Topology – week 8

Math3061

Daniel Tubbenhauer, University of Sydney

Eulerian circuits and graphs

A Eulerian circuit is a circuit that passes through every edge exactly once

Eulerian circuits and graphs

A Eulerian circuit is a circuit that passes through every edge exactly once

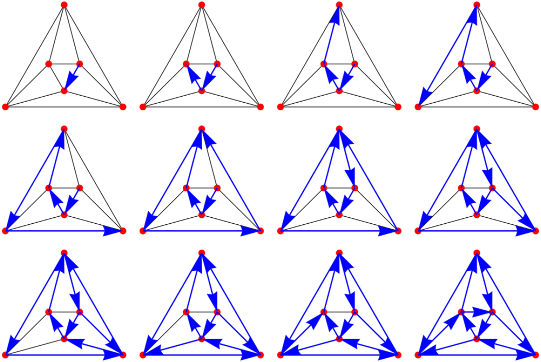
A graph is Eulerian if it has a Eulerian circuit

Eulerian circuits and graphs

A Eulerian circuit is a circuit that passes through every edge exactly once

A graph is Eulerian if it has a Eulerian circuit

Example



Warning Eulerian graphs do not need to be connected because they may have vertices of degree 0!

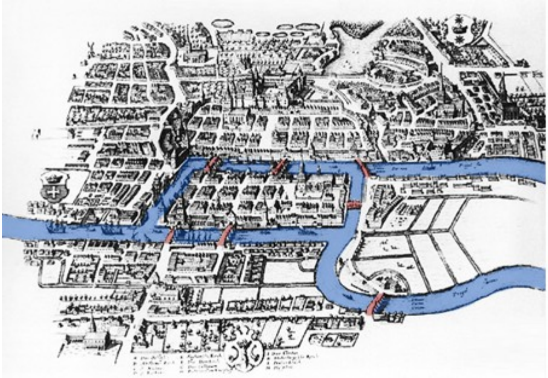
Finding Eulerian circuits

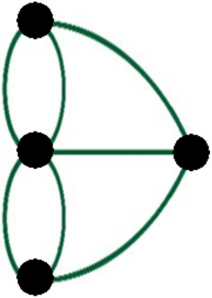
In 1736 Euler asked when graphs have Eulerian circuits (without having this terminology)

Finding Eulerian circuits

In 1736 Euler asked when graphs have Eulerian circuits (without having this terminology)

The motivation was that they wanted to know if it was possible to walk around the city of Königsberg crossing each bridge exactly once

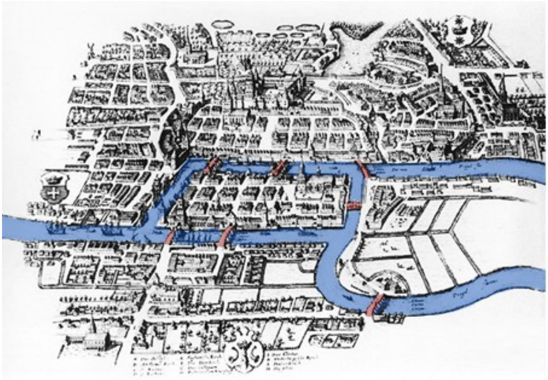


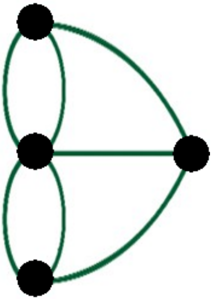


Finding Eulerian circuits

In 1736 Euler asked when graphs have Eulerian circuits (without having this terminology)

The motivation was that they wanted to know if it was possible to walk around the city of Königsberg crossing each bridge exactly once





In answering this question Euler laid the foundations of graph theory

Classifying Eulerian graphs

Theorem

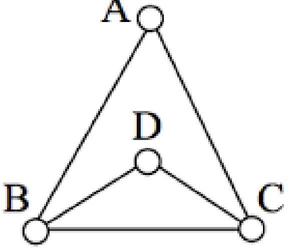
Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree

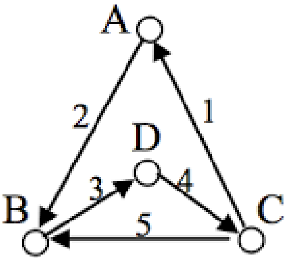
Theorem

Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree

Proof

Assume that there is at least one vertex v of odd degree. Since we want to visit every edge exactly once we will eventually get stuck in v or another vertex of odd degree while trying to create an Eulerian cycle. Hence, G can not have an Eulerian cycle





Classifying Eulerian graphs

Proof continued

Conversely, if every vertex has even degree, then G is not a tree so contains some circuit C. If C is an Euler circuit we are done, and if not remove all edges of C from G. The resulting (potentially disconnected) graph G' has still even degrees for all of its vertices but fewer edges than G

Classifying Eulerian graphs

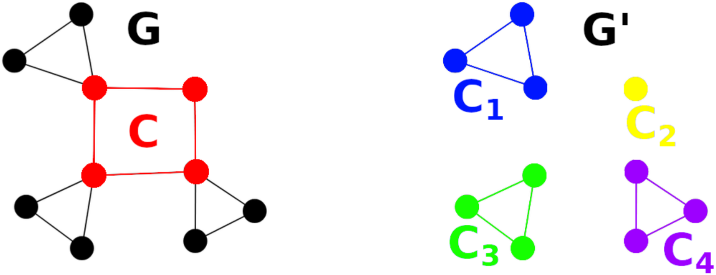
Proof continued

Conversely, if every vertex has even degree, then G is not a tree so contains some circuit C. If C is an Euler circuit we are done, and if not remove all edges of C from G. The resulting (potentially disconnected) graph G' has still even degrees for all of its vertices but fewer edges than G

So we can argue by induction on the number of edges (the base case has no edges and is thus clear), and inductively we can assume that the connected components of G' have Euler circuits $C_1, \ldots, C_n$

Proof continued

We piece C and $C_1, \dots, C_n$ together into an Euler cycle: we walk along C and whenever we hit a vertex of C_i we take a detour over C_i



Eulerian paths

A Eulerian path is a path that is not a circuit and which passes through every edge exactly once

Corollary

Let $G = (V, E)$ be a connected graph that is not Eulerian. Then G has a Eulerian path if and only if it has exactly two vertices of odd degree

Eulerian paths

A Eulerian path is a path that is not a circuit and which passes through every edge exactly once

Corollary

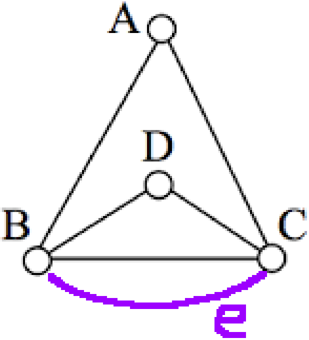
Let $G = (V, E)$ be a connected graph that is not Eulerian. Then G has a Eulerian path if and only if it has exactly two vertices of odd degree

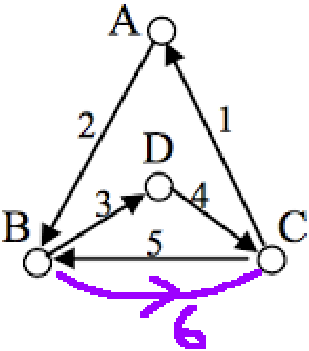
Proof

Only vertices of odd degree can be a start or an end vertex, so we need precisely two of them (all other must be of even degree by the same argument as before)

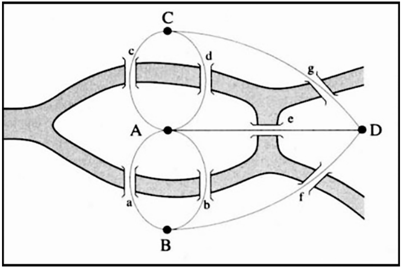
Proof continued

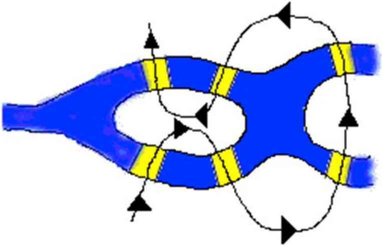
Conversely, if v and w are the two vertices of even degree, then we put an additional edge e between them. We get a graph $G' = G \cup \{e\}$ and the previous theorem gives us an Euler circuit C in G'. Then $C \setminus \{e\}$ is an Euler path



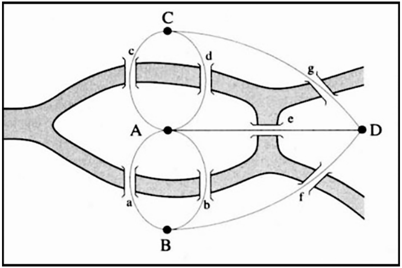


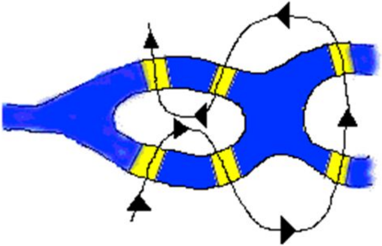
What about Königsberg?





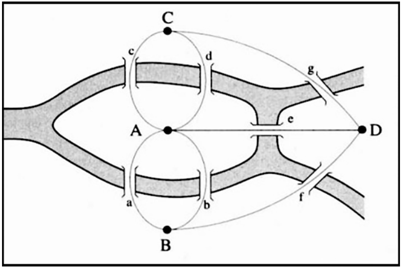
What about Königsberg?

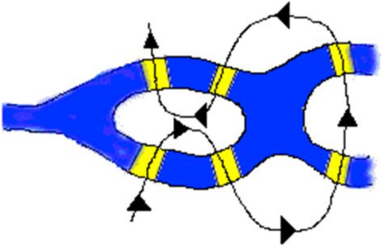




There is no Eulerian circuit since all vertices have odd degree

What about Königsberg?

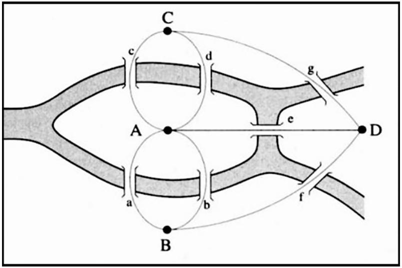


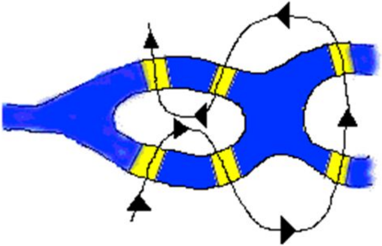


There is no Eulerian circuit since all vertices have odd degree

There is no Eulerian path since all vertices have odd degree

What about Königsberg?





There is no Eulerian circuit since all vertices have odd degree

There is no Eulerian path since all vertices have odd degree

Solution: Destroy bridge e ;-)

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Remarks

- Homeomorphism is the higher dim analog of isomorphism for graphs
  We treat two spaces as being "equal" if they are homeomorphic

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Remarks

- Homeomorphism is the higher dim analog of isomorphism for graphs
  We treat two spaces as being "equal" if they are homeomorphic
- The maps f and g are both bijections with continuous inverses

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Remarks

- Homeomorphism is the higher dim analog of isomorphism for graphs
  We treat two spaces as being "equal" if they are homeomorphic
- The maps f and g are both bijections with continuous inverses
- We have $X \cong X$

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Remarks

- Homeomorphism is the higher dim analog of isomorphism for graphs
  We treat two spaces as being "equal" if they are homeomorphic
- The maps f and g are both bijections with continuous inverses
- We have $X \cong X$
- If $X \cong Y$, then $Y \cong X$

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

Remarks

- Homeomorphism is the higher dim analog of isomorphism for graphs We treat two spaces as being "equal" if they are homeomorphic
- The maps f and g are both bijections with continuous inverses
- We have $X \cong X$
- If $X \cong Y$, then $Y \cong X$
- If $X \cong Y$ and $Y \cong Z$, then $X \cong Z$

Examples of homeomorphisms

Proposition

If $a < b$ *and* $c < d$*, then* $[a, b] \cong [c, d]$

Examples of homeomorphisms

Proposition

If $a < b$ *and* $c < d$*, then* $[a, b] \cong [c, d]$

Proof

Define maps $f : [a, b] \longrightarrow [c, d]; x \mapsto c + \frac{d-c}{b-a}(x - a)$

Examples of homeomorphisms

Proposition

If $a < b$ and $c < d$, then $[a, b] \cong [c, d]$

Proof

Define maps $f : [a, b] \longrightarrow [c, d]; x \mapsto c + \frac{d-c}{b-a}(x - a)$

$$g : [c, d] \longrightarrow [a, b]; x \mapsto a + \frac{b-a}{d-c}(x - c)$$

Examples of homeomorphisms

Proposition

If $a < b$ and $c < d$, then $[a, b] \cong [c, d]$

Proof

Define maps $f : [a, b] \longrightarrow [c, d]; x \mapsto c + \frac{d-c}{b-a}(x - a)$

$$g : [c, d] \longrightarrow [a, b]; x \mapsto a + \frac{b-a}{d-c}(x - c)$$

Exercise Show that $(a, b) \cong (c, d)$ and $(a, b] \cong (c, d]$ $\overset{!!!}{\cong}$ $[a, b) \cong [c, d)$

Examples of homeomorphisms

Proposition

If $a < b$ and $c < d$, then $[a, b] \cong [c, d]$

Proof

Define maps $f : [a, b] \longrightarrow [c, d]; x \mapsto c + \frac{d-c}{b-a}(x - a)$

$$g : [c, d] \longrightarrow [a, b]; x \mapsto a + \frac{b-a}{d-c}(x - c)$$

Exercise Show that $(a, b) \cong (c, d)$ and $(a, b] \cong (c, d] \overset{!!!}{\cong} [a, b) \cong [c, d)$

Proposition

If $a < b$, then $(a, b) \cong \mathbb{R}$

Examples of homeomorphisms

Proposition

If $a < b$ and $c < d$, then $[a, b] \cong [c, d]$

Proof

Define maps $f : [a, b] \longrightarrow [c, d]; x \mapsto c + \frac{d-c}{b-a}(x - a)$

$$g : [c, d] \longrightarrow [a, b]; x \mapsto a + \frac{b-a}{d-c}(x - c)$$

Exercise Show that $(a, b) \cong (c, d)$ and $(a, b] \cong (c, d] \overset{!!!}{\cong} [a, b) \cong [c, d)$

Proposition

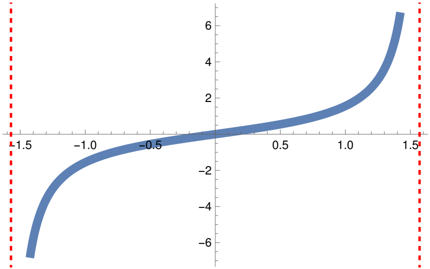
If $a < b$, then $(a, b) \cong \mathbb{R}$

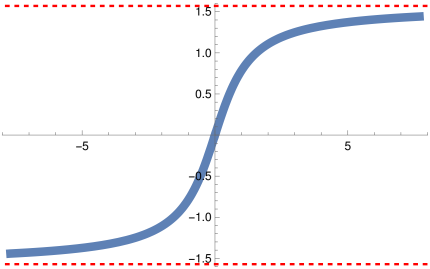
Proof It is enough to show that $(-\frac{\pi}{2}, \frac{\pi}{2}) \cong \mathbb{R}$

Examples of homeomorphisms

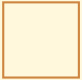
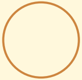
Proof continued

Homeomorphisms are given by $f(x) = \tan(x)$ and $g(x) = \tan^{-1}(x)$

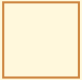
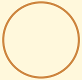
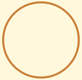




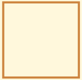
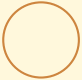
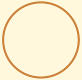
Proposition

 $\cong$  $= S^1$

Proposition

 $\cong$  $= S^1$ *We show that*  $\cong$ 

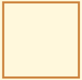
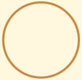
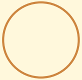
Proposition

 $\cong$  $= S^1$ *We show that*  $\cong$ 

Proof

The square is $\{\,(x,y)\,|\,|x|+|y|=1\,\}$ and $S^1=\{\,(x,y)\,|\,x^2+y^2=1\,\}$

Proposition

 $\cong$  $= S^1$ *We show that*  $\cong$ 

Proof

The square is $\{\,(x,y)\,|\,|x|+|y|=1\,\}$ and $S^1=\{\,(x,y)\,|\,x^2+y^2=1\,\}$

Define: $f:$  $\longrightarrow S^1; (x,y)\mapsto\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)$

Proposition

$\square \cong \bigcirc = S^1$ *We show that* $\Diamond \cong \bigcirc$

Proof

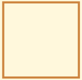
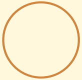
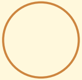
The square is $\{\, (x,y) \mid |x| + |y| = 1 \,\}$ and $S^1 = \{\, (x,y) \mid x^2 + y^2 = 1 \,\}$

Define: $f : \square \longrightarrow S^1; (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$g : S^1 \longrightarrow \square; (x,y) \mapsto \left(\frac{x}{|x|+|y|}, \frac{y}{|x|+|y|}\right)$

Examples of homeomorphisms...

Proposition

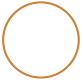
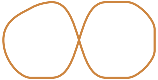
 $\cong$  $= S^1$ *We show that*  $\cong$ 

Proof

The square is $\{ (x,y) \mid |x| + |y| = 1 \}$ and $S^1 = \{ (x,y) \mid x^2 + y^2 = 1 \}$

Define: $f :$  $\longrightarrow S^1; (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$g : S^1 \longrightarrow$ $; (x,y) \mapsto \left(\frac{x}{|x|+|y|}, \frac{y}{|x|+|y|}\right)$

Note that  $\not\cong$ 

Examples of homeomorphisms...

Proposition

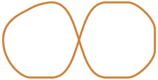
$\square \cong \bigcirc = S^1$ *We show that* $\Diamond \cong \bigcirc$

Proof

The square is $\{\,(x,y) \mid |x| + |y| = 1\,\}$ and $S^1 = \{\,(x,y) \mid x^2 + y^2 = 1\,\}$

Define: $f : \square \longrightarrow S^1; (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$g : S^1 \longrightarrow \square; (x,y) \mapsto \left(\frac{x}{|x|+|y|}, \frac{y}{|x|+|y|}\right)$

Note that $\bigcirc \not\cong$ 

For free we see that the square and disk are homeomorphic:

Corollary

$\blacksquare \cong \blacklozenge \cong \bullet$

Stereographic projection in two dimensions

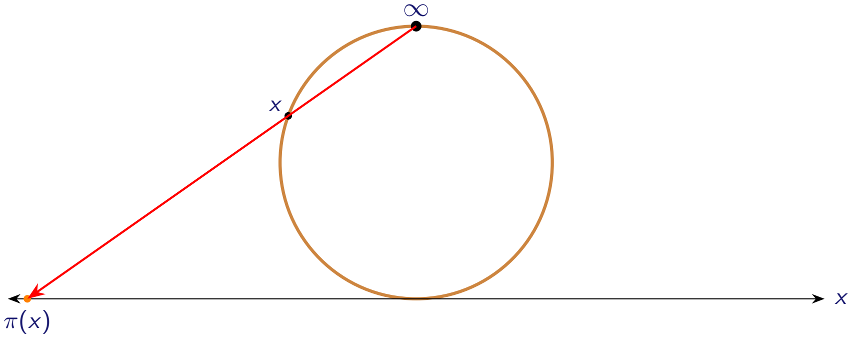
Think of the north pole of the circle S^1 as ∞

Stereographic projection gives a homeomorphism $\pi\colon S^1 \setminus \{\infty\} \to \mathbb{R}$:

Stereographic projection in two dimensions

Think of the north pole of the circle S^1 as ∞

Stereographic projection gives a homeomorphism $\pi\colon S^1 \setminus \{\infty\} \to \mathbb{R}$:



Stereographic projection in three dimensions

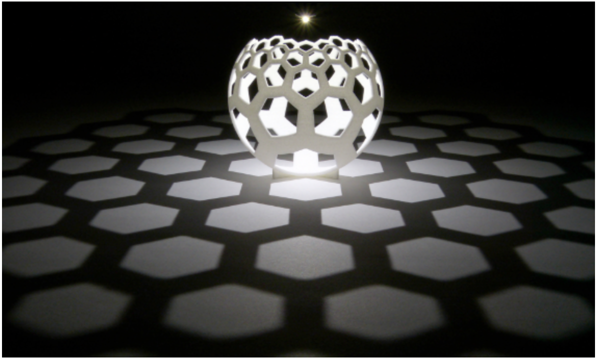
Think of the north pole of the circle S^2 as ∞

Stereographic projection gives a homeomorphism $\pi \colon S^2 \setminus \{\infty\} \to \mathbb{R}^2$:

Stereographic projection in three dimensions

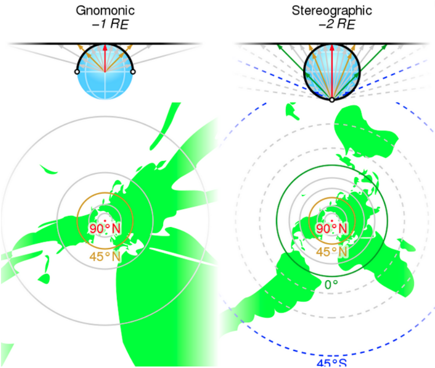
Think of the north pole of the circle S^2 as ∞

Stereographic projection gives a homeomorphism $\pi\colon S^2 \setminus \{\infty\} \to \mathbb{R}^2$:



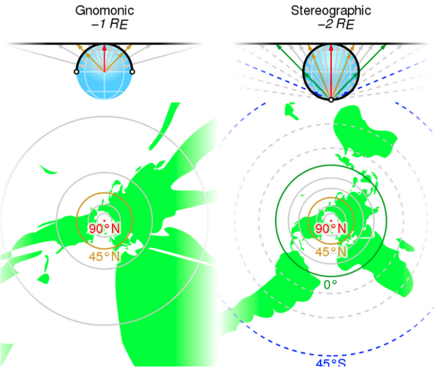
Maps

Stereographic projection is used to draw maps:



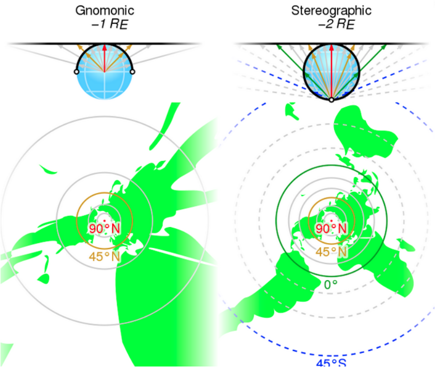
Maps

Stereographic projection is used to draw maps:



Other projections are also used such as gnomonic projections, conic projections and the Mercator projection, which is a cylindrical projection

Stereographic projection is used to draw maps:



Other projections are also used such as gnomonic projections, conic projections and the Mercator projection, which is a cylindrical projection

Now that we have seen homeomorphisms we are ready to define surfaces

Surfaces — informal definition

Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $f(x, y) = z$ / alternatively to a disc

Surfaces — informal definition

Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $f(x, y) = z$ / alternatively to a disc

Here "locally" means that we can find a "local neighborhood" of every point where the function looks like the plane $f(x, y) = z$ / a disc

Surfaces — informal definition

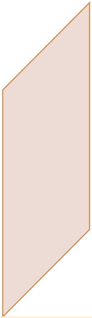
Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $f(x, y) = z$ / alternatively to a disc

Here "locally" means that we can find a "local neighborhood" of every point where the function looks like the plane $f(x, y) = z$ / a disc

Examples

- A standard xyz-plane in $\mathbb{R}^3$



Surfaces — informal definition

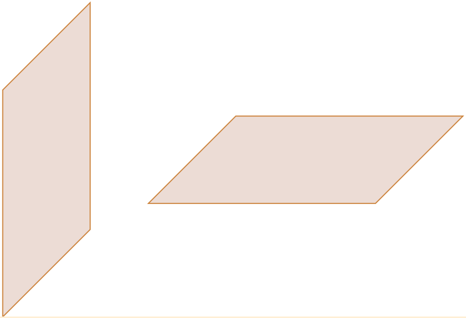
Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $f(x,y)=z$ / alternatively to a disc

Here "locally" means that we can find a "local neighborhood" of every point where the function looks like the plane $f(x,y)=z$ / a disc

Examples

- A standard xyz-plane in $\mathbb{R}^3$



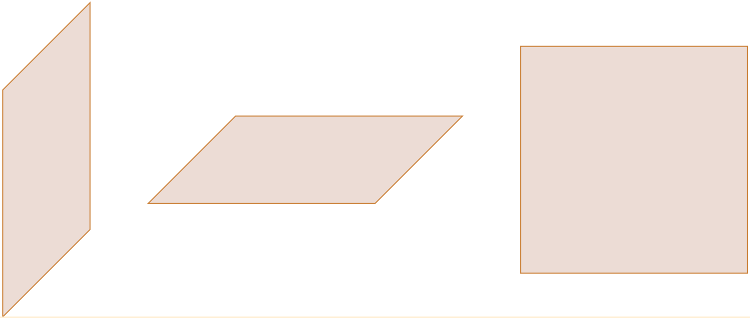
Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ given by $f(x,y)=z$ / alternatively to a disc

Here "locally" means that we can find a "local neighborhood" of every point where the function looks like the plane $f(x,y)=z$ / a disc

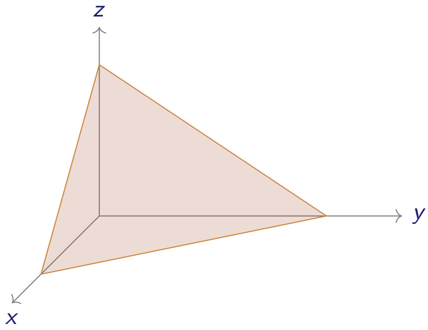
Examples

- A standard xyz-plane in $\mathbb{R}^3$



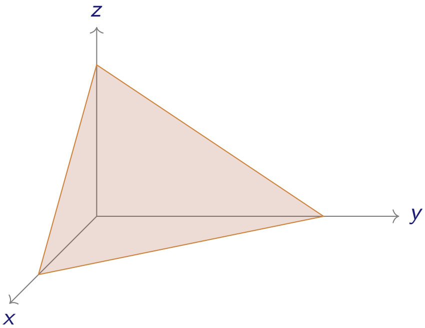
Surfaces — examples...

- Non-standard planes in $\mathbb{R}^3$

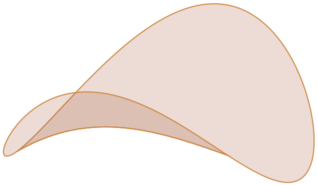


Surfaces — examples...

- Non-standard planes in $\mathbb{R}^3$

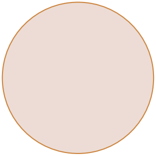


- Curved surfaces in $\mathbb{R}^3$



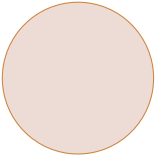
Surfaces — examples...

- A disk $\mathbb{D}^2$

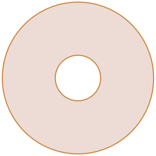


Surfaces — examples...

- A disk $\mathbb{D}^2$



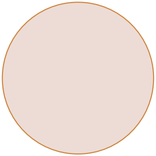
- An annulus $\mathbb{A}$



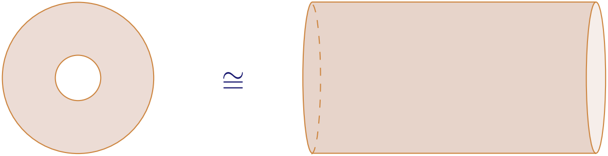
$\cong$

Surfaces — examples...

- A disk $\mathbb{D}^2$



- An annulus $\mathbb{A} \cong$ cylinder

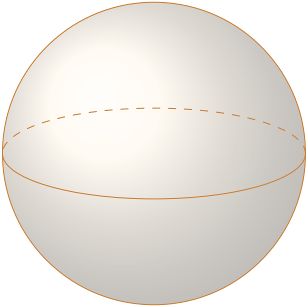


Strictly speaking, these are not surfaces according to our definition because they have a boundary, whereas planes in $\mathbb{R}^2$ do not have boundaries.

Our rigorous definition of a surface will allow surfaces with boundaries

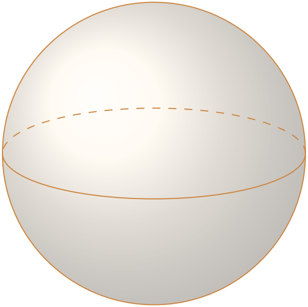
Surfaces — examples...

- A sphere S^2

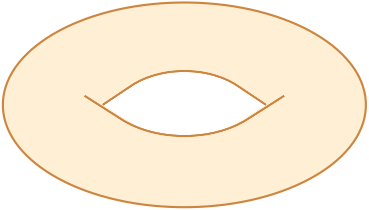


Surfaces — examples...

- A sphere S^2



- A torus $\mathbb{T}$



Surfaces — real world examples...

- A sphere $S^2 \cong$ soccer ball

Surfaces — real world examples...

- A sphere $S^2 \cong$ soccer ball

- A torus $\mathbb{T} \cong$ swim ring

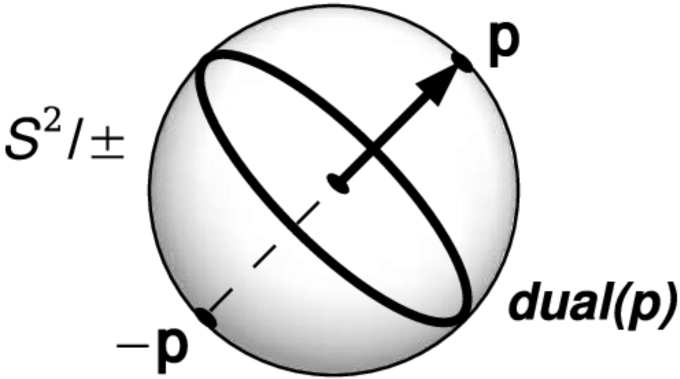
Surfaces — real world example...

- Here is a surface with boundary:

The patches are examples of neighborhoods which are discs

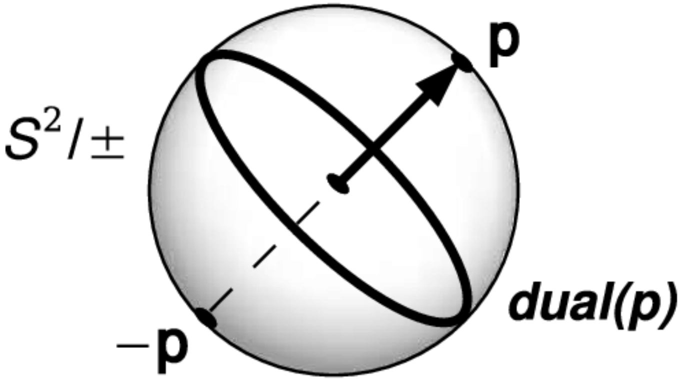
Surfaces — examples...

- The real projective plane $\mathbb{P}^2 = S^2/\text{antipode}$



Surfaces — examples...

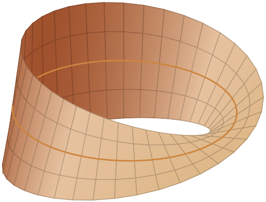
- The real projective plane $\mathbb{P}^2 = S^2/\text{antipode}$



We will see other ways to describe $\mathbb{P}^2$ later

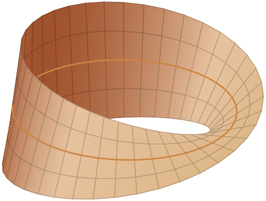
Surfaces — examples...

- A Möbius band, or Möbius strip, $\mathbb{M}$

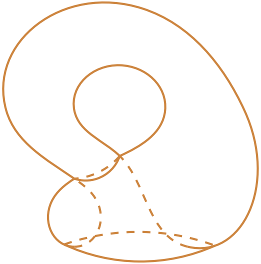


Surfaces — examples...

- A Möbius band, or Möbius strip, $\mathbb{M}$

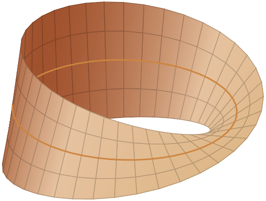


- A Klein bottle $\mathbb{K}$, also Klein surface

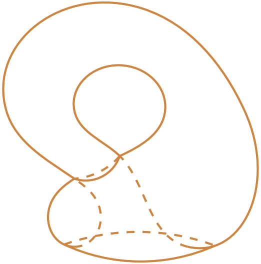


Surfaces — examples...

- A Möbius band, or Möbius strip, $\mathbb{M}$



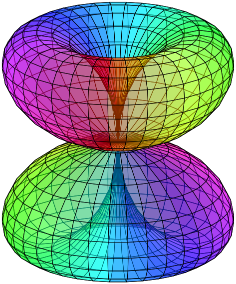
- A Klein bottle $\mathbb{K}$, also Klein surface



This is a three dimensional "shadow" of a four dimensional object

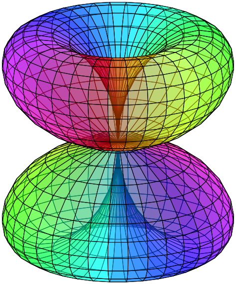
Surfaces — non-examples

- This is not a surface because of the cusp at the origin

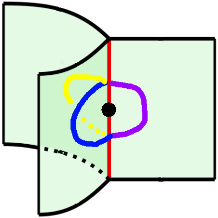


Surfaces — non-examples

- This is not a surface because of the cusp at the origin



- This is not a surface because the indicated point has not a disc neighborhood



Identification spaces

A partition of a surface $S \subseteq \mathbb{R}^m$ is a collection $X_1, \ldots, X_r$ of subsets of S such that $S = X_1 \cup X_2 \cup \cdots \cup X_r$

Identification spaces

A partition of a surface $S \subseteq \mathbb{R}^m$ is a collection $X_1, \ldots, X_r$ of subsets of S such that $S = X_1 \cup X_2 \cup \cdots \cup X_r$

The space S is an identification space for $Y \subseteq \mathbb{R}^n$ if there exists a continuous surjective map $f : S \longrightarrow Y$

Identification spaces

A partition of a surface $S \subseteq \mathbb{R}^m$ is a collection $X_1, \ldots, X_r$ of subsets of S such that $S = X_1 \cup X_2 \cup \cdots \cup X_r$

The space S is an identification space for $Y \subseteq \mathbb{R}^n$ if there exists a continuous surjective map $f : S \longrightarrow Y$

Note $Y = f(X_1) \cup f(X_2) \cup \cdots \cup f(X_r)$ and that the map f implicitly identifies the points in $f(X_{i_1}) \cap \cdots \cap f(X_{i_s})$, for $1 \leq i_1, \ldots, i_s \leq r$

Identification spaces

A partition of a surface $S \subseteq \mathbb{R}^m$ is a collection $X_1, \ldots, X_r$ of subsets of S such that $S = X_1 \cup X_2 \cup \cdots \cup X_r$

The space S is an identification space for $Y \subseteq \mathbb{R}^n$ if there exists a continuous surjective map $f : S \longrightarrow Y$

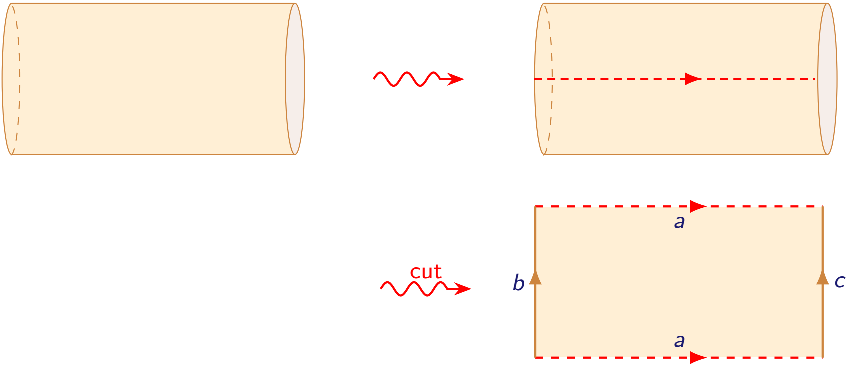
Note $Y = f(X_1) \cup f(X_2) \cup \cdots \cup f(X_r)$ and that the map f implicitly identifies the points in $f(X_{i_1}) \cap \cdots \cap f(X_{i_s})$, for $1 \leq i_1, \ldots, i_s \leq r$

This makes is possible to understand Y in terms of, often, easier spaces $X_1, \ldots, X_r$, which we think of as covering Y like a patchwork quilt

Identification space for a cylinder

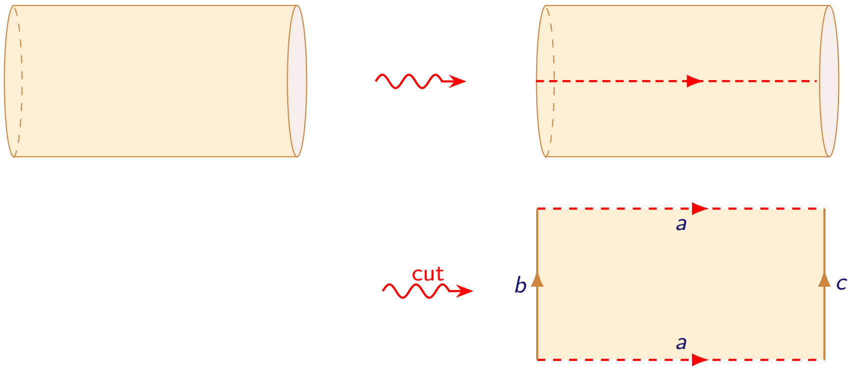
Identification space for a cylinder

Identification space for a cylinder

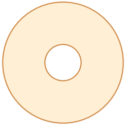


That is, the cylinder is the identification space obtained by identifying the top and bottom edges of a suitably sized rectangle

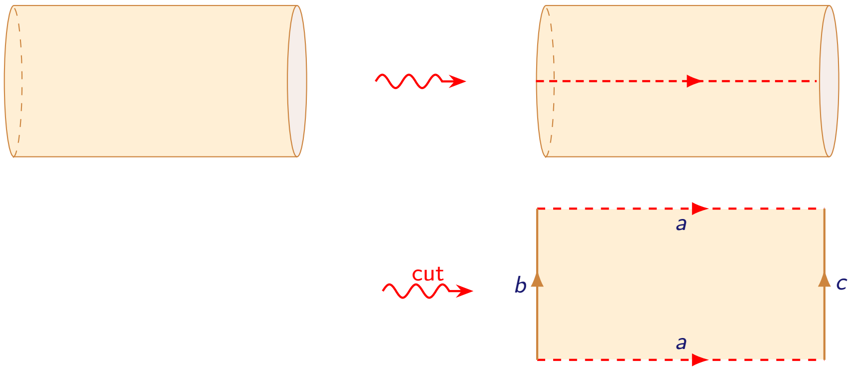
Identification space for a cylinder



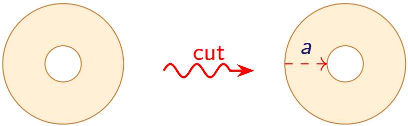
That is, the cylinder is the identification space obtained by identifying the top and bottom edges of a suitably sized rectangle

$\mathbb{A} =$ 

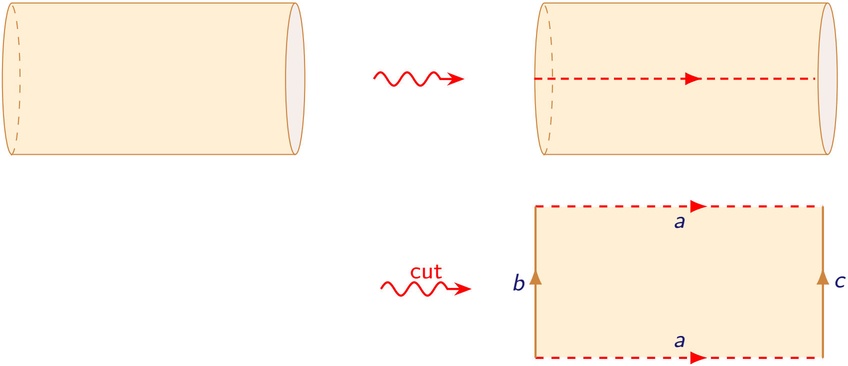
Identification space for a cylinder



That is, the cylinder is the identification space obtained by identifying the top and bottom edges of a suitably sized rectangle

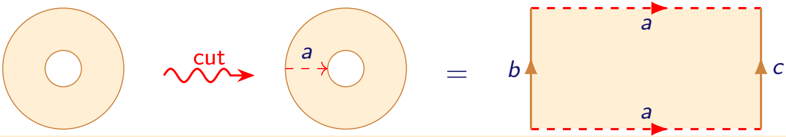
$\mathbb{A} =$ 

Identification space for a cylinder

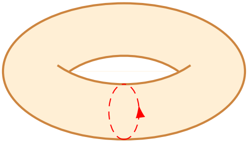


That is, the cylinder is the identification space obtained by identifying the top and bottom edges of a suitably sized rectangle

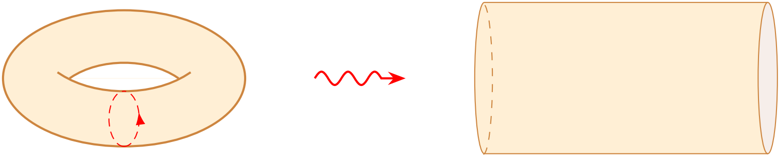
$\mathbb{A} =$



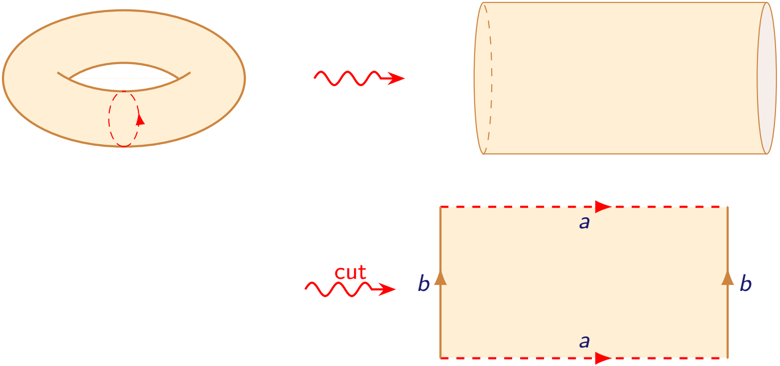
Identification space for a torus



Identification space for a torus

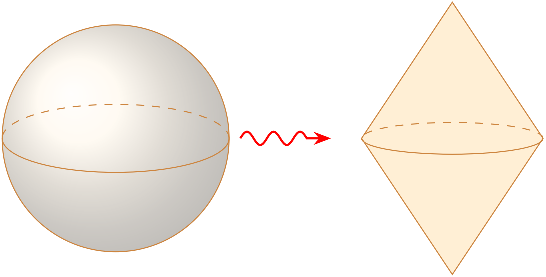


Identification space for a torus

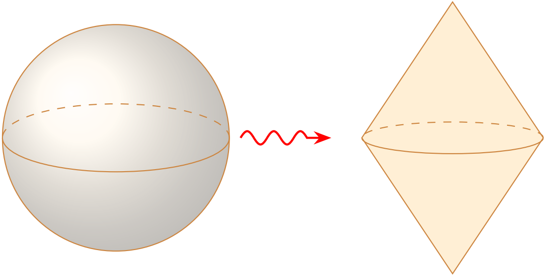


So, the torus $\mathbb{T}$ is obtained by identifying the top and bottom, and the left and right, edges of a rectangle

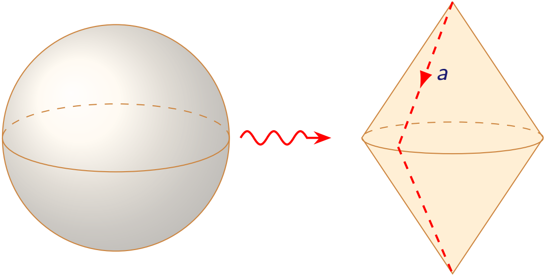
Identification space for a sphere



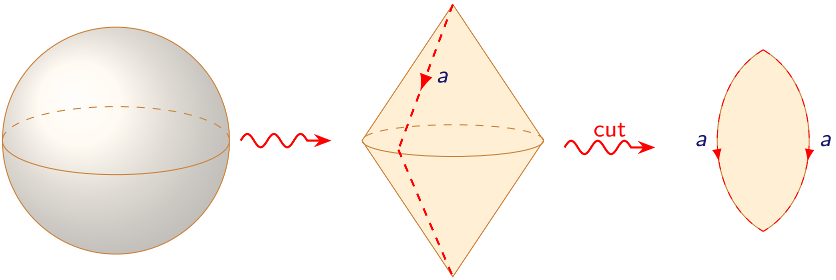
Identification space for a sphere



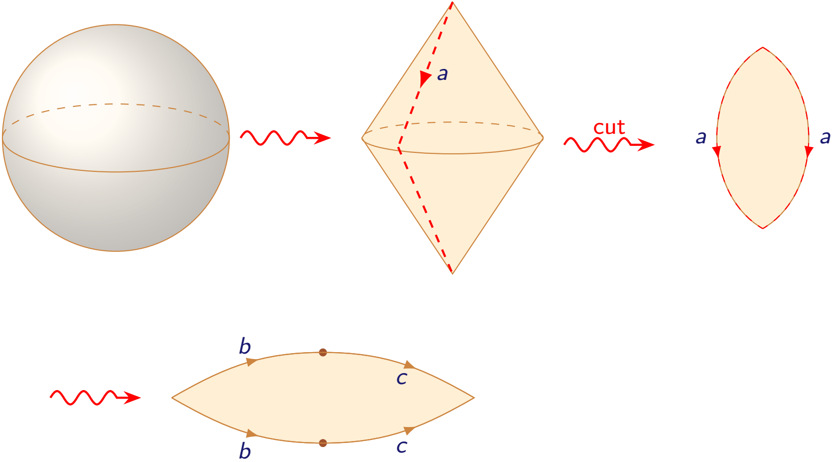
Identification space for a sphere



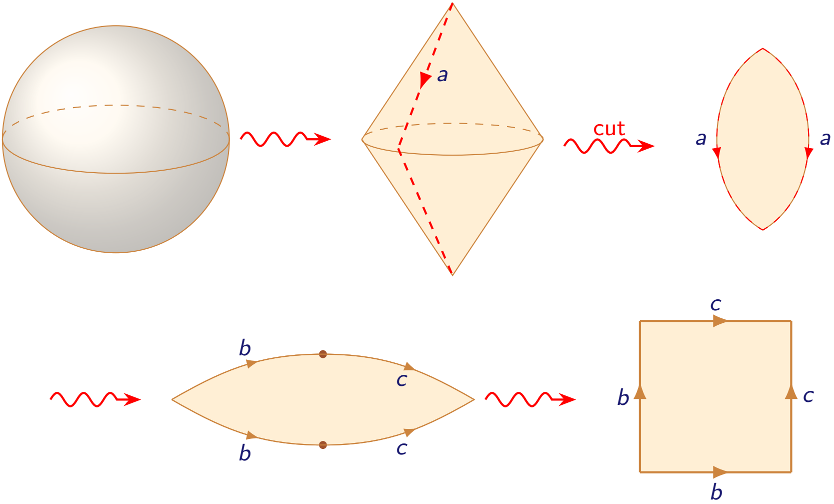
Identification space for a sphere



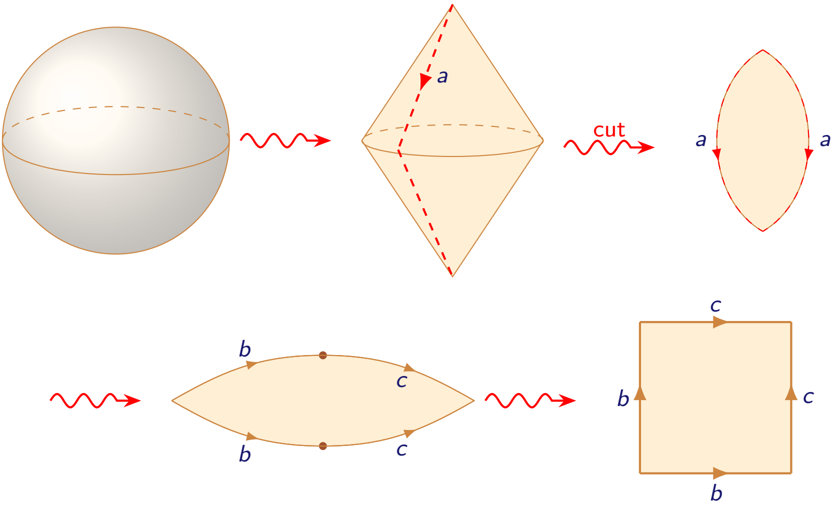
Identification space for a sphere



Identification space for a sphere

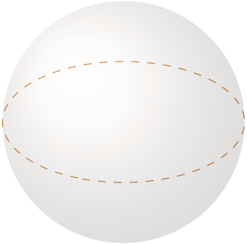


Identification space for a sphere

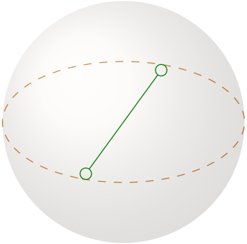


The sphere S^2 is obtained by identifying adjacent sides of a rectangle, or a 2-gon (a polygon with two sides)

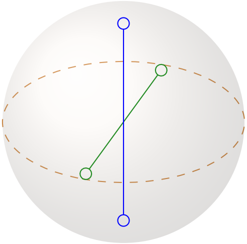
Identification space for the projective plane $\mathbb{P}^2$



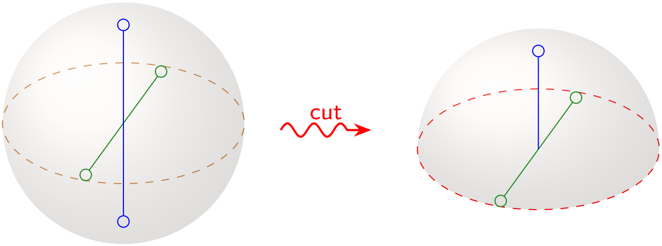
Identification space for the projective plane $\mathbb{P}^2$



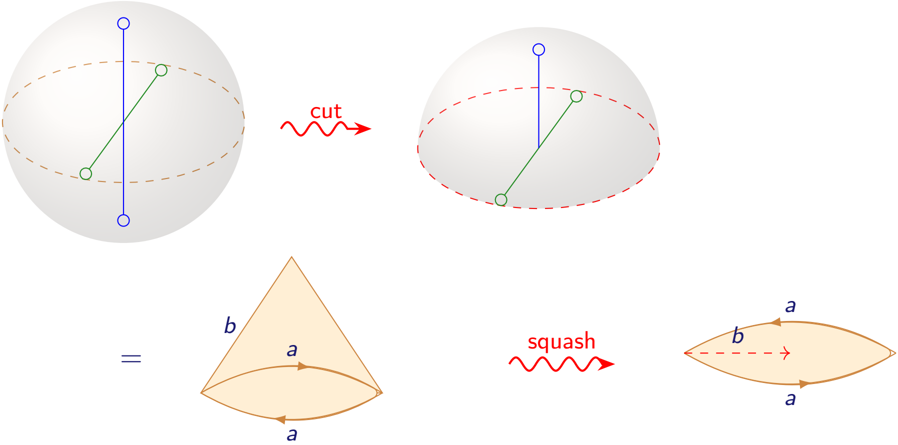
Identification space for the projective plane $\mathbb{P}^2$



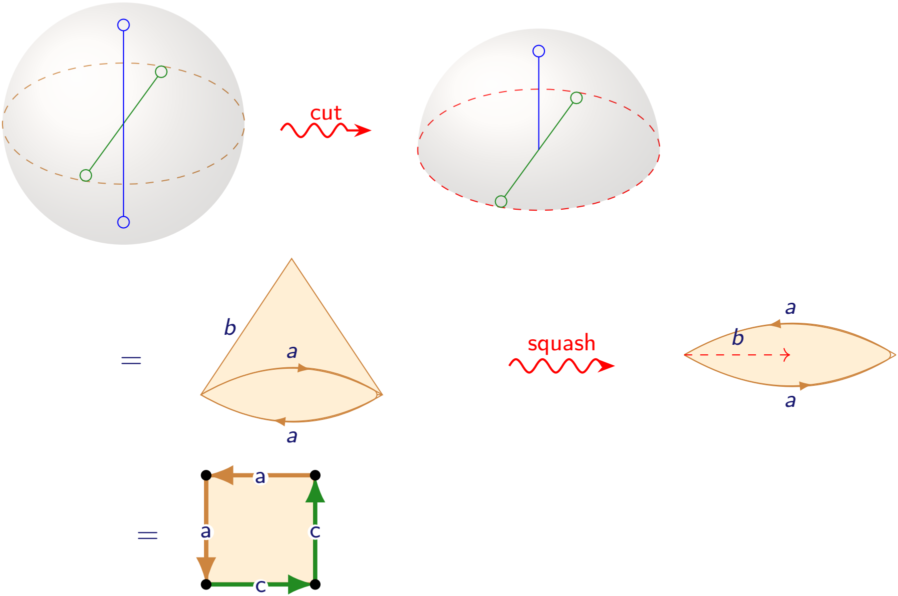
Identification space for the projective plane $\mathbb{P}^2$



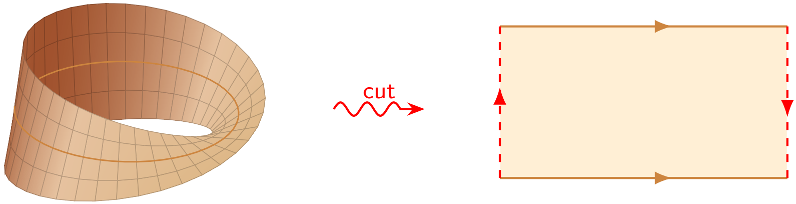
Identification space for the projective plane $\mathbb{P}^2$



Identification space for the projective plane $\mathbb{P}^2$



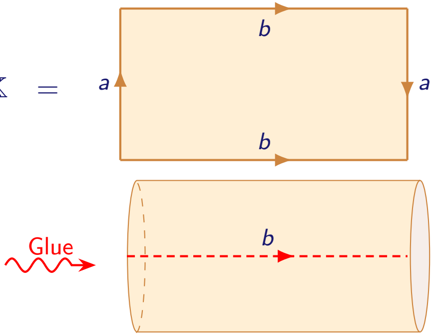
Identification space for a Möbius strip



Identification space for a Klein bottle

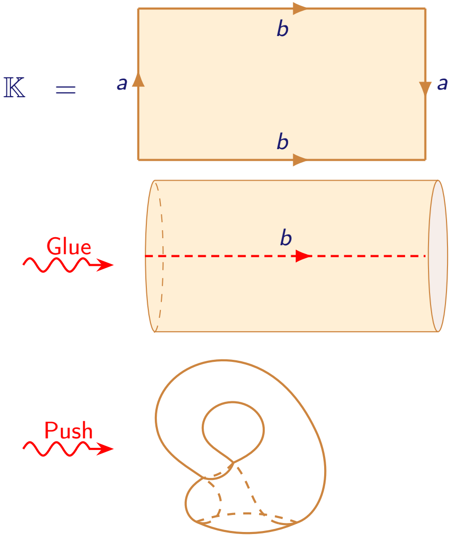
The Klein bottle is defined to be the identification space

$\mathbb{K} =$



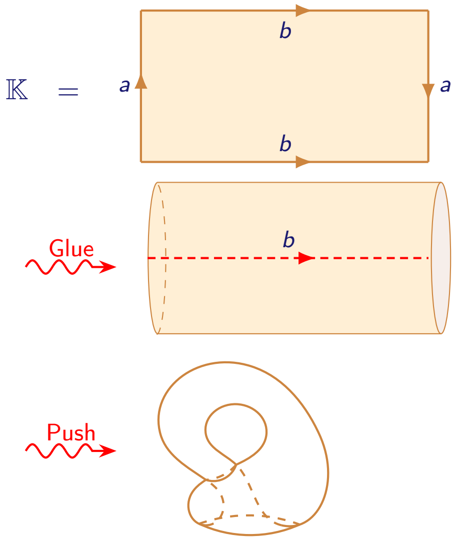
Identification space for a Klein bottle

The Klein bottle is defined to be the identification space



Identification space for a Klein bottle

The Klein bottle is defined to be the identification space



It is not clear how we to do the last step in $\mathbb{R}^3$ and, in fact, we can't!

Polygons in $\mathbb{R}^n$

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

Polygons in $\mathbb{R}^n$

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

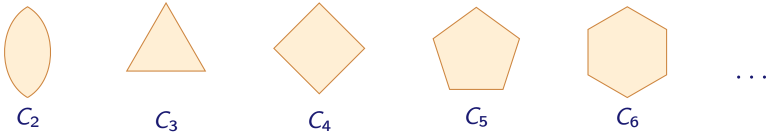
A polygon is an embedding of the cyclic graph C_m into $\mathbb{R}^2$, together with its face, such that such that the vertices of C_m map to distinct points in $\mathbb{R}^2$ and the images of the edges do not intersect in $\mathbb{R}^2$

Polygons in $\mathbb{R}^n$

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

A polygon is an embedding of the cyclic graph C_m into $\mathbb{R}^2$, together with its face, such that such that the vertices of C_m map to distinct points in $\mathbb{R}^2$ and the images of the edges do not intersect in $\mathbb{R}^2$

$\Longrightarrow$ The image of C_n in $\mathbb{R}^2$ is homeomorphic to the closed disc $\mathbb{D}^2$

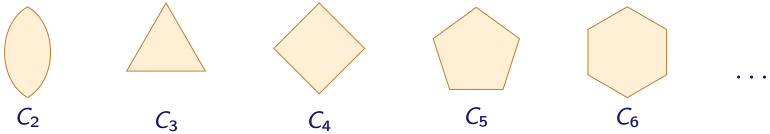


Polygons in $\mathbb{R}^n$

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

A polygon is an embedding of the cyclic graph C_m into $\mathbb{R}^2$, together with its face, such that such that the vertices of C_m map to distinct points in $\mathbb{R}^2$ and the images of the edges do not intersect in $\mathbb{R}^2$

$\implies$ The image of C_n in $\mathbb{R}^2$ is homeomorphic to the closed disc $\mathbb{D}^2$



Remarks

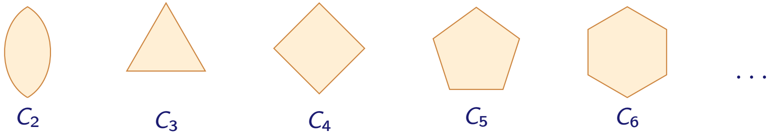
- The image of C_m in $\mathbb{R}^2$ is an m-gon, or a polygon with m sides

Polygons in $\mathbb{R}^n$

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

A polygon is an embedding of the cyclic graph C_m into $\mathbb{R}^2$, together with its face, such that such that the vertices of C_m map to distinct points in $\mathbb{R}^2$ and the images of the edges do not intersect in $\mathbb{R}^2$

$\implies$ The image of C_n in $\mathbb{R}^2$ is homeomorphic to the closed disc $\mathbb{D}^2$



Remarks

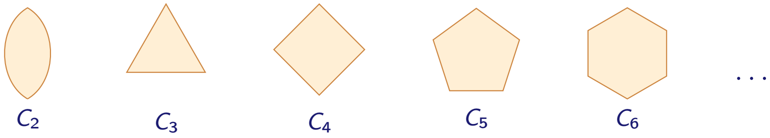
- The image of C_m in $\mathbb{R}^2$ is an m-gon, or a polygon with m sides
- Polygons are surfaces in $\mathbb{R}^2$. They are different from cyclic graphs because they have vertices, edges and one face

Polygons in $\mathbb{R}^n$

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

A polygon is an embedding of the cyclic graph C_m into $\mathbb{R}^2$, together with its face, such that such that the vertices of C_m map to distinct points in $\mathbb{R}^2$ and the images of the edges do not intersect in $\mathbb{R}^2$

$\implies$ The image of C_n in $\mathbb{R}^2$ is homeomorphic to the closed disc $\mathbb{D}^2$



Remarks

- The image of C_m in $\mathbb{R}^2$ is an m-gon, or a polygon with m sides
- Polygons are surfaces in $\mathbb{R}^2$. They are different from cyclic graphs because they have vertices, edges and one face
- The graph C_2 has only one edge. When working with surfaces we think of C_2 as having two edges so that its image in $\mathbb{R}^2$ is a 2-gon

Surfaces and polygonal decompositions

Definition

A surface S is an identification space in $\mathbb{R}^n$ that is obtained by gluing together polygons along their edges in such a way that at most two edges meet along any edge

The polygons give a polygonal decomposition of the surface S

Surfaces and polygonal decompositions

Definition

A surface S is an identification space in $\mathbb{R}^n$ that is obtained by gluing together polygons along their edges in such a way that at most two edges meet along any edge
The polygons give a polygonal decomposition of the surface S

Remarks

- A surface is an identification space where we identify pairs of edges in polygons. Informally, a surface is a patchwork quilt of polygons
- This essentially agrees with our earlier definition of surfaces because every polygon is homeomorphic to a closed disc $\mathbb{D}^2$ so, locally, surfaces look like planes / like discs

Surfaces and polygonal decompositions

Definition

A surface S is an identification space in $\mathbb{R}^n$ that is obtained by gluing together polygons along their edges in such a way that at most two edges meet along any edge

The polygons give a polygonal decomposition of the surface S

Remarks

- A surface is an identification space where we identify pairs of edges in polygons. Informally, a surface is a patchwork quilt of polygons
- This essentially agrees with our earlier definition of surfaces because every polygon is homeomorphic to a closed disc $\mathbb{D}^2$ so, locally, surfaces look like planes / like discs
- A surface can have many seemingly different polygonal decompositions

Surfaces and polygonal decompositions

Definition

A surface S is an identification space in $\mathbb{R}^n$ that is obtained by gluing together polygons along their edges in such a way that at most two edges meet along any edge

The polygons give a polygonal decomposition of the surface S

Remarks

- A surface is an identification space where we identify pairs of edges in polygons. Informally, a surface is a patchwork quilt of polygons
- This essentially agrees with our earlier definition of surfaces because every polygon is homeomorphic to a closed disc $\mathbb{D}^2$ so, locally, surfaces look like planes / like discs
- A surface can have many seemingly different polygonal decompositions
- A surface with a polygonal decomposition has vertices, edges and faces

Surfaces and polygonal decompositions

Definition

A surface S is an identification space in $\mathbb{R}^n$ that is obtained by gluing together polygons along their edges in such a way that at most two edges meet along any edge

The polygons give a polygonal decomposition of the surface S

Remarks

- A surface is an identification space where we identify pairs of edges in polygons. Informally, a surface is a patchwork quilt of polygons
- This essentially agrees with our earlier definition of surfaces because every polygon is homeomorphic to a closed disc $\mathbb{D}^2$ so, locally, surfaces look like planes / like discs
- A surface can have many seemingly different polygonal decompositions
- A surface with a polygonal decomposition has vertices, edges and faces
- We sometimes write $S = (V, E, F)$, where V is the vertex set, edge set E, and face set F

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \ldots$

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \ldots$
- Use the same color for edges that have the same label

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \ldots$
- Use the same color for edges that have the same label
- Fix a direction of every edge (this is important!)

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \dots$
- Use the same color for edges that have the same label
- Fix a direction of every edge (this is important!)

Remarks

- Identifying edges implicitly identifies vertices

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \ldots$
- Use the same color for edges that have the same label
- Fix a direction of every edge (this is important!)

Remarks

- Identifying edges implicitly identifies vertices
- Colouring the edges is not strictly necessary but makes it easier to see how the edges are identified in the polygonal decomposition

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \ldots$
- Use the same color for edges that have the same label
- Fix a direction of every edge (this is important!)

Remarks

- Identifying edges implicitly identifies vertices
- Colouring the edges is not strictly necessary but makes it easier to see how the edges are identified in the polygonal decomposition
- You do not need to color the edges in your work, but you can if you want to

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \ldots$
- Use the same color for edges that have the same label
- Fix a direction of every edge (this is important!)

Remarks

- Identifying edges implicitly identifies vertices
- Colouring the edges is not strictly necessary but makes it easier to see how the edges are identified in the polygonal decomposition
- You do not need to color the edges in your work, but you can if you want to

• It is important to give the correct orientation, or direction, for the paired edges because changing the direction of a paired edge will usually change the surface

Identifying edges in polygonal decompositions

Whenever we draw polygonal decompositions we will usually:

- Label all of the edges with letters: $a, b, c, \ldots$
- Use the same color for edges that have the same label
- Fix a direction of every edge (this is important!)

Remarks

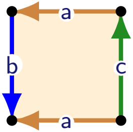
- Identifying edges implicitly identifies vertices
- Colouring the edges is not strictly necessary but makes it easier to see how the edges are identified in the polygonal decomposition
- You do not need to color the edges in your work, but you can if you want to

• It is important to give the correct orientation, or direction, for the paired edges because changing the direction of a paired edge will usually change the surface

- When doing surgery always double check that you do not accidentally change the orientation of an edge

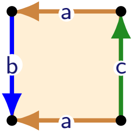
Examples of polygonal decompositions

We have already seen that:

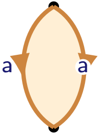
- Annulus $\mathbb{A} \cong$ 

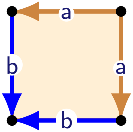
Examples of polygonal decompositions

We have already seen that:

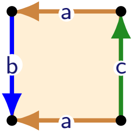
- Annulus $\mathbb{A} \cong$ 

- Sphere

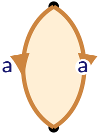
$S^2 \cong$ 

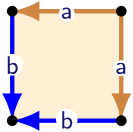
$\cong$ 

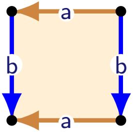
We have already seen that:

- Annulus $\mathbb{A} \cong$ 

- Sphere

$S^2 \cong$ 

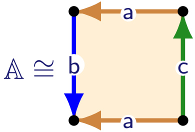
$\cong$ 

- Torus $\mathbb{T} \cong$ 

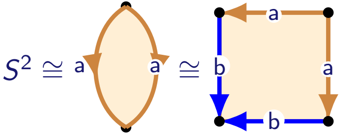
Examples of polygonal decompositions

We have already seen that:

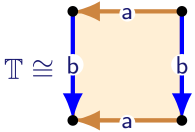
- Annulus



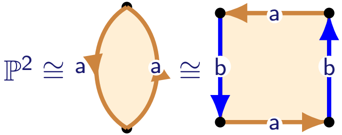
- Sphere



- Torus

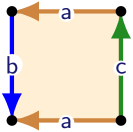


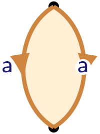
- Projective plane

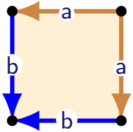


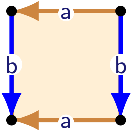
Examples of polygonal decompositions

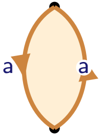
We have already seen that:

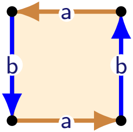
- Annulus $\mathbb{A} \cong$ 

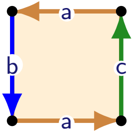
- Sphere $S^2 \cong$ 

$\cong$ 

- Torus $\mathbb{T} \cong$ 

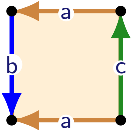
- Projective plane $\mathbb{P}^2 \cong$ 

$\cong$ 

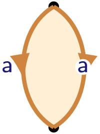
- Möbius strip $\mathbb{M} \cong$ 

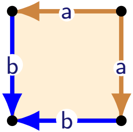
Examples of polygonal decompositions

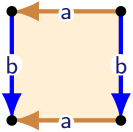
We have already seen that:

- Annulus $\mathbb{A} \cong$ 

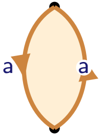
- Sphere

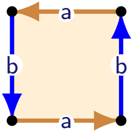
$S^2 \cong$ 

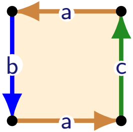
$\cong$ 

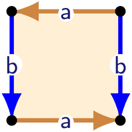
- Torus $\mathbb{T} \cong$ 

- Projective plane

$\mathbb{P}^2 \cong$ 

$\cong$ 

- Möbius strip $\mathbb{M} \cong$ 

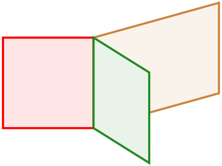
- Klein bottle $\mathbb{K} \cong$ 

Important facts about polygonal decompositions

- Every polygon is homeomorphic to a closed disk $\mathbb{D}^2$

Important facts about polygonal decompositions

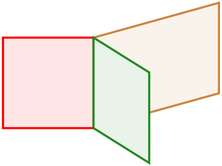
- Every polygon is homeomorphic to a closed disk $\mathbb{D}^2$
- At most two polygons meet in any edge, so



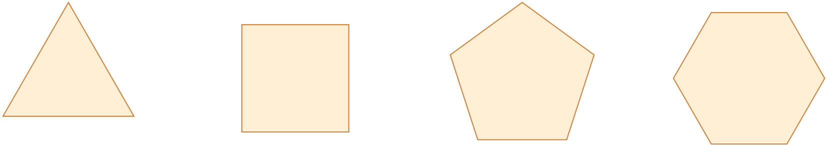
is not polygonal decomposition of a surface

Important facts about polygonal decompositions

- Every polygon is homeomorphic to a closed disk $\mathbb{D}^2$
- At most two polygons meet in any edge, so

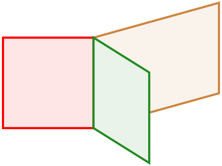


  is not polygonal decomposition of a surface
- Any polygonal decomposition can be replaced with one that only uses 3-gons:

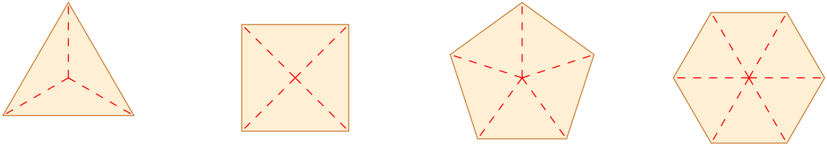


Important facts about polygonal decompositions

- Every polygon is homeomorphic to a closed disk $\mathbb{D}^2$
- At most two polygons meet in any edge, so

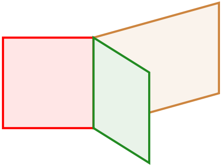


  is not polygonal decomposition of a surface
- Any polygonal decomposition can be replaced with one that only uses 3-gons:

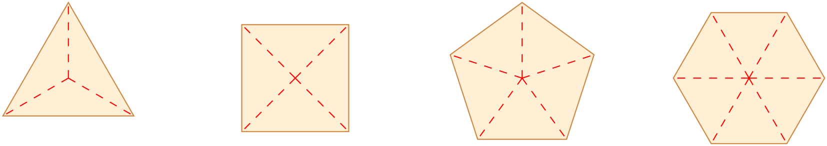


Important facts about polygonal decompositions

- Every polygon is homeomorphic to a closed disk $\mathbb{D}^2$
- At most two polygons meet in any edge, so



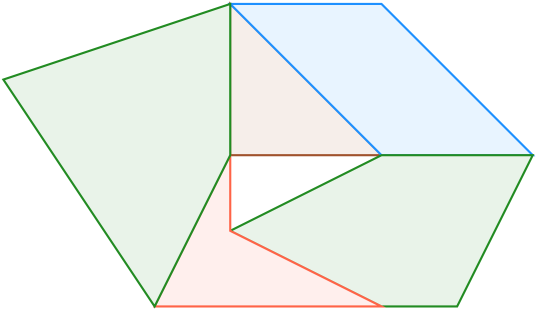
  is not polygonal decomposition of a surface
- Any polygonal decomposition can be replaced with one that only uses 3-gons:



$\Longrightarrow$ Iterating this process, shows that any surface has infinitely many different polygonal decompositions!

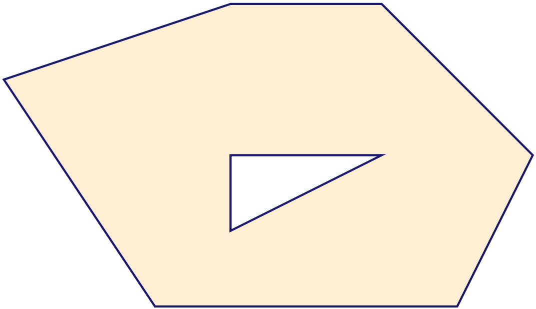
Important facts about polygonal decompositions...

- Every connected surface has a polygonal decomposition with one polygon — with identified edges
  (A polygonal surface is connected if the underlying graph is connected)



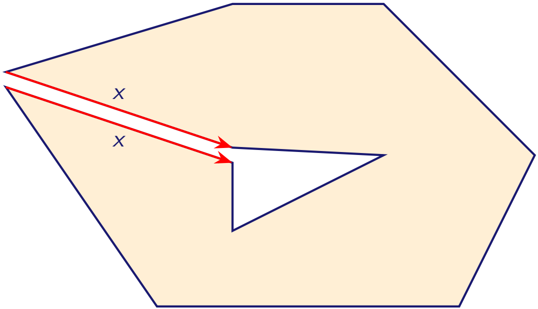
Important facts about polygonal decompositions...

- Every connected surface has a polygonal decomposition with one polygon — with identified edges
  (A polygonal surface is connected if the underlying graph is connected)



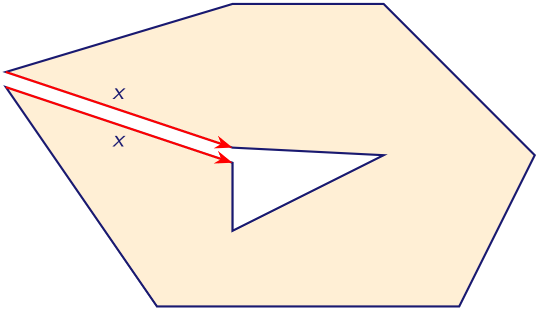
Important facts about polygonal decompositions...

- Every connected surface has a polygonal decomposition with one polygon — with identified edges
  (A polygonal surface is connected if the underlying graph is connected)



Important facts about polygonal decompositions...

- Every connected surface has a polygonal decomposition with one polygon — with identified edges
  (A polygonal surface is connected if the underlying graph is connected)



- We have to check that what we are doing does not depend on the choice of polygonal decomposition

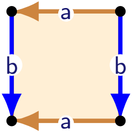
Surgery: cutting and gluing

Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing

$\cong$

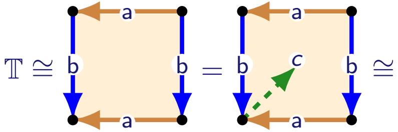
Surgery: cutting and gluing

Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing

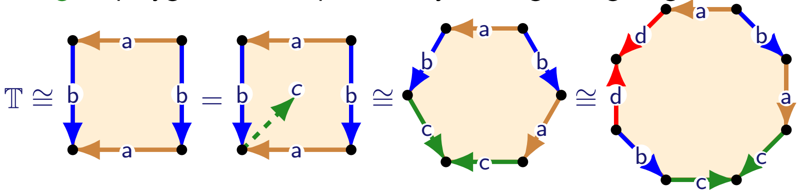
$\mathbb{T} \cong$  $\cong$

Surgery: cutting and gluing

Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing

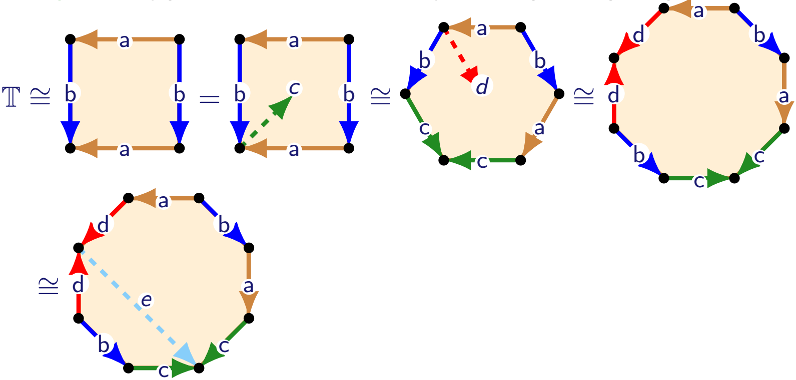


Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing



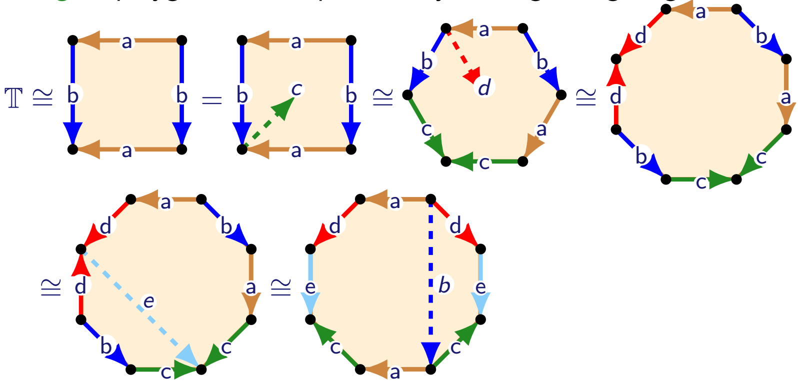
Surgery: cutting and gluing

Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing

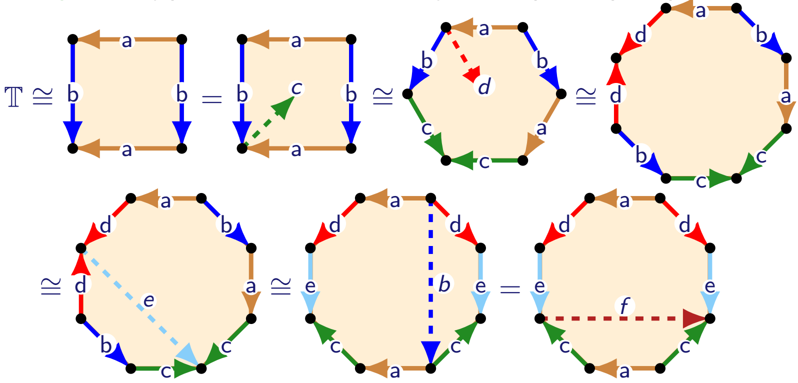


Surgery: cutting and gluing

Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing

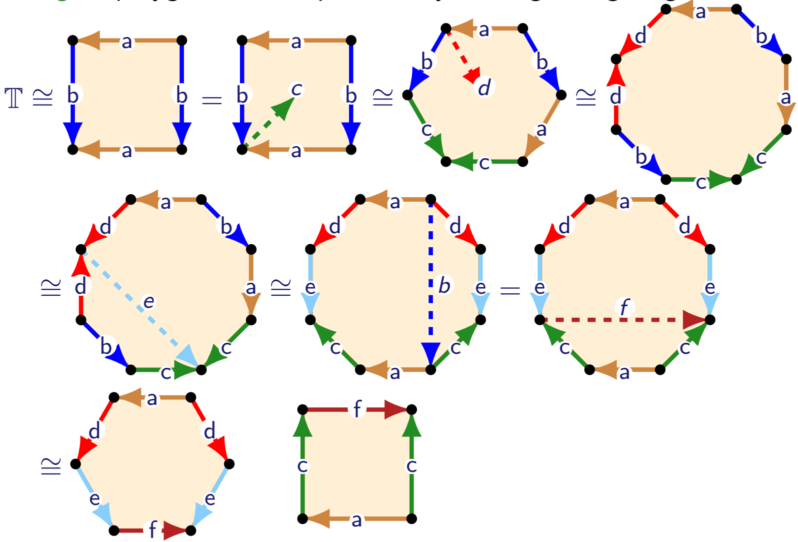


Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing

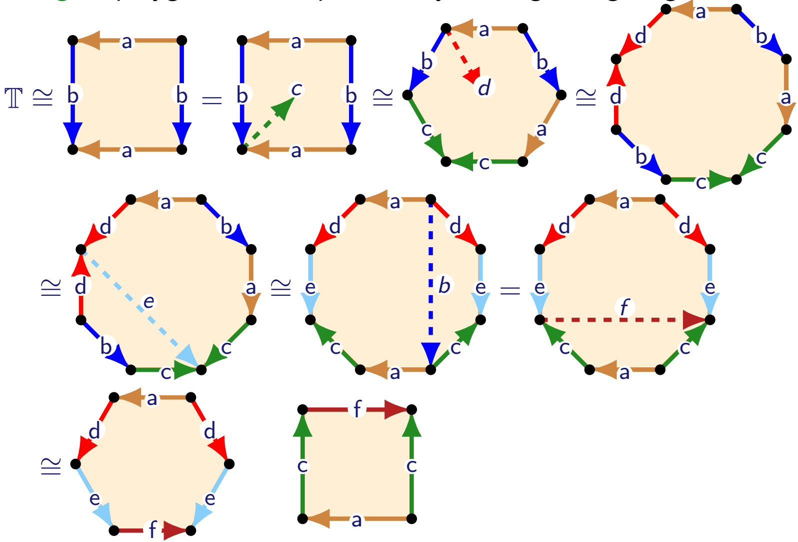


Surgery: cutting and gluing

Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing



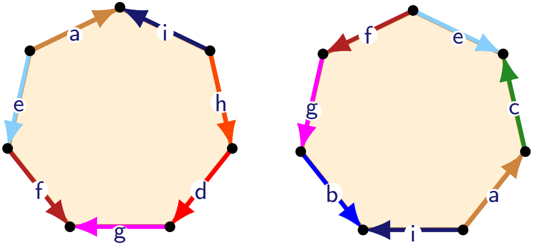
Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing



We want an easy way to identify surfaces from polygonal decompositions

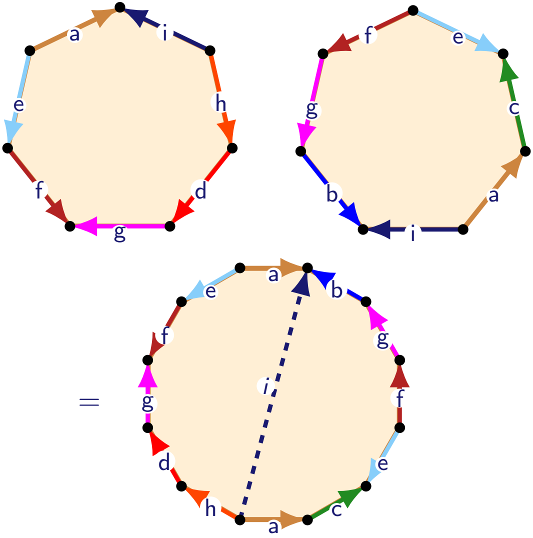
Example surface

Exercise Can we describe the following surface?



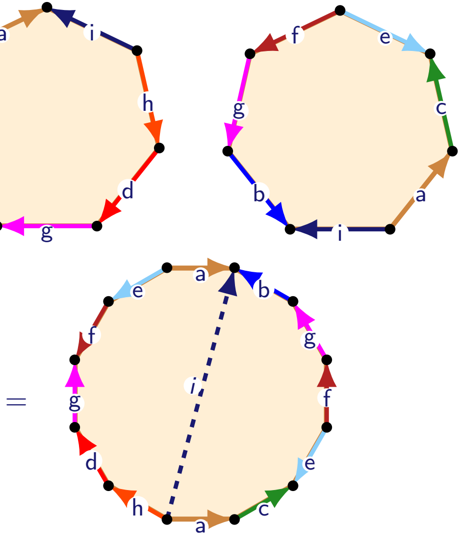
Example surface

Exercise Can we describe the following surface?



Example surface

Exercise Can we describe the following surface?



Answer Not yet! First we need more language and technology.

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice
- The boundary of S is the union of the free edges

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice
- The boundary of S is the union of the free edges
- A boundary circle is a cycle in the polygonal decomposition in which every edge is free

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice
- The boundary of S is the union of the free edges
- A boundary circle is a cycle in the polygonal decomposition in which every edge is free

  We will show that boundary of S is a disjoint union of boundary circles

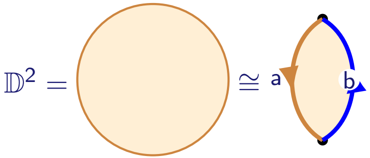
Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice
- The boundary of S is the union of the free edges
- A boundary circle is a cycle in the polygonal decomposition in which every edge is free

  We will show that boundary of S is a disjoint union of boundary circles

Example



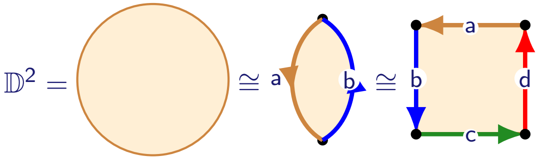
Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice
- The boundary of S is the union of the free edges
- A boundary circle is a cycle in the polygonal decomposition in which every edge is free

  We will show that boundary of S is a disjoint union of boundary circles

Example



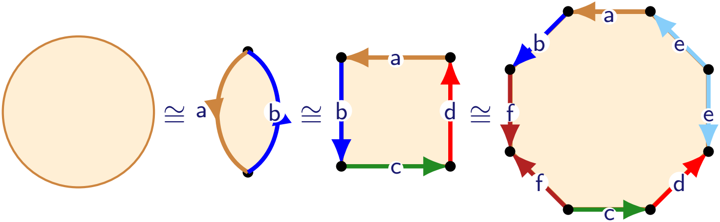
Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

- An edge is free if it occurs only once in the polygonal decomposition
- An edge is paired if it occurs twice
- The boundary of S is the union of the free edges
- A boundary circle is a cycle in the polygonal decomposition in which every edge is free

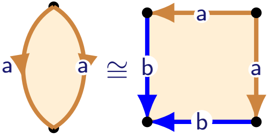
  We will show that boundary of S is a disjoint union of boundary circles

Example

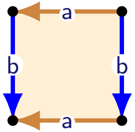
$\mathbb{D}^2 =$ 

Example boundary circles...

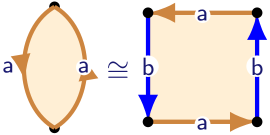
- Sphere

$S^2 \cong$ 

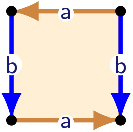
- Torus

$\mathbb{T} \cong$ 

- Projective plane

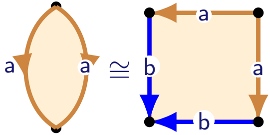
$\mathbb{P}^2 \cong$ 

- Klein bottle

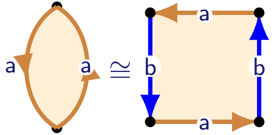
$\mathbb{K} \cong$ 

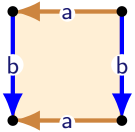
Example boundary circles...

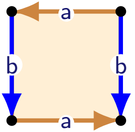
- Sphere

$S^2 \cong$ 

- Projective plane

$\mathbb{P}^2 \cong$ 

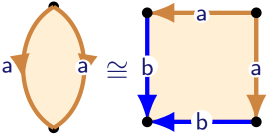
- Torus $\mathbb{T} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

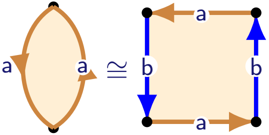
All edges paired $\implies$ no boundary

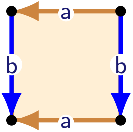
Example boundary circles...

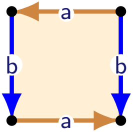
- Sphere

$S^2 \cong$ 

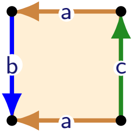
- Projective plane

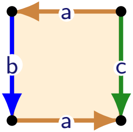
$\mathbb{P}^2 \cong$ 

- Torus $\mathbb{T} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

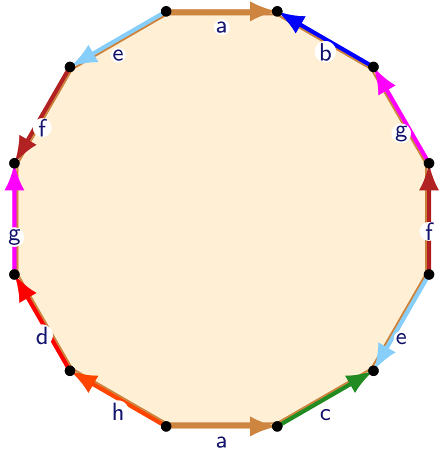
All edges paired $\Longrightarrow$ no boundary

- Annulus $\mathbb{A} \cong$ 

- Möbius $\mathbb{M} \cong$ 

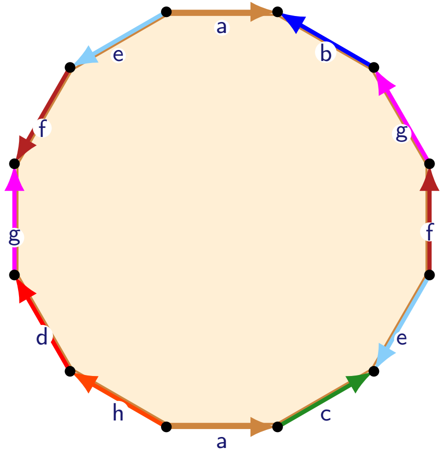
Example boundary circles...

Exercise What is the boundary of the surface?



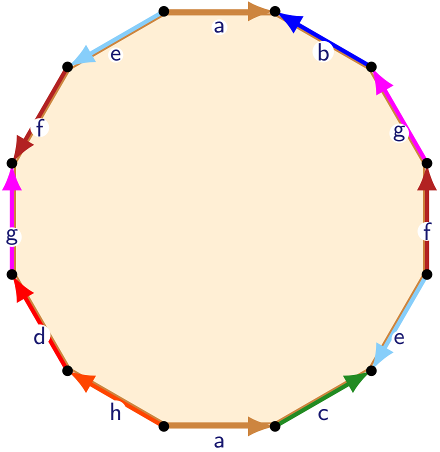
Example boundary circles...

Exercise What is the boundary of the surface?



Free edges: b, c, d, h

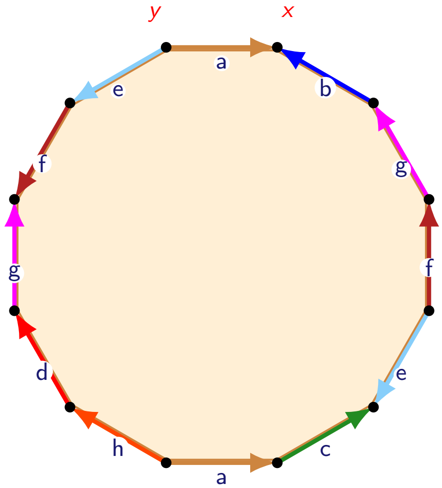
Exercise What is the boundary of the surface?



Free edges: b, c, d, h

Key observation
Paired edges imply that some vertices are equal

Exercise What is the boundary of the surface?



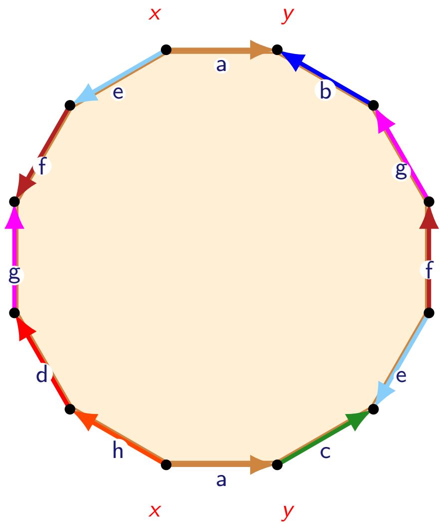
Free edges: b, c, d, h

Key observation

Paired edges imply that some vertices are equal

Example boundary circles...

Exercise What is the boundary of the surface?

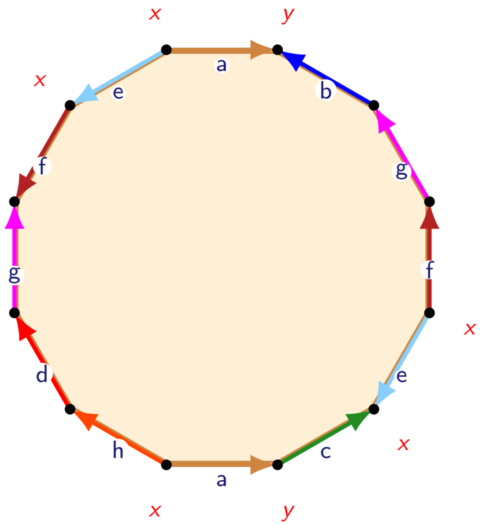


Free edges: b, c, d, h

Key observation

Paired edges imply that some vertices are equal

Exercise What is the boundary of the surface?

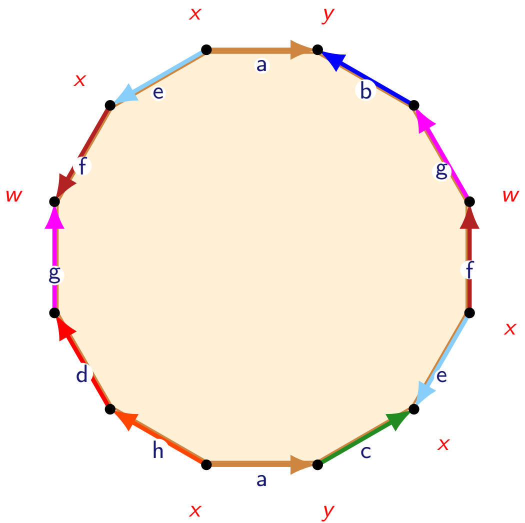


Free edges: b, c, d, h

Key observation

Paired edges imply that some vertices are equal

Exercise What is the boundary of the surface?

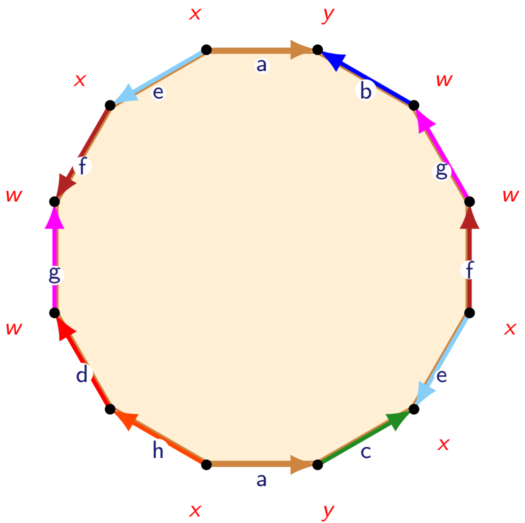


Free edges: b, c, d, h

Key observation

Paired edges imply that some vertices are equal

Exercise What is the boundary of the surface?

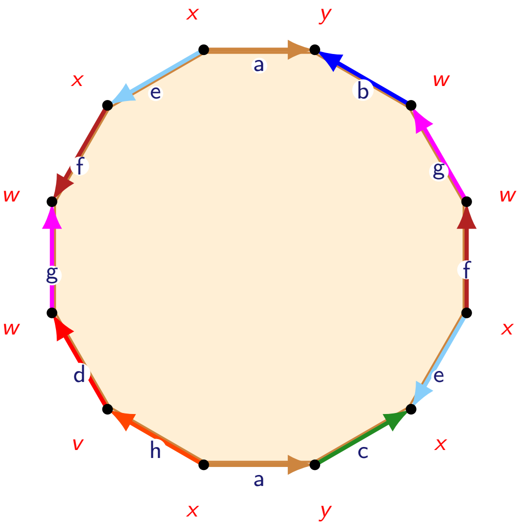


Free edges: b, c, d, h

Key observation

Paired edges imply that some vertices are equal

Exercise What is the boundary of the surface?



Free edges: b, c, d, h

Key observation

Paired edges imply that some vertices are equal

The Euler characteristic of a surface

Let $S = (V, E, F)$ be a surface with a polygonal decomposition

Definition

The Euler characteristic of S is $\chi(S) = |V| - |E| + |F|$

The Euler characteristic of a surface

Let $S = (V, E, F)$ be a surface with a polygonal decomposition

Definition

The Euler characteristic of S is $\chi(S) = |V| - |E| + |F|$

Remarks

- The Euler characteristic $\chi(S) = |V| - |E| + |F|$ of S is a higher dimensional generalization of the Euler characteristic of a graph $G = (V, E)$, which is $\chi(G) = |V| - |E|$

The Euler characteristic of a surface

Let $S = (V, E, F)$ be a surface with a polygonal decomposition

Definition

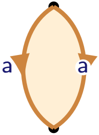
The Euler characteristic of S is $\chi(S) = |V| - |E| + |F|$

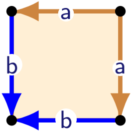
Remarks

- The Euler characteristic $\chi(S) = |V| - |E| + |F|$ of S is a higher dimensional generalization of the Euler characteristic of a graph $G = (V, E)$, which is $\chi(G) = |V| - |E|$
- The definition of $\chi(S)$ appears to depend on the choice of polygonal decomposition (V, E, F) of S. In fact, we will soon see that $\chi(S)$ is independent of this choice

Euler characteristic of basic surfaces.

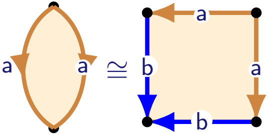
- Sphere

$S^2 \cong$


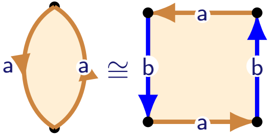
$\cong$


$, \chi = 2$

- Sphere

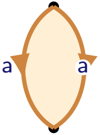
$S^2 \cong$  $, \chi = 2$

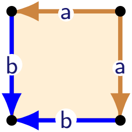
- Projective plane

$\mathbb{P}^2 \cong$  $, \chi = 1$

Euler characteristic of basic surfaces.

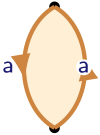
- Sphere

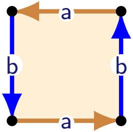
$S^2 \cong$ 

$\cong$ 

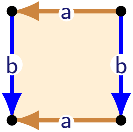
$, \chi = 2$

- Projective plane

$\mathbb{P}^2 \cong$ 

$\cong$ 

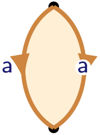
$, \chi = 1$

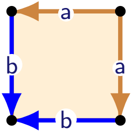
- Torus $\mathbb{T} \cong$ 

$, \chi = 0$

Euler characteristic of basic surfaces.

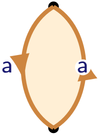
- Sphere

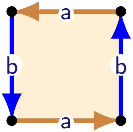
$S^2 \cong$ 

$\cong$ 

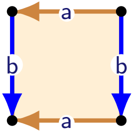
$, \chi = 2$

- Projective plane

$\mathbb{P}^2 \cong$ 

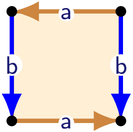
$\cong$ 

$, \chi = 1$

- Torus $\mathbb{T} \cong$ 

$, \chi = 0$

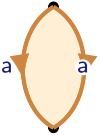
- Klein bottle

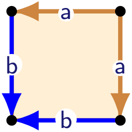
$\mathbb{K} \cong$ 

$, \chi = 0$

Euler characteristic of basic surfaces.

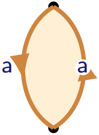
- Sphere

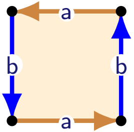
$S^2 \cong$ 

$\cong$ 

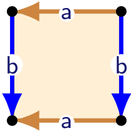
$, \chi = 2$

- Projective plane

$\mathbb{P}^2 \cong$ 

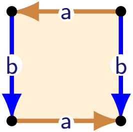
$\cong$ 

$, \chi = 1$

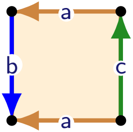
- Torus $\mathbb{T} \cong$ 

$, \chi = 0$

- Klein bottle

$\mathbb{K} \cong$ 

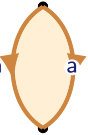
$, \chi = 0$

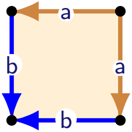
- Annulus $\mathbb{A} \cong$ 

$, \chi = 0$

Euler characteristic of basic surfaces.

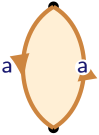
- Sphere

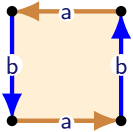
$S^2 \cong$ 

$\cong$ 

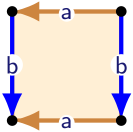
$, \chi = 2$

- Projective plane

$\mathbb{P}^2 \cong$ 

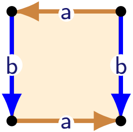
$\cong$ 

$, \chi = 1$

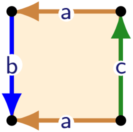
- Torus $\mathbb{T} \cong$ 

$, \chi = 0$

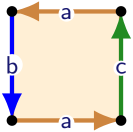
- Klein bottle

$\mathbb{K} \cong$ 

$, \chi = 0$

- Annulus $\mathbb{A} \cong$ 

$, \chi = 0$

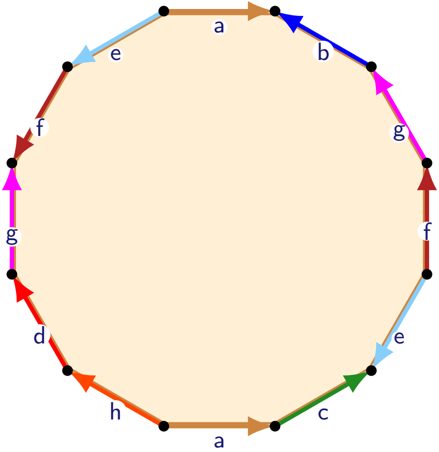
- Möbius $\mathbb{M} \cong$ 

$, \chi = 0$

Euler characteristic example

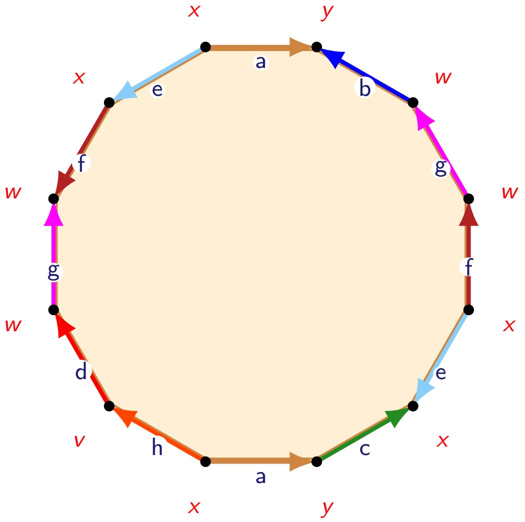
Example What is the Euler characteristic of the surface:

$S \quad =$



Euler characteristic example

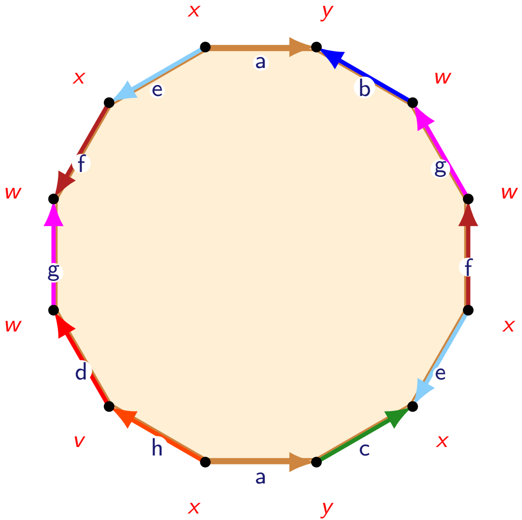
Example What is the Euler characteristic of the surface:

$S =$ 

Euler characteristic example

Example What is the Euler characteristic of the surface:

$S \quad =$



$\Longrightarrow \quad \chi(S) = -3$

Subdivision of a surface

Let S be a surface with a polygonal decomposition

A subdivision of S is any polygonal decomposition that is obtained from S by successively applying the following operations:

Subdivision of a surface

Let S be a surface with a polygonal decomposition

A subdivision of S is any polygonal decomposition that is obtained from S by successively applying the following operations:

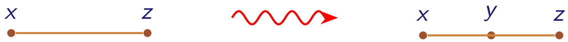
- Subdividing an edge by adding a new vertex

Subdivision of a surface

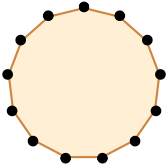
Let S be a surface with a polygonal decomposition

A subdivision of S is any polygonal decomposition that is obtained from S by successively applying the following operations:

- Subdividing an edge by adding a new vertex



- Subdividing a face by adding a new edge

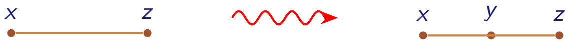


Subdivision of a surface

Let S be a surface with a polygonal decomposition

A subdivision of S is any polygonal decomposition that is obtained from S by successively applying the following operations:

- Subdividing an edge by adding a new vertex



- Subdividing a face by adding a new edge

Remarks

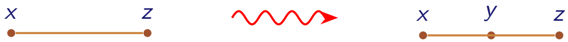
- The subdivision of a subdivision of S is a subdivision of S

Subdivision of a surface

Let S be a surface with a polygonal decomposition

A subdivision of S is any polygonal decomposition that is obtained from S by successively applying the following operations:

- Subdividing an edge by adding a new vertex



- Subdividing a face by adding a new edge

Remarks

- The subdivision of a subdivision of S is a subdivision of S
- If $\dot{S}$ has a polygonal decomposition that is a subdivision of a polygonal decomposition of S then $S \cong \dot{S}$

Subdividing and Euler characteristic

Proposition

Let $\dot{S}$ *be a subdivision of* S*. Then* $\chi(S) = \chi(\dot{S})$

Subdividing and Euler characteristic

Proposition

Let $\dot{S}$ be a subdivision of S. Then $\chi(S) = \chi(\dot{S})$

Proof It is enough to check this for the two subdivision operations:

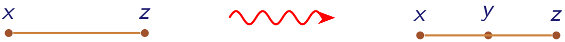
Subdividing and Euler characteristic

Proposition

Let $\dot{S}$ be a subdivision of S. Then $\chi(S) = \chi(\dot{S})$

Proof It is enough to check this for the two subdivision operations:

- Subdividing an edge:



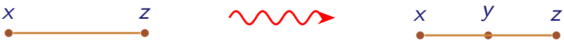
Subdividing and Euler characteristic

Proposition

Let $\dot{S}$ be a subdivision of S. Then $\chi(S) = \chi(\dot{S})$

Proof It is enough to check this for the two subdivision operations:

- Subdividing an edge:



- Subdividing a face:

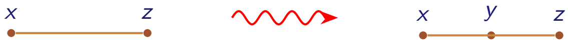
Subdividing and Euler characteristic

Proposition

Let $\dot{S}$ be a subdivision of S. Then $\chi(S) = \chi(\dot{S})$

Proof It is enough to check this for the two subdivision operations:

- Subdividing an edge:



- Subdividing a face:

Both operations preserve χ

Subdividing and boundary circles

Proposition

Let $\dot{S}$ be a subdivision of S. Then S and $\dot{S}$ have the same number of boundary circles

Subdividing and boundary circles

Proposition

Let $\dot{S}$ be a subdivision of S. Then S and $\dot{S}$ have the same number of boundary circles

Proof It is enough to check this for the two subdivision operations:

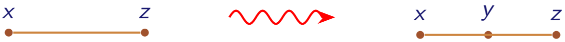
Subdividing and boundary circles

Proposition

Let $\dot{S}$ be a subdivision of S. Then S and $\dot{S}$ have the same number of boundary circles

Proof It is enough to check this for the two subdivision operations:

- Subdividing an edge:



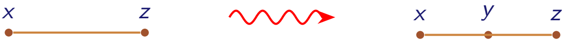
Subdividing and boundary circles

Proposition

Let $\dot{S}$ be a subdivision of S. Then S and $\dot{S}$ have the same number of boundary circles

Proof It is enough to check this for the two subdivision operations:

- Subdividing an edge:



- Subdividing a face:

Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

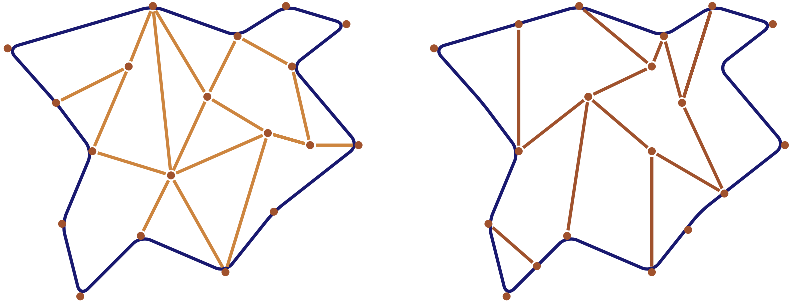
Proof Merge the two subdivisions

Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

Proof Merge the two subdivisions

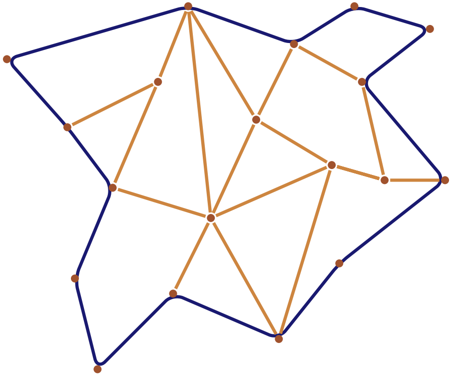


Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

Proof Merge the two subdivisions

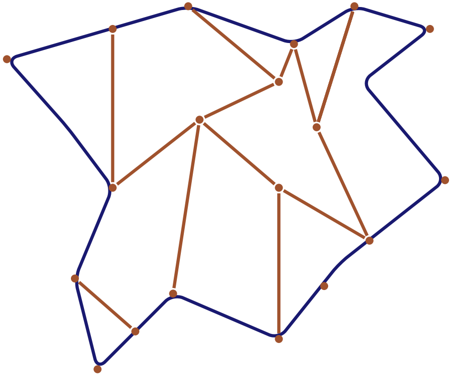


Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

Proof Merge the two subdivisions

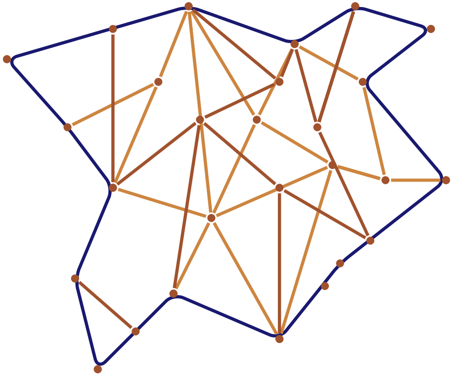


Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

Proof Merge the two subdivisions

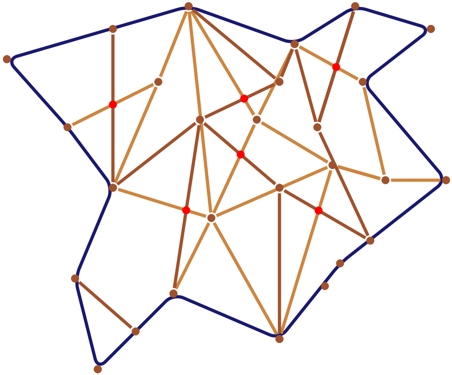


Common subdivisions

Theorem

Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

Proof Merge the two subdivisions — adding extra vertices as necessary



Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Observe that if P is a polygonal decomposition of S then $f(P)$ is a polygonal decomposition of T. Similarly, if Q is a polygonal subdivision of T then $g(T)$ is a polygonal decomposition of S

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Observe that if P is a polygonal decomposition of S then $f(P)$ is a polygonal decomposition of T. Similarly, if Q is a polygonal subdivision of T then $g(T)$ is a polygonal decomposition of S

By the theorem we can assume that S and T have the same polygonal decomposition in the sense that $P = g(Q)$ and $Q = f(P)$

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Observe that if P is a polygonal decomposition of S then $f(P)$ is a polygonal decomposition of T. Similarly, if Q is a polygonal subdivision of T then $g(T)$ is a polygonal decomposition of S

By the theorem we can assume that S and T have the same polygonal decomposition in the sense that $P = g(Q)$ and $Q = f(P)$

$$\implies \quad \chi(S) = \chi_P(S)$$

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Observe that if P is a polygonal decomposition of S then $f(P)$ is a polygonal decomposition of T. Similarly, if Q is a polygonal subdivision of T then $g(T)$ is a polygonal decomposition of S

By the theorem we can assume that S and T have the same polygonal decomposition in the sense that $P = g(Q)$ and $Q = f(P)$

$$\implies \quad \chi(S) = \chi_P(S) = \chi_{f(P)}$$

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Observe that if P is a polygonal decomposition of S then $f(P)$ is a polygonal decomposition of T. Similarly, if Q is a polygonal subdivision of T then $g(T)$ is a polygonal decomposition of S

By the theorem we can assume that S and T have the same polygonal decomposition in the sense that $P = g(Q)$ and $Q = f(P)$

$$\implies \quad \chi(S) = \chi_P(S) = \chi_{f(P)} = \chi_Q(T) = \chi(T).$$

Two invariants

Corollary

Suppose that S and T are homeomorphic surfaces that have polygonal decompositions. Then $\chi(S) = \chi(T)$ and S and T have the same number of boundary circles.

Proof Since $S \cong T$ there is a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Observe that if P is a polygonal decomposition of S then $f(P)$ is a polygonal decomposition of T. Similarly, if Q is a polygonal subdivision of T then $g(T)$ is a polygonal decomposition of S

By the theorem we can assume that S and T have the same polygonal decomposition in the sense that $P = g(Q)$ and $Q = f(P)$

$$\implies \quad \chi(S) = \chi_P(S) = \chi_{f(P)} = \chi_Q(T) = \chi(T).$$

Similarly, S and T have the same number of boundary circles

Why are invariants useful?

Question

Let S and T be surfaces. Is $S \cong T$?

To show that S and T are homeomorphic is, in principle, easy: we find a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Why are invariants useful?

Question

Let S and T be surfaces. Is $S \cong T$?

To show that S and T are homeomorphic is, in principle, easy: we find a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Showing that $S \ncong T$ is harder as we need to show that no such maps exist

Why are invariants useful?

Question

Let S and T be surfaces. Is $S \cong T$?

To show that S and T are homeomorphic is, in principle, easy: we find a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Showing that $S \ncong T$ is harder as we need to show that no such maps exist

Using invariants makes this easier because $S \cong T$ only if $\chi(S) = \chi(T)$ and if S and T have the same number of boundary circles

Why are invariants useful?

Question

Let S and T be surfaces. Is $S \cong T$?

To show that S and T are homeomorphic is, in principle, easy: we find a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Showing that $S \ncong T$ is harder as we need to show that no such maps exist

Using invariants makes this easier because $S \cong T$ only if $\chi(S) = \chi(T)$ and if S and T have the same number of boundary circles

$\Longrightarrow$ if $\chi(S) \neq \chi(T)$, or if S and T have a different number of boundary circles, then $S \ncong T$

Why are invariants useful?

Question

Let S and T be surfaces. Is $S \cong T$?

To show that S and T are homeomorphic is, in principle, easy: we find a continuous map $f : S \longrightarrow T$ with a continuous inverse $g : T \longrightarrow S$

Showing that $S \ncong T$ is harder as we need to show that no such maps exist

Using invariants makes this easier because $S \cong T$ only if $\chi(S) = \chi(T)$ and if S and T have the same number of boundary circles

$\Longrightarrow$ if $\chi(S) \neq \chi(T)$, or if S and T have a different number of boundary circles, then $S \ncong T$

Exercise Using what we know so far, deduce that the surfaces

$$S^2,\ \mathbb{A},\ \mathbb{D}^2,\ \mathbb{K},\ \mathbb{M},\ \mathbb{P}^2$$

are pairwise non-homeomorphic (see Tutorial 9)