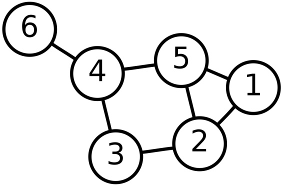
Topology – recollection

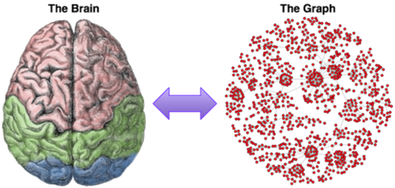
Math3061

Daniel Tubbenhauer, University of Sydney

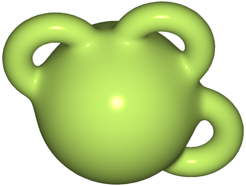
The three main topics

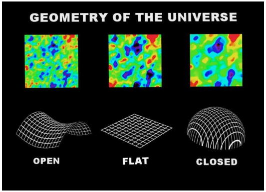
Topic 1: graphs!



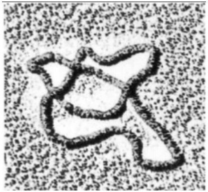


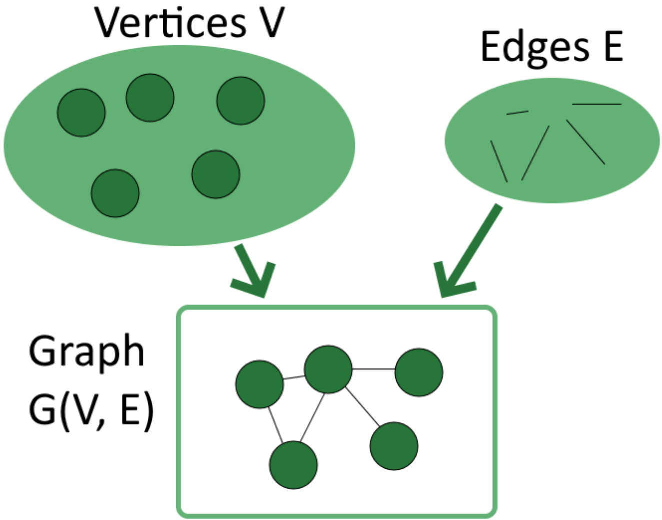
Topic 2: surfaces!





Topic 3: knots!




Vertices V
Edges E
Graph
G(V, E)

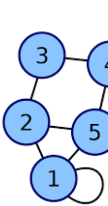
Questions we ask about a graph G

1. Have we seen G before? Is it one of the standard ones (lines, cycles, complete graphs, complete bipartite graphs)?
2. How many vertices and edges does G have?
3. What is its Euler characteristic?
4. Is G connected? How many connected components does G have?
5. Is G a tree? If not, then can we find a spanning tree?
6. What are its paths (start and endpoint might be different)? What are its circuits?
7. Does G have an Eulerian circuit? Does G have an Eulerian path?
8. Is G planar, i.e. does it embed into the plane = the disc = S^2?
9. Does G embed into other surfaces?
10. How many colors do we need to color maps defined by G?

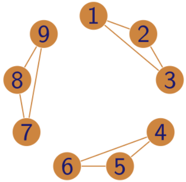
Let us answer 1-10 for the Pappus graph

But before, let us recall what the above are!

A connected graph with $|V| = 6$, $|E| = 8$, $\chi = -2$ and one loop:



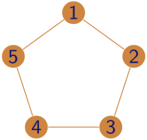
A non-connected graph with $|V| = 9$, $|E| = 9$, $\chi = 0$:



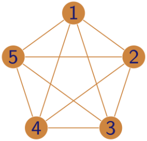
Standard graphs

Line:

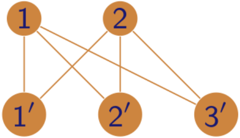
Cycle:



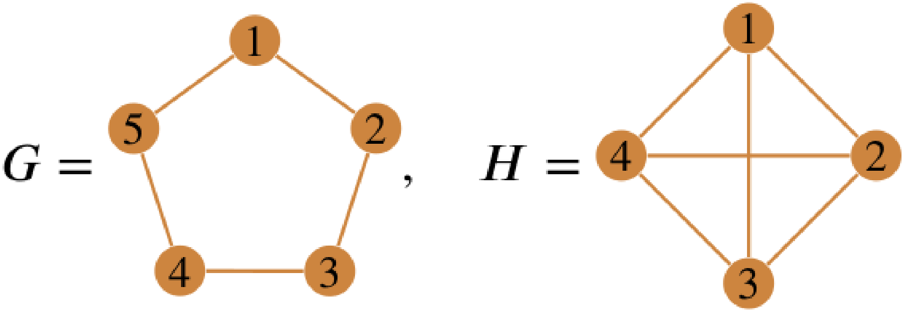
Complete:



Complete bipartite:

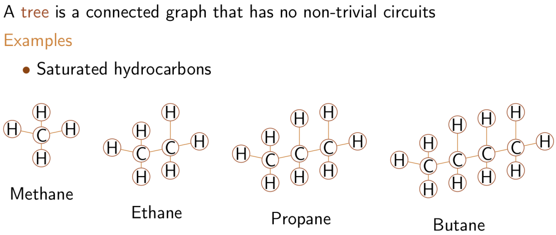


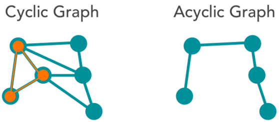
Exercise Check whether you understand how the various standard graphs are related and what properties they have. For example, which ones are subgraphs, which ones are planar etc.



Trees

Trees are acyclic, so only the right graph below is a tree:





Trees satisfy many properties and are always amenable for induction, e.g. prove the following as an exercise:

Corollary

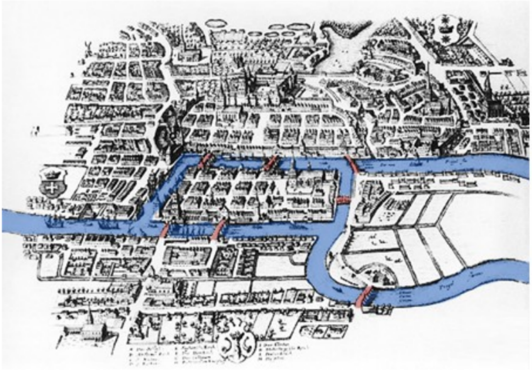
Suppose that $T = (V, E)$ *is a tree. Then* $|V| = |E| + 1$.

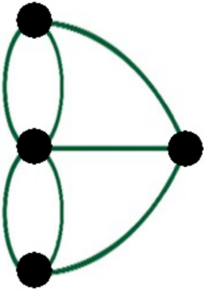
Euler and cycles

Euler's famous criterion:

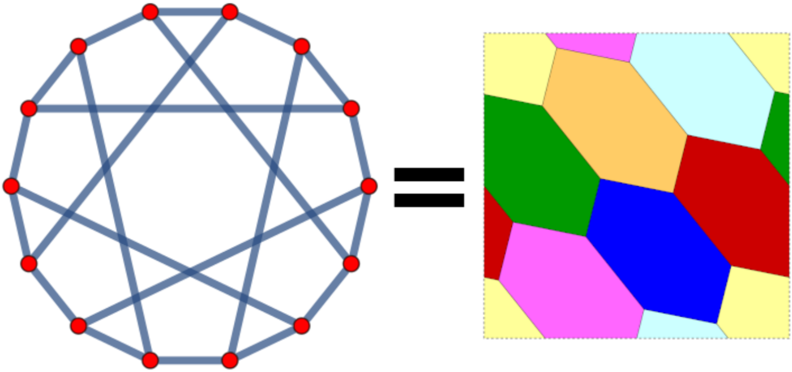
Theorem

Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree





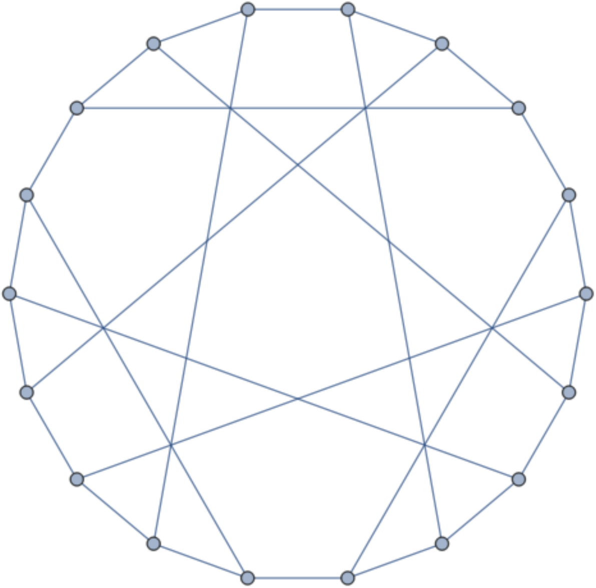
Embeddings on surfaces

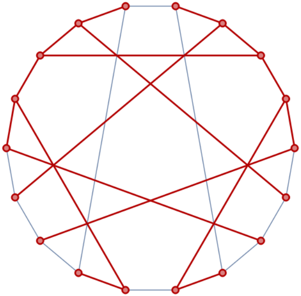


Heawood's coloring formula:

$$c = \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor$$

The Pappus graph G



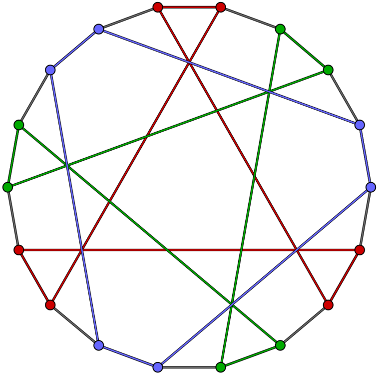


The Pappus graph is a not a standard graph – it is neither a line nor a cycle nor complete nor complete bipartite

We clearly have $|V| = 18$ and $|E| = 27$, so that $\chi(G) = |V| - |E| = -9$, and G is connected

The Pappus graph is not a tree and a spanning tree is illustrated above (there are many more spanning trees)

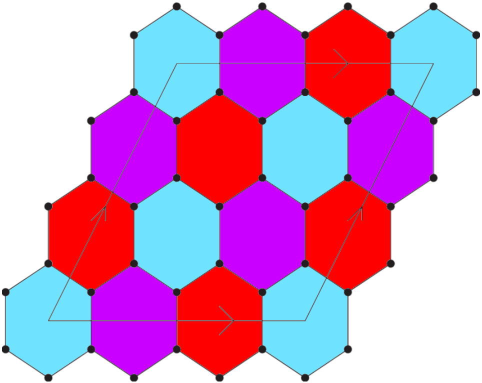
The Pappus graph G – answering 1–10, part 2



The Pappus graph has many cycles that are hexagons, as illustrated above. In fact, one checks that the length of the smallest cycle is 6

Every vertex in the Pappus graph is of degree 3, so there are neither Eulerian circuits nor paths

The Pappus graph G – answering 1–10, part 3

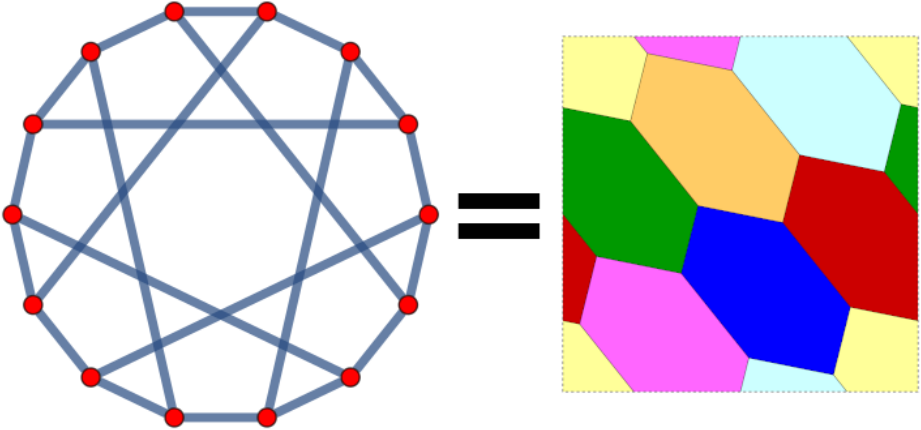


The Pappus graph does not embed into S^2

G embeds onto the torus and then needs 3 colors to color it, see above

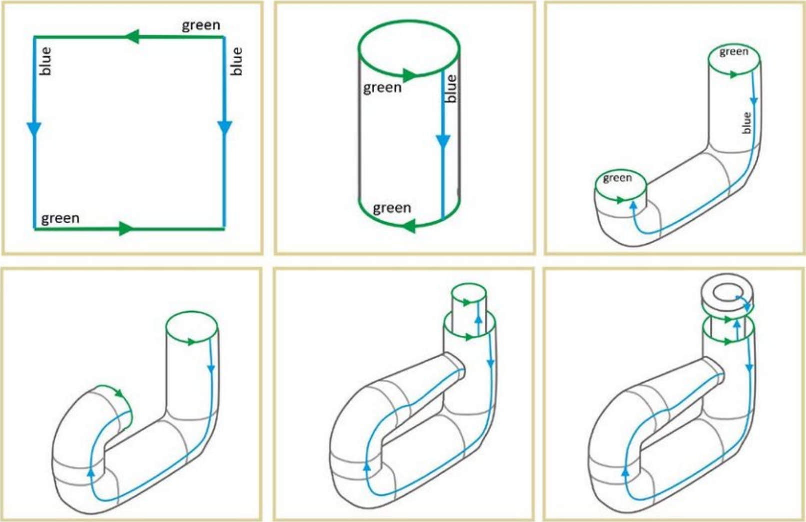
Heawood's theorem for the torus would give $\lfloor\frac{7+\sqrt{49-24\cdot 0}}{2}\rfloor = 7$ as the number of colorings needed in the worst case, so G does better

The Heawood graph – answer 1–10 as an exercise



The above graph is called the Heawood graph – **try yourself!**

Topic 2: surfaces!



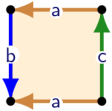
Questions we ask about a surface S

1. Have we seen S before? Is it one of the standard ones (sphere, torus, Klein bottle, projective plane etc.)?
2. How many boundary cycles = punctures does S have?
3. What is its Euler characteristic?
4. Is S connected? How many connected components does S have?
5. Is S orientable?
6. Can we find a polygonal form of S?
7. What is its standard form?
8. How many cross-caps are there in standard form?
9. How many handles are there in standard form?
10. If $d = 0$, then what is the chromatic number of S?

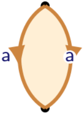
Let us answer 1-10 for a randomly generated polygonal form

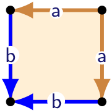
But before, let us recall what the above are!

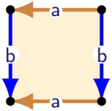
The standard surfaces in polygonal form

- Annulus $\mathbb{A} \cong$ 

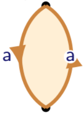
- Sphere

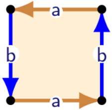
$S^2 \cong$ 

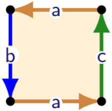
$\cong$ 

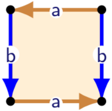
- Torus $\mathbb{T} \cong$ 

- Projective plane

$\mathbb{P}^2 \cong$ 

$\cong$ 

- Möbius strip $\mathbb{M} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

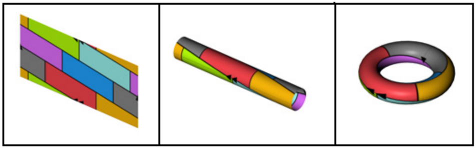
These are 2 dimensional objects, e.g. the torus is hollow:

$\mathbb{T} =$ 

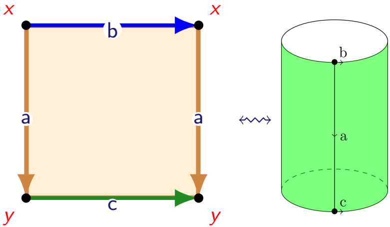
From polygons to surfaces

Recall that one goes from a polygon to a surface by identifying paired edges

For the torus that means e.g.



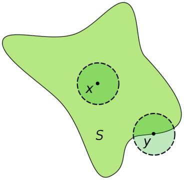
For an annulus one gets

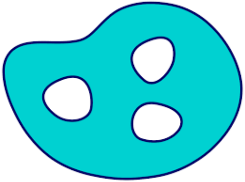


One can build of most these, e.g. a Möbius, strip out of paper

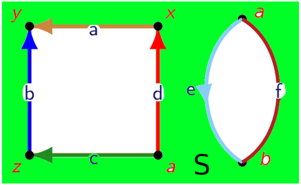
The boundary

Boundary points have neighborhoods that are half-discs; all other point have disc neighborhoods





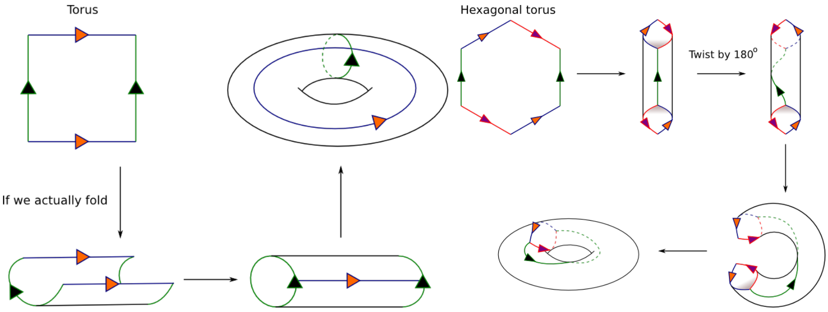
In a polygonal form, the free edges wrap around boundary components:



Note that the surface S is on the outside in these pictures

Euler characteristic

Every surface S has infinitely many polygonal forms and they might look wildly different, e.g.:

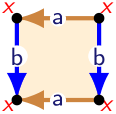


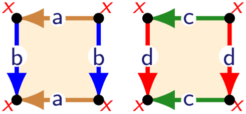
The Euler characteristic $\chi(S) = |V| - |E| + |F|$ is the same for any polygonal form

$$\text{left: } \chi(\mathbb{T}) = 1 - 2 + 1 = 0, \quad \text{right: } \chi(\mathbb{T}) = 2 - 3 + 1 = 0$$

Euler characteristic – only almost perfect

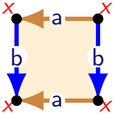
We have $\chi(S) \neq \chi(T) \Rightarrow S \ncong T$ but the converse is not true:

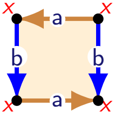
$\mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T}) = 0$, $\mathbb{T} \cup \mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T} \cup \mathbb{T}) = 0$

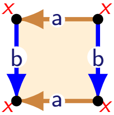
Fix: check connectivity

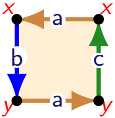
$\mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T}) = 0$, $\mathbb{K} \leftrightsquigarrow$ 

$, \chi(\mathbb{K}) = 0$

Fix: check orientability

$\mathbb{T} \leftrightsquigarrow$ 

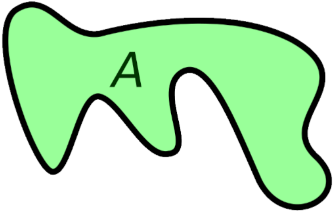
$, \chi(\mathbb{T}) = 0$, $\mathbb{A} \leftrightsquigarrow$ 

$, \chi(\mathbb{A}) = 0$

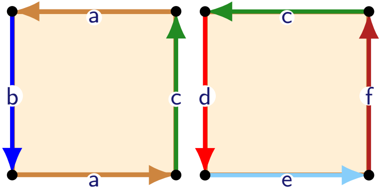
Fix: check boundary

Connected = we can go from every point of S to any other point of S

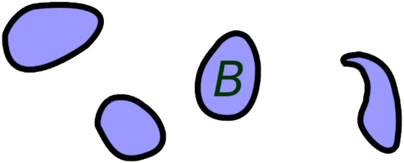
Connected:



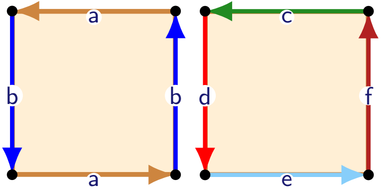
or



Not connected:



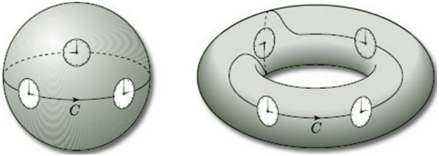
or

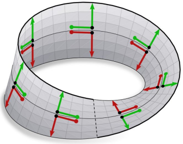


Orientability – we can tell on the words

Orientable = consistent choice of a coordinate system

Top: orientable, bottom: not orientable



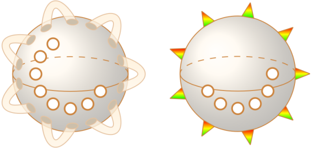


This is hard to check on the surface itself but:

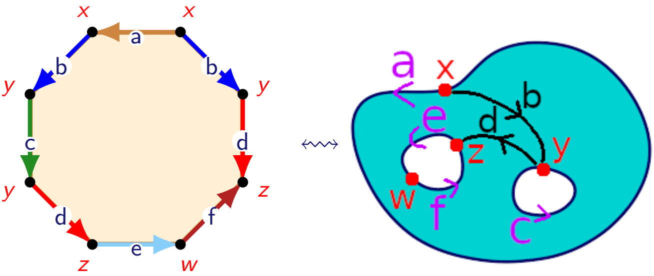
- Words encode orientability
  - Orientable: $\ldots a \ldots \bar{a} \ldots$ or $\ldots \bar{a} \ldots a \ldots$
  - Non-orientable: $\ldots a \ldots a \ldots$ or $\ldots \bar{a} \ldots \bar{a} \ldots$

Boundary = punctures = holes

Eight and six boundary components, respectively:



On the polygon this is the free-edge game: identify free edges, and check what cycles they form, e.g.:



The classification theorem

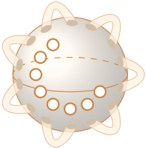
Theorem

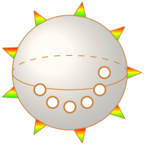
Let S be a connected surface. Then there exist non-negative integers d, p and t such that

1. $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$
2. *the boundary of S is the disjoint union of d circles*
3. *S is orientable if and only if $p = 0$*

Moreover, we can assume that $pt = 0$, in which case S is uniquely determined up to homeomorphism by (d, p, t)

Thus, every surfaces is of either of the following two forms, called standard:

$\#^8 \mathbb{D}^2 \# \#^7 \mathbb{T} \quad \cong$ 

$\#^6 \mathbb{D}^2 \# \#^9 \mathbb{P}^2 \quad \cong$ 

$d \leftrightsquigarrow$ punctures=boundary=holes

$p \leftrightsquigarrow$ projective planes=cross caps

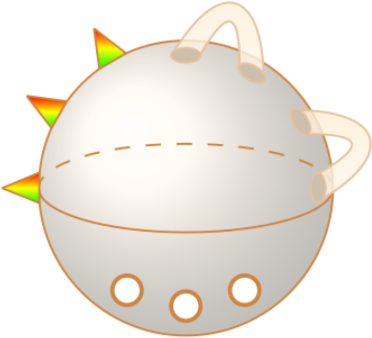
$t \leftrightsquigarrow$ handles=tori

Handles and cross-caps do not want to go along

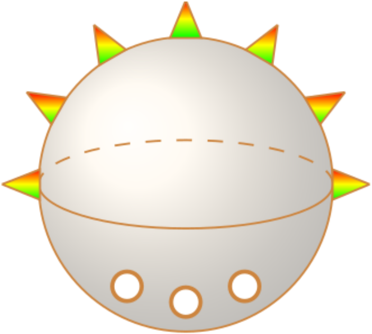
$$\mathbb{T} \# \mathbb{P}^2 \cong \mathbb{K} \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \leftrightsquigarrow \text{“}t = 2p\text{”}$$

Not true: $\mathbb{T} \cong \mathbb{K}$

We can use this to always get rid of all tori in the presence of $\mathbb{P}^2$, e.g.:



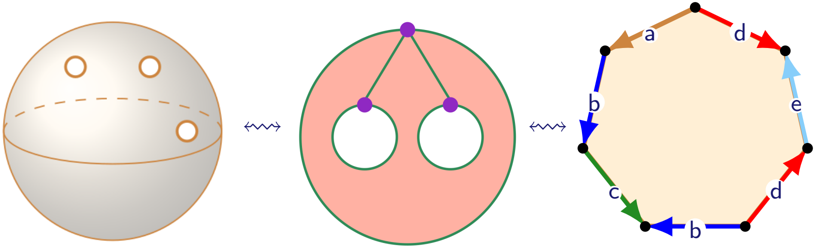
$\cong$



The left-hand surface is not in standard form

From a surface to a polygon

Here is an example how to find a word for the 3-times punctured sphere:



In general, using the classification theorem, we had standard words that we can paste together:

$$\#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \dots\ a_t\ b_t\overline{a_t}\ \overline{b_t}$$

$$\#^p\mathbb{P}^2 = a_1\ a_1\ a_2\ a_2\ \dots\ a_p\ a_p$$

$$\#^d\mathbb{D}^2 = a_1\ b_1\ a_2\ b_2\ \dots\ b_{d-1}a_d\ \overline{b}_{d-1}\ \dots\overline{b}_2\ \overline{b}_1$$

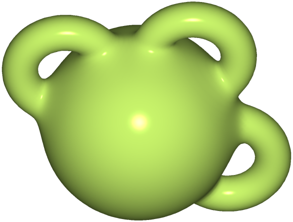
Heawood's exciting theorem

For a connected closed surface $S \ncong \mathbb{K}$ we have that the chromatic number $C(S)$ is

$$C(S) = \left\lfloor \tfrac{1}{2}(7 + \sqrt{49 - 24\chi(S)}) \right\rfloor$$

Additionally $C(\mathbb{K}) = 6$

Example

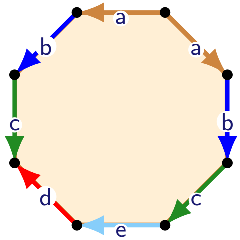
$S =$  $, \chi(S) = -4, C(S) = \lfloor 9.5208 \rfloor = 9$

recall the formula:

$$\chi(S) = 2 - d - p - 2t$$

for $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$

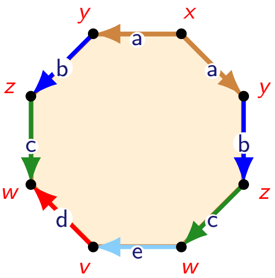
A random example



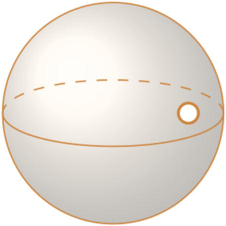
To find (d, p, t) for S we go through a list of steps:

1. Identify vertices and count them $\Rightarrow |V|$
2. Count edges and faces $\Rightarrow |E|$ and $|F|$
3. Compute $\chi(S) = |V| - |E| + |F|$
4. Check how free edges arrange themselves in cycles $\Rightarrow d$
5. Check for $a...a$ and $\bar{a}...\bar{a}$; if we find them, then $t = 0$ otherwise $p = 0$ $\Rightarrow$ we get either p or t
6. Use $\chi(S) = 2 - d - p - 2t$ to determine the remaining entry t or p

A random example – part 2

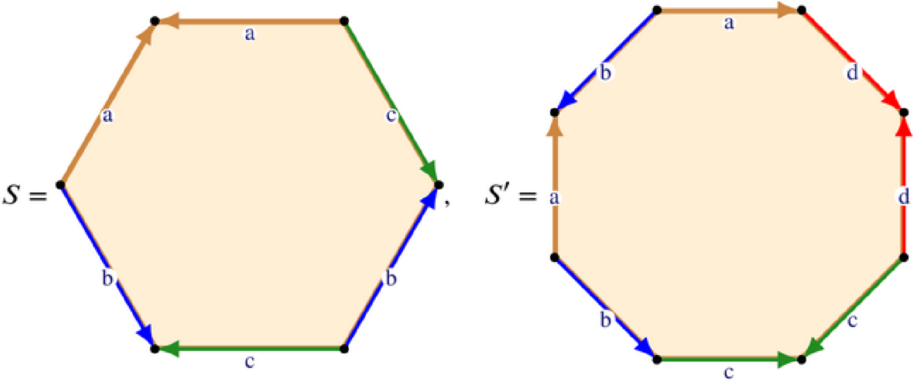


$\leftrightsquigarrow$



Lets do it!

1. From the above we get $|V| = 5$
2. Counting edges and faces gives $|E| = 5$ and $|F| = 1$
3. We get $\chi(S) = |V| - |E| + |F| = 1$
4. The only free edges $d\colon v \to w$ and $e\colon w \to v$ form one cycle, so $d = 1$
5. No pairs $a...a$ or $\bar{a}...\bar{a}$, so $p = 0$
6. $1 = \chi(S) = 2 - 1 - 0 - 2t$ gives $t = 0$



These two surfaces are well-known and want to be identified – **try yourself!**

Exercise Write down some word representing a polygonal form and identify its corresponding standard form, meaning (d, p, t)

Topic 3: knots!

Questions we ask about a knot K

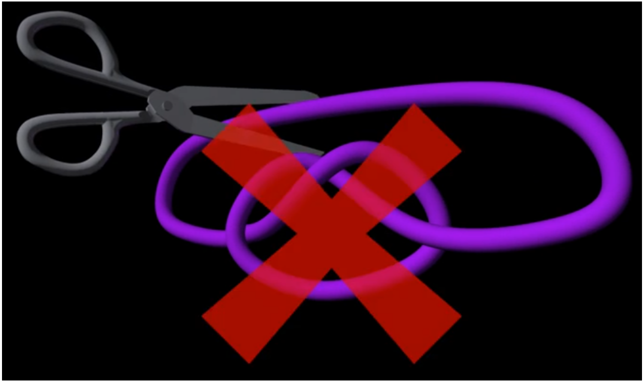
1. Have we seen K before? Is it one of the standard ones, i.e. for low crossing number?
2. Can the diagram(=projection) of K that we see be simplified?
3. Is K the unknot a.k.a. trivial?
4. What is the crossing number of K?
5. Is K alternating?
6. Is K three colorable?
7. Is K p-colorable for $p > 3$?
8. What is the knot determinant of K?
9. Can we explicitly compute a Seifert surface for a diagram of K?
10. What is the genus of K?

Let us answer 1-10 for the knot 5_1

But before, let us recall what the above are!

Knots

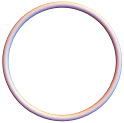
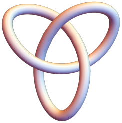
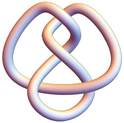
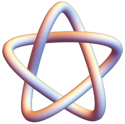
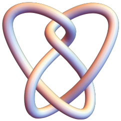
A knot is an embedding of S^1 into $\mathbb{R}^3$ and we study these up to equivalence, i.e. continuous deformation without cutting



Note that all knots are homeomorphic, so this is the wrong notion of equivalence for knots

The periodic table of knots

A main point of knot theory is to have a table of knots up to mirror images:

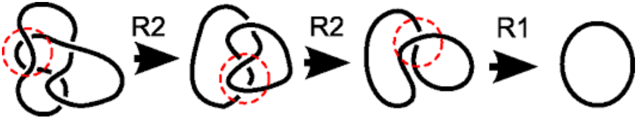
knot					
name	unknot	trefoil	figure 8	cinquefoil	three-twist
notation	0_1	3_1	4_1	5_1	5_2
$cross(K)$	0	3	4	5	5
$\det(K)$	1	3	5	5	7
$g(K)$	0	1	1	2	1
prime?	yes	yes	yes	yes	yes
alternating?	yes	yes	yes	yes	yes

Google The Rolfsen Knot Table or use e.g. KnotData of Mathematica

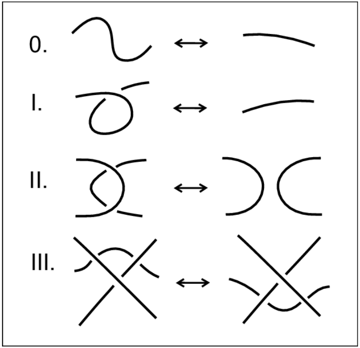
Mirror images (=flipped crossings) cannot be detected by our invariants

Simplify diagrams using Reidemeister moves

A first step is to check whether there are any "obvious" simplifications:

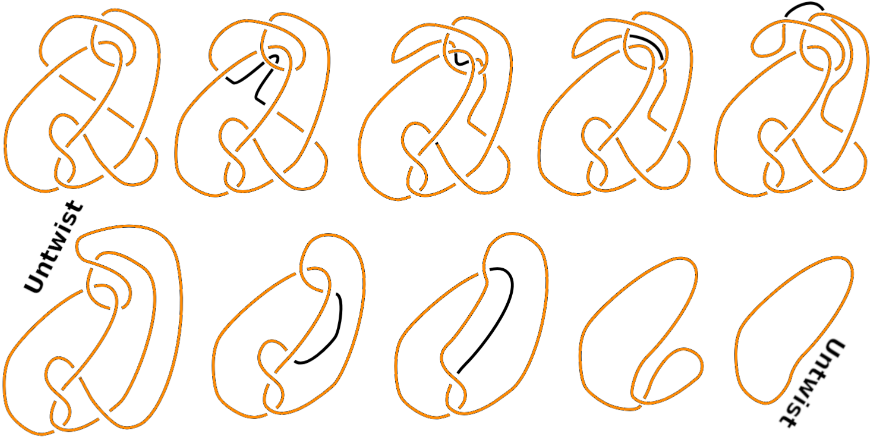


Recall that two knot diagrams represent the same knot if and only if we can relate them by the Reidemeister moves:



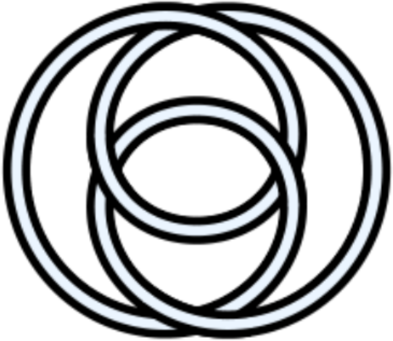
The culprit

Sometimes diagrams drastically simplify:



Exercise Check whether you understand the Reidemeister moves used for the culprit on the previous slide

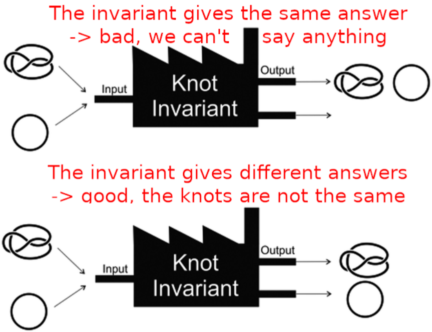
Exercise Check using isotopies and Reidemeister moves whether these two beasts are the same knot:

$K =$  , $K' =$ 

The main question...

...is always: are two knot diagrams representing the same knot?

We want knot invariants to do this!

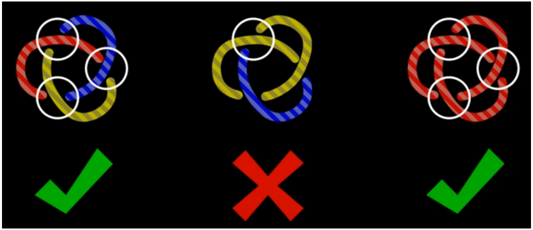


We had essentially two ways to decide that

- Knot invariant 1: colorability
- Knot invariant 2: genus

p-colorable; here only $p = 3$

Coloring = each segment gets a color such that we have 3-colored crossings or monochromatic crossings



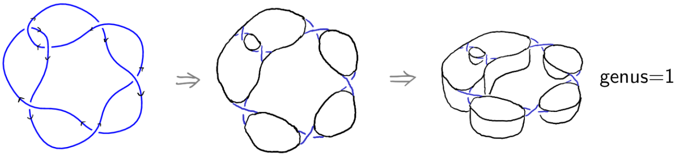
A knot is 3-colorable if it admits a non-monochromatic coloring

Trefoil knot: **tricolorable**

Figure-eight knot: **NOT tricolorable**

The genus: great to check whether a knot is trivial

Genus = the minimal t of all Seifert surfaces; to compute it for an alternating knot run Seifert's algorithm:

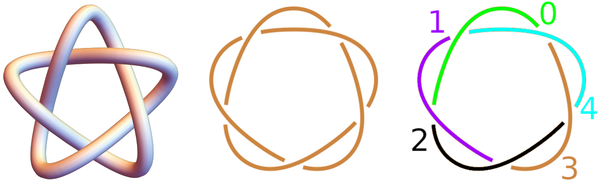


Then $t = \frac{1}{2}(1 + c - s)$ where c is the number of crossings and s the number of Seifert circles

Cool fact (verify "$\Leftarrow$" as an exercise):

$$g(K) = 0 \Leftrightarrow K \cong \text{unknot}$$

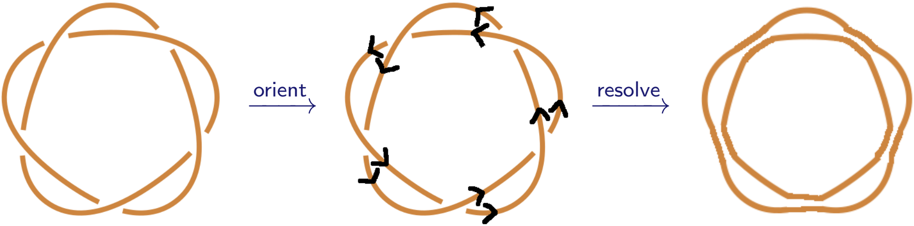
The knot $K = 5_1$



Let us go through the list of steps:

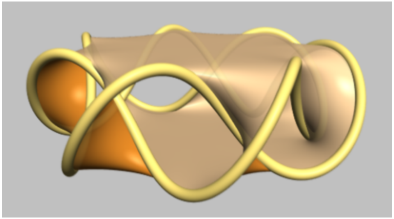
1. We have seen it before, it is 5_1
2. The diagram cannot be made simpler in any obvious way
3. The knot is not trivial, see next slide or coloring above
4. Since the diagram is alternating $cross(K) = 5$
5. The diagram is clearly alternating
6. No, K is not 3-colorable see above
7. Yes, K is 5-colorable, see above
8. We have $\det(K) = 5$ by computation
9. Yes, Seifert surfaces are easy to get, see next slide
10. $g(K) = 1$, see next slide

The knot $K = 5_1$ – Seifert surfaces and genus

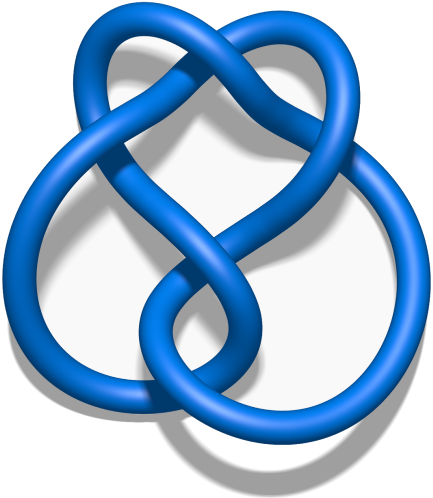


Thus, $c = 5$ and $s = 2$ gives $\chi(S) = s - c = -3$ and
$g = \frac{1}{2}(1 - \chi(S)) = \frac{1}{2}(1 + c - s) = 2$

Putting in the twists gives:



Another knot – answer 1–10 as an exercise



This is knot 5_2 – **try yourself!**

I hope you enjoyed topology!