standard ones, i.e. for low crossing number?
2. Can the diagram(=projection) of K that we see be simplified?
3. Is K the unknot a.k.a. trivial?
4. What is the crossing number of K?
5. Is K alternating?
6. Is K three colorable?
7. Is K p-colorable for $p > 3$?
8. What is the knot determinant of K?
9. Can we explicitly compute a Seifert surface for a diagram of K?
10. What is the genus of K?

Let us answer 1-10 for the knot 5_1

But before, let us recall what the above are!

Knots

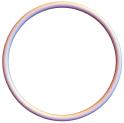
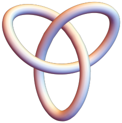
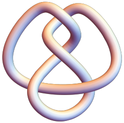
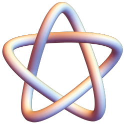
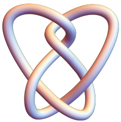
A knot is an embedding of S^1 into $\mathbb{R}^3$ and we study these up to equivalence, i.e. continuous deformation without cutting



Note that all knots are homeomorphic, so this is the wrong notion of equivalence for knots

The periodic table of knots

A main point of knot theory is to have a table of knots up to mirror images:

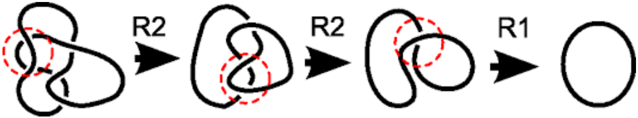
knot					
name	unknot	trefoil	figure 8	cinquefoil	three-twist
notation	0_1	3_1	4_1	5_1	5_2
$cross(K)$	0	3	4	5	5
$\det(K)$	1	3	5	5	7
$g(K)$	0	1	1	2	1
prime?	yes	yes	yes	yes	yes
alternating?	yes	yes	yes	yes	yes

Google The Rolfsen Knot Table or use e.g. KnotData of Mathematica

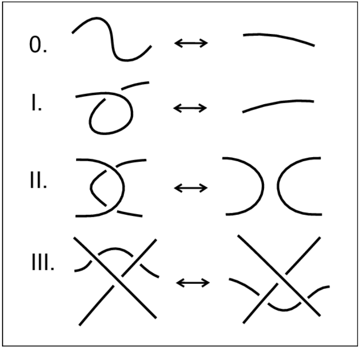
Mirror images (=flipped crossings) cannot be detected by our invariants

Simplify diagrams using Reidemeister moves

A first step is to check whether there are any “obvious” simplifications:

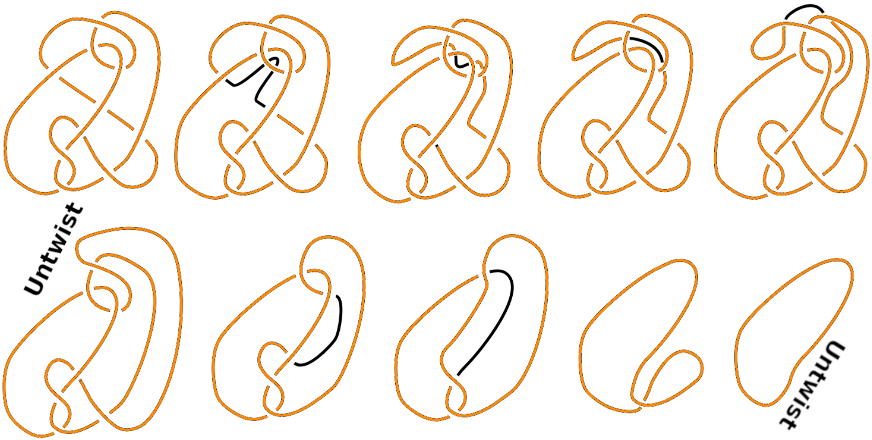


Recall that two knot diagrams represent the same knot if and only if we can relate them by the Reidemeister moves:



The culprit

Sometimes diagrams drastically simplify:



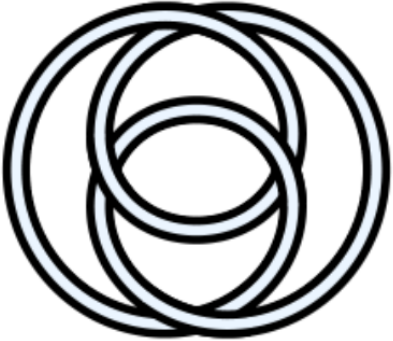
Reidemeister moves – practise makes perfect

Exercise Check whether you understand the Reidemeister moves used for the culprit on the previous slide

Exercise Check using isotopies and Reidemeister moves whether these two beasts are the same knot:

$K =$

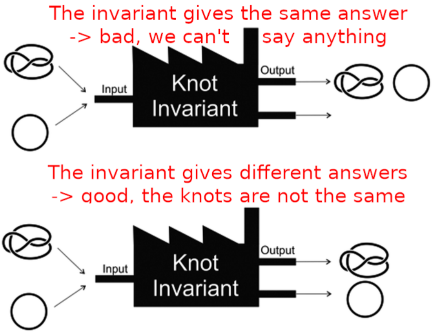
, $K' =$



The main question...

...is always: are two knot diagrams representing the same knot?

We want knot invariants to do this!

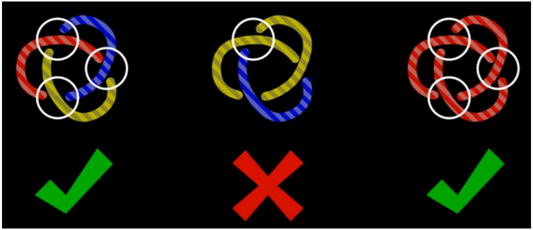


We had essentially two ways to decide that

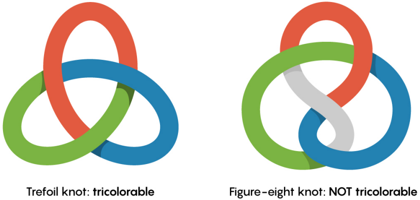
- Knot invariant 1: colorability
- Knot invariant 2: genus

p-colorable; here only $p = 3$

Coloring = each segment gets a color such that we have 3-colored crossings or monochromatic crossings

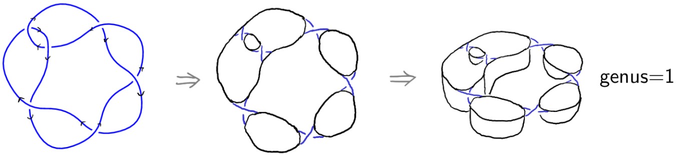


A knot is 3-colorable if it admits a non-monochromatic coloring



The genus: great to check whether a knot is trivial

Genus = the minimal t of all Seifert surfaces; to compute it for an alternating knot run Seifert's algorithm:

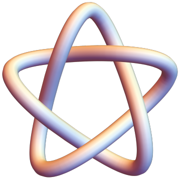


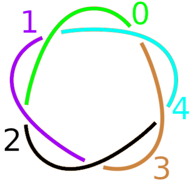
Then $t = \frac{1}{2}(1 + c - s)$ where c is the number of crossings and s the number of Seifert circles

Cool fact (verify "$\Leftarrow$" as an exercise):

$$g(K) = 0 \Leftrightarrow K \cong \text{unknot}$$

The knot $K = 5_1$

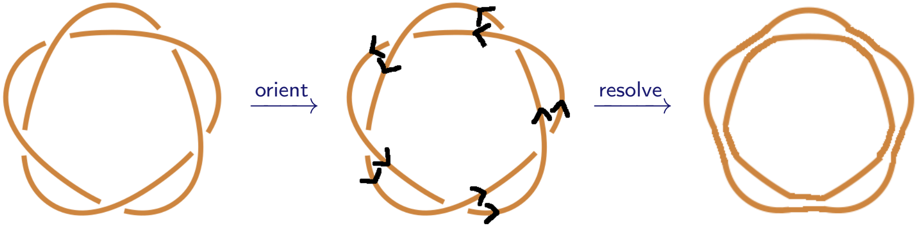




Let us go through the list of steps:

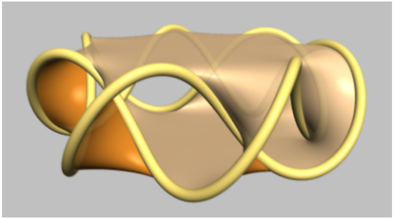
1. We have seen it before, it is 5_1
2. The diagram cannot be made simpler in any obvious way
3. The knot is not trivial, see next slide or coloring above
4. Since the diagram is alternating $cross(K) = 5$
5. The diagram is clearly alternating
6. No, K is not 3-colorable see above
7. Yes, K is 5-colorable, see above
8. We have $\det(K) = 5$ by computation
9. Yes, Seifert surfaces are easy to get, see next slide
10. $g(K) = 1$, see next slide

The knot $K = 5_1$ – Seifert surfaces and genus

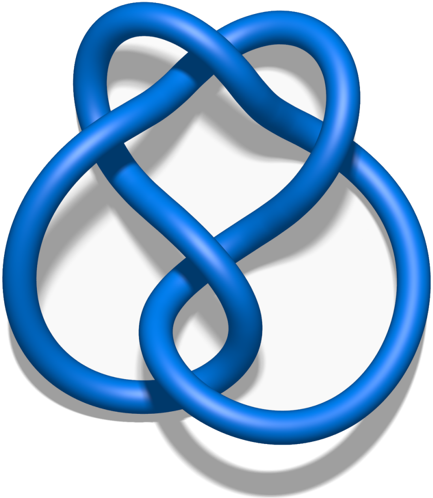


Thus, $c = 5$ and $s = 2$ gives $\chi(S) = s - c = -3$ and
$g = \frac{1}{2}(1 - \chi(S)) = \frac{1}{2}(1 + c - s) = 2$

Putting in the twists gives:



Another knot – answer 1–10 as an exercise



This is knot 5_2 – **try yourself!**

I hope you enjoyed topology!