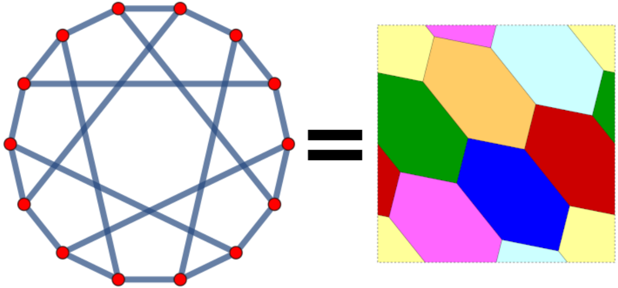
Topology – week 7

Math3061

Daniel Tubbenhauer, University of Sydney

Lecturer	Daniel Tubbenhauer
Office hour	Zoom (https://uni-sydney.zoom.us/j/89436493625) Monday 4:30pm-5:30pm or by appointment (an informal email suffices)
Contact	daniel.tubbenhauer@sydney.edu.au
Web	www.dtubbenhauer.com/teaching.html

- I apologize in advances for any typos or other errors on these slides!
- I'd be grateful for any corrections that you send to me or post on Ed



Unit outline

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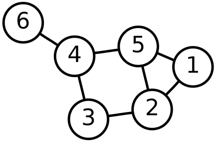
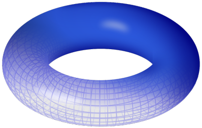
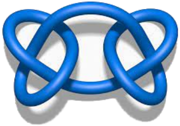
We will study:

- Graphs
- Surfaces
- Knots







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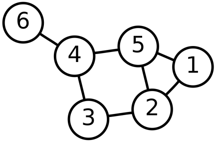

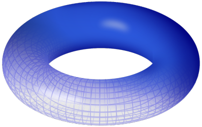

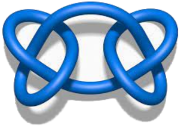

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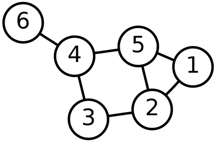

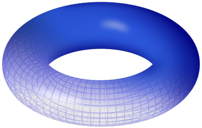

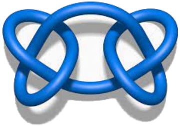

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- In topology we are allowed to bend and stretch
- We are not allowed to cut, tear or join surfaces together

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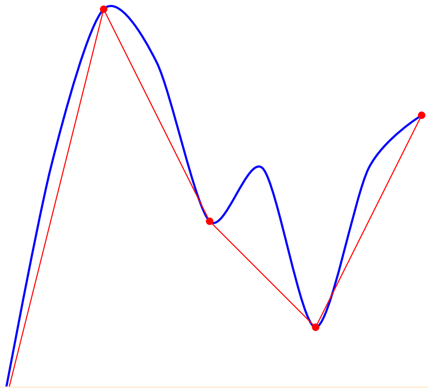

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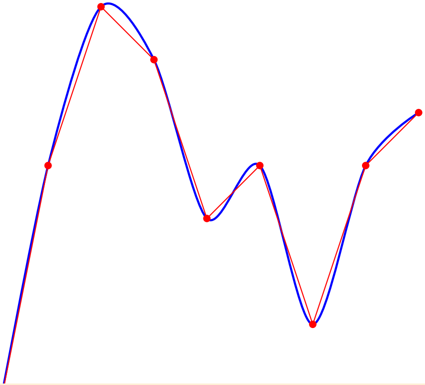
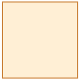


Topological equivalences



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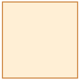
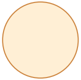
 $\equiv$ 

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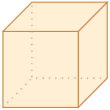
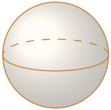
 $\equiv$ 

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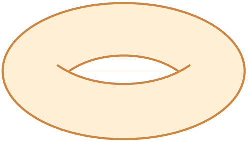
...as well as looking at more exotic surfaces

 $=$  $\equiv$??

A torus is the same as a coffee mug

Source https://en.wikipedia.org/wiki/Topology

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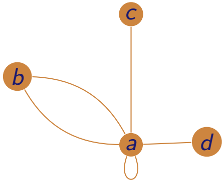

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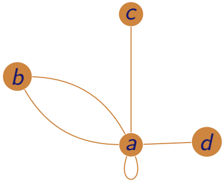

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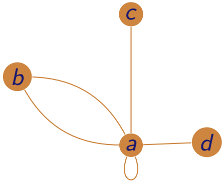

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As shown, we allow loops and duplicate edges

Warning: drawings can be misleading

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There are many ways to draw the same graph so we always need to check that whatever are doing does not depend on how the graph is drawn!

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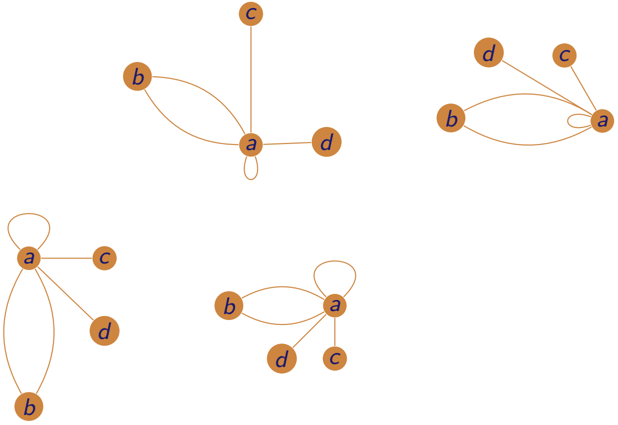
Here are four different ways to draw the same graph



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Path graphs P_n, for $n \geq 1$ (also called line graphs)

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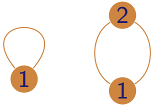

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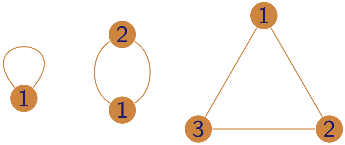

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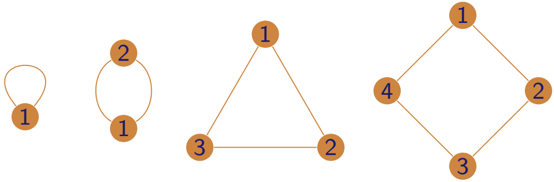

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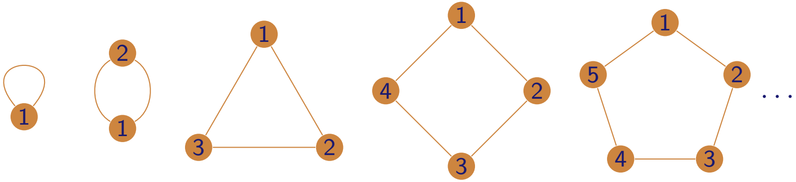

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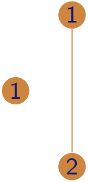

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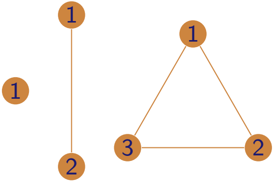

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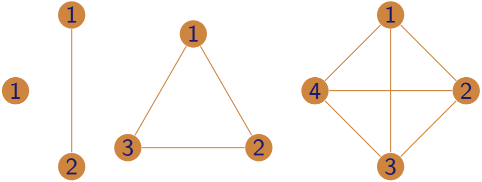

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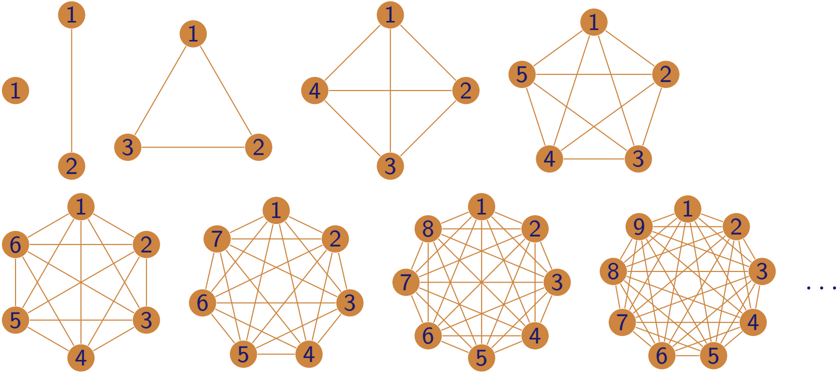
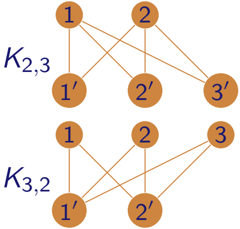


Complete bipartite graphs $K_{n,m}$, for $n, m \geq 1$

Vertex set $V = \{1, 2, \ldots, n, 1', 2', \ldots, m'\}$

Edge set $E = \left\{ \{i, j'\} \,\middle|\, 1 \leq i \leq n, 1 \leq j \leq m \right\}$



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Examples

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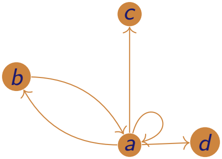

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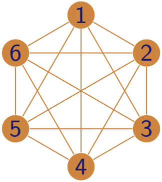

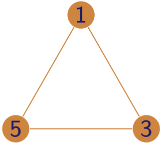

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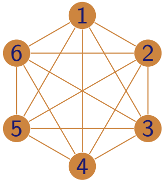

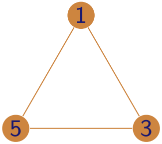

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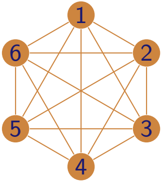

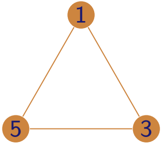

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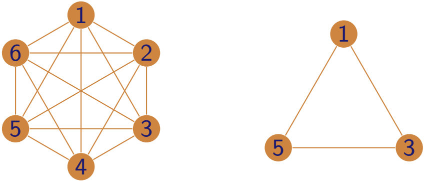

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...but what does it mean for graphs to be "the same "?

Isomorphic graphs

Two graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic, written $G \cong H$, if there is a bijection $f : V \longrightarrow W$ such that the induced map on edges, which sends an edge $\{v, v'\} \in E$ to $\{f(v), f(v')\}$, is also a bijection.

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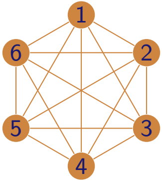

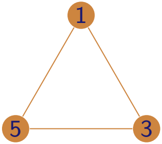

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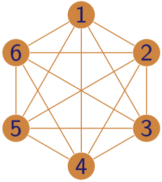

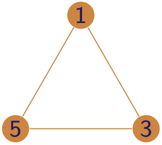

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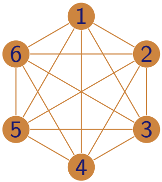

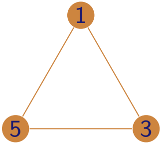

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For example, define f by

$f(1) = 1$,
$f(3) = 2$, and
$f(5) = 3$

Subgraphs of complete graphs

Proposition

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Write $V = \{v_1, v_2, \ldots, v_n\}$.

Let $N = \{1, 2, \ldots, n\}$ be the vertex set of K_n and let

$$E_n = \{\, \{i, j\} \mid 1 \le i < j \le n \,\}$$

be its edge set.

Define $H = (N, E_V)$ to be the subgraph of K_n with

$$E_V = \{\, \{i, j\} \mid \{v_i, v_j\} \in E \,\}.$$

Then the map $f : N \longrightarrow V$ given by $f(i) = v_i \in V$ is a graph isomorphism.

Planar graphs

A planar graph is a graph that can be drawn in the $\mathbb{R}^2$ in such a way that no edges cross.

This gives a planar embedding of the graph

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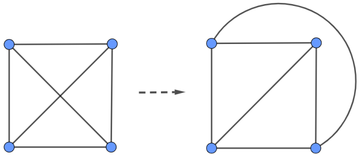
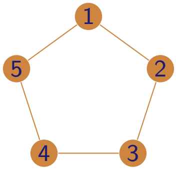
Examples

- Graphs can have planar embeddings and other non-planar realizations
- Every path graph P_n is planar
- Every cyclic graph C_n is planar



Planar graph (K_4)

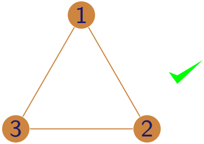
Planar embedding of K_4

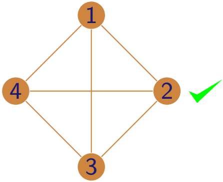


Complete graphs are rarely planar

- K_1 

- K_2 

- K_3 

- K_4 

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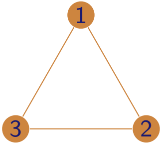
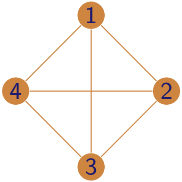
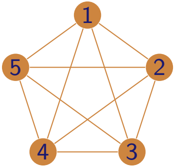
- K_3 

- K_4 

- K_5 

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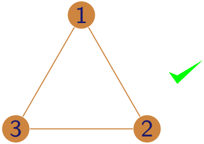
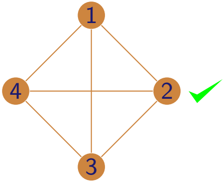
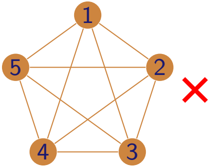
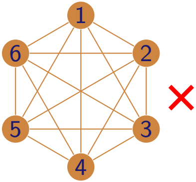
- K_3 

- K_4 

- K_5 

- K_6 

Graph embeddings in $\mathbb{R}^3$

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Every graph can be embedded (i.e. without edge crossings) in $\mathbb{R}^3$

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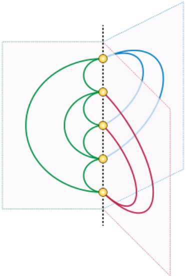
Moral Graphs are "low dimensional" objects

Proof First, loops and duplicate edges are easy to treat, so we ignore them. Next, use a book embedding:

A 3-page embedding of K_5:



In general, one can embed K_n into a book with $\lceil n/2 \rceil$ pages. Since every graph is a subgraph of some K_n, so we are done since books $\subset \mathbb{R}^3$

The degree of a vertex

Let $G = (V, E)$ be a graph. The degree of a vertex $v \in V$ is

$$\deg(v) = \#\Big\{\text{number of edges in } E \text{ that have } v \text{ as an endpoint}\Big\}$$

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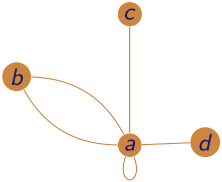
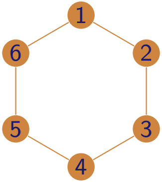
- 

deg(a)=5
- P_n 

deg(4)=2

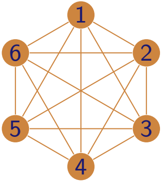
Degrees of vertices in standard graphs; examples

- C_n



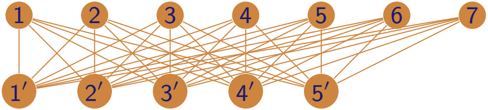
deg(4)=2

- K_n



deg(4)=5

- $K_{n,m}$



deg(4)=5

The handshaking lemma

Proposition (Vertex-degree equation $=$ handshaking lemma)

Let $G = (V, E)$ be a finite graph. Then

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Proof

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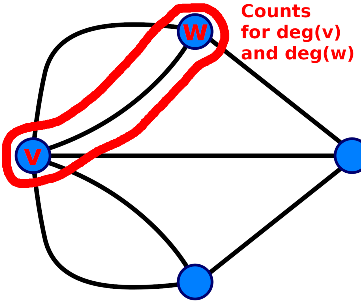
Let $G = (V, E)$ *be a finite graph. Then*

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof If I shake your hand, then you shake mine: every edges is adjacent to two vertices, hence each edges contributes twice



The handshaking lemma

Proposition (Vertex-degree equation = handshaking lemma)

Let $G = (V, E)$ be a finite graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof

Strictly speaking, we would use induction on $|E|$:

There is nothing to show if there is no edge, and if $|E| > 0$ remove any edge e use induction for $E' = E \setminus \{e\}$, and add e using the previous observation

The Euler characteristic of a graph

Let $G = (V, E)$ be a graph. The Euler characteristic of G is the integer

$$\chi(G) = |V| - |E|$$

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Examples

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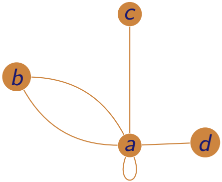
$\chi(G) = -1$

- 

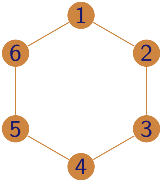
$\chi(G) = -1$

- P_n 

$\chi(G) = 1$

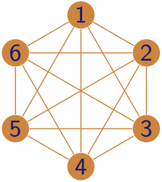
The Euler characteristic of standard graphs

- C_n



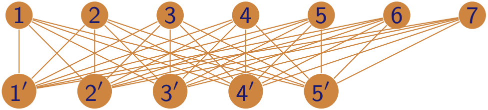
$\chi(G) = 0$

- K_n



$\chi(G) = n - \frac{1}{2}n(n-1) = -\frac{1}{2}n(n-3)$

- $K_{n,m}$



$\chi(G) = n + m - nm$

Subdividing graphs

Let $G = (V, E)$. A subdivision of G is any graph $\dot{G}$ that is obtained from G by successively replacing V with $V \cup \{u\}$, for $u \notin V$, and E with $E \cup \big\{\{v, u\}, \{u, w\}\big\} \setminus \big\{\{v, w\}\big\}$, for an edge $\{v, w\} \in E$

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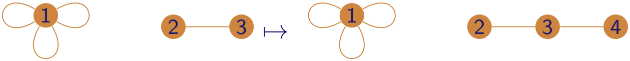
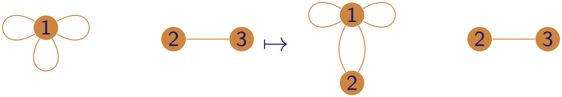
- 

- 

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Subdivision and Euler characteristic

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The operation

clearly increases V and E by one, so their difference does not change.

Paths in graphs

Let $G = (V, E)$ be a graph and $v, w \in V$. A path in G of length n from v to w is a sequence of vertices $v = v_0, v_1, \ldots, v_n = w$ such that $\{v_i, v_{i+1}\} \in E$, for $0 \leq i < n$.

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That is, the path looks like $v_0 — v_1 — v_2 — \cdots — v_n$

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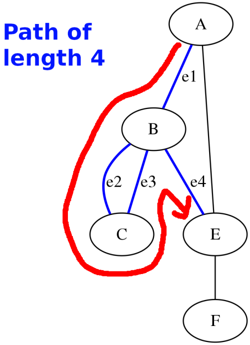

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The connected components of a graph G are the maximal connected subgraphs of G. That is, $H = (W, F)$ is a connected component of $G = (V, E)$ if H is
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Example

Not connected, two connected components

Connected examples

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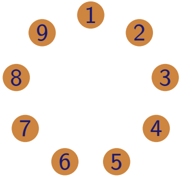

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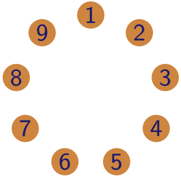

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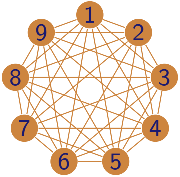

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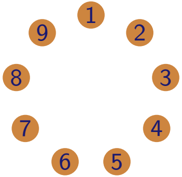

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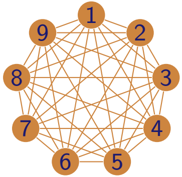

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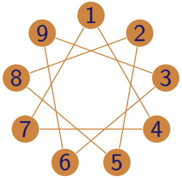

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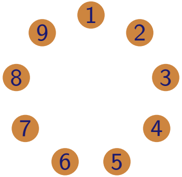

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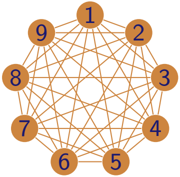

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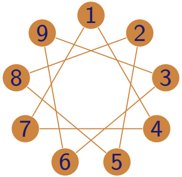

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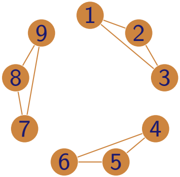

Circuits

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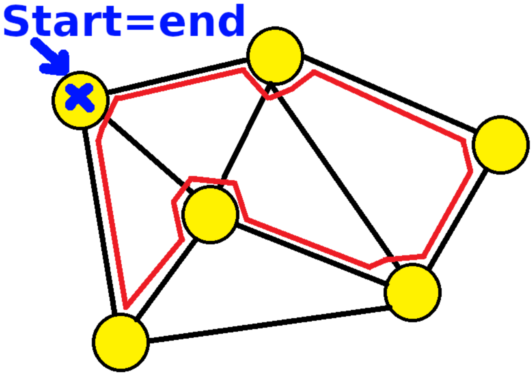

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- We will soon see that the Euler characteristic is closed related to the number of "reduced" circuits in a graph

Contractible circuits

A circuit $v = v_0 \to v_1 \to \cdots \to v_n = v$ is contractible if it contains two consecutive repeated edges $\{v_i, v_{i+1}\} = \{v_{i+1}, v_{i+2}\}$, for some $0 \leq i \leq n-2$

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A circuit is reduced if it is not contractible

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Observations

- Reduced circuits are "efficient" in the sense that they do not backtrack
- A reduced circuit of length n is not necessarily isomorphic to the cycle graph C_{n+1} because it could, for example, be a figure 8 graph

Leaves and trees

A non-trivial circuit is a reduced circuit of length $n > 0$

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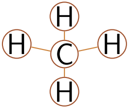
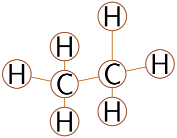
A tree is a connected graph that has no non-trivial circuits

- Saturated hydrocarbons



Methane



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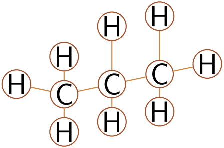

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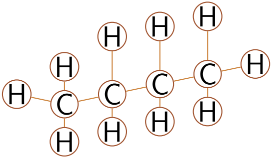

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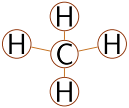

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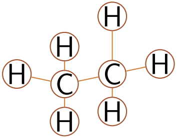

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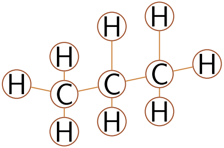

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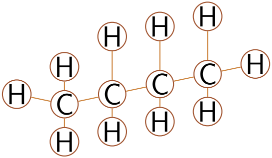

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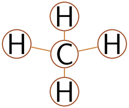

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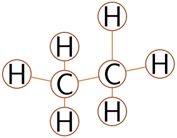

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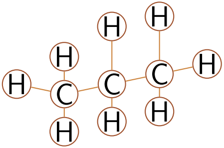

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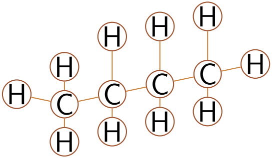
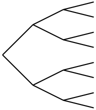
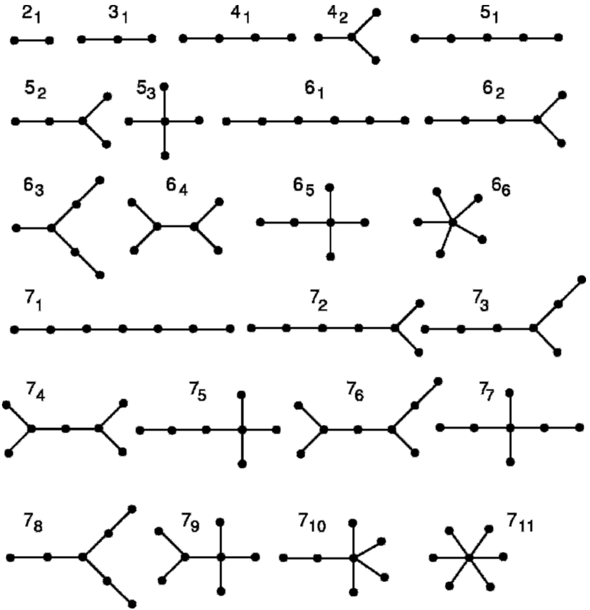


Butane

- A tournament tree



A catalog of small (connected) trees



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Proof Take a longest reduced path P in T, then both endpoints of P are leaves

Why? Say the endpoints are v and w. WLOG suppose v is not a leaf; then v has at least two neighbors and one of them is not in P. (Otherwise we would have a circuit.) Thus one can make P longer. Contradiction

The Euler characteristic of a tree

Theorem

Suppose that T is a tree. Then $\chi(T) = 1$

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Proof Argue by induction on the number of edges $|E|$

For $|E|$ small use the previous table.

Otherwise, remove one leave (which exists by the previous statement). The resulting tree has $\chi(T) = 1$, and adding the leave back increases V and E by one, so χ remains constant

Number of edges and vertices in a tree

Corollary

Suppose that $T = (V, E)$ *is a tree. Then* $|V| = |E| + 1$.

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Proof By the previous statement

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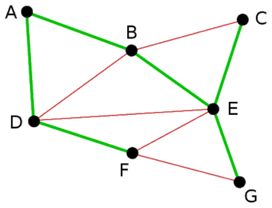
Example



Spanning trees continued

Proposition

Suppose that $G = (V, E)$ is a connected graph.
Then G has a spanning tree $T = (V, F)$ (same vertices)

Proof Remove edges from nontrivial circuit of G to break them; the result is a spanning tree

(Formally, use induction on the number of nontrivial circuit of G)

An upper bound on $\chi(G)$

Corollary

Suppose that G is a connected graph. Then $\chi(G) \leq 1$ with equality if and only if G is a tree.

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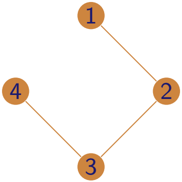
Corollary

Let G be a connected graph. The number of independent cycles (defined via example on the next slide) in G is $1 - \chi(G)$

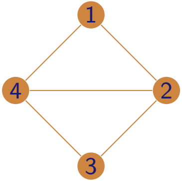
Proof By the previous statements

Independent cycles

Examples



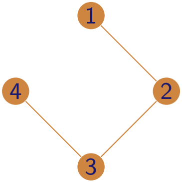
$\chi = 1$, no independent cycles



$\chi = -1$, two independent cycles

We have $\{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\} =$
$\{\{1,2\},\{2,4\},\{1,4\}\} + \{\{2,3\},\{3,4\},\{2,4\}\} \bmod 2$

Examples



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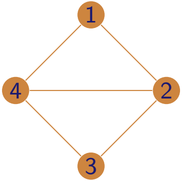

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Remark It is possible to construct a vector space of "cycles" that has dimension $1 - \chi(G)$, which shows that the number of independent cycles makes sense. This is beyond the scope of this course.

Topology – week 8

Math3061

Daniel Tubbenhauer, University of Sydney

Eulerian circuits and graphs

A Eulerian circuit is a circuit that passes through every edge exactly once

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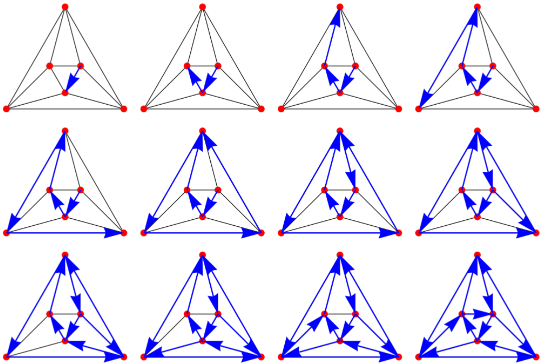

Warning Eulerian graphs do not need to be connected because they may have vertices of degree 0!

Finding Eulerian circuits

In 1736 Euler asked when graphs have Eulerian circuits (without having this terminology)

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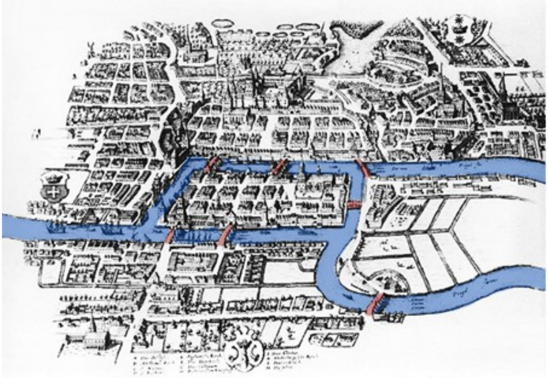

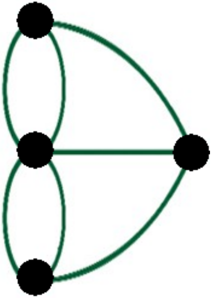

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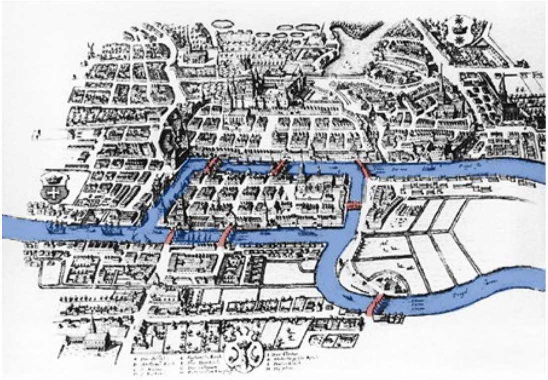

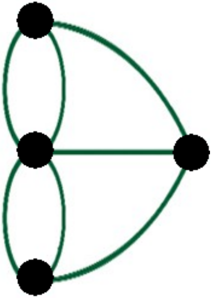

In answering this question Euler laid the foundations of graph theory

Classifying Eulerian graphs

Theorem

Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree

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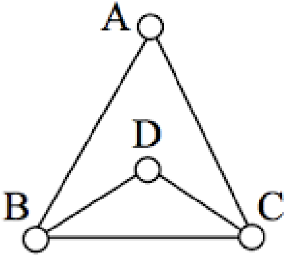
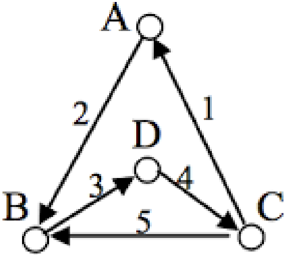
Proof

Assume that there is at least one vertex v of odd degree. Since we want to visit every edge exactly once we will eventually get stuck in v or another vertex of odd degree while trying to create an Eulerian cycle. Hence, G can not have an Eulerian cycle





Proof continued

Conversely, if every vertex has even degree, then G is not a tree so contains some circuit C. If C is an Euler circuit we are done, and if not remove all edges of C from G. The resulting (potentially disconnected) graph G' has still even degrees for all of its vertices but fewer edges than G

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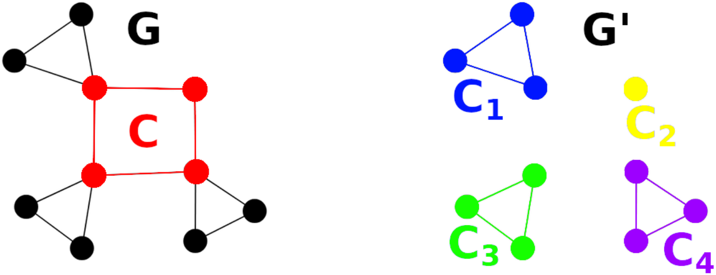
So we can argue by induction on the number of edges (the base case has no edges and is thus clear), and inductively we can assume that the connected components of G' have Euler circuits $C_1, \ldots, C_n$

Proof continued

We piece C and $C_1, \ldots, C_n$ together into an Euler cycle: we walk along C and whenever we hit a vertex of C_i we take a detour over C_i



Eulerian paths

A Eulerian path is a path that is not a circuit and which passes through every edge exactly once

Corollary

Let $G = (V, E)$ be a connected graph that is not Eulerian. Then G has a Eulerian path if and only if it has exactly two vertices of odd degree

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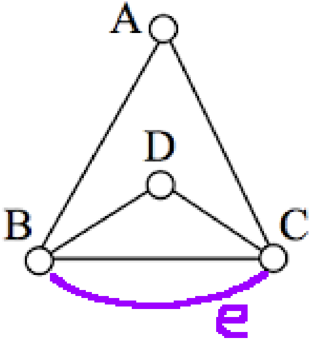
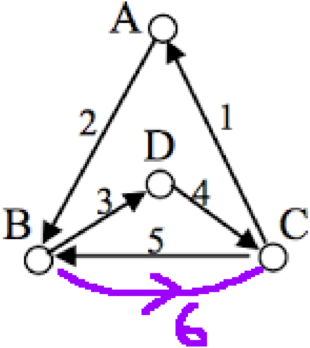
Proof

Only vertices of odd degree can be a start or an end vertex, so we need precisely two of them (all other must be of even degree by the same argument as before)

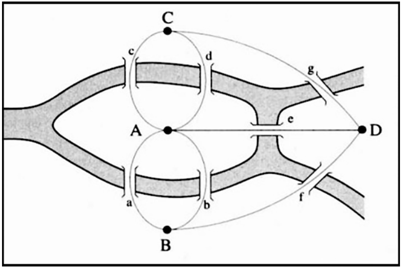
Proof continued

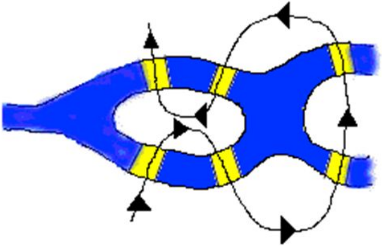
Conversely, if v and w are the two vertices of even degree, then we put an additional edge e between them. We get a graph $G' = G \cup \{e\}$ and the previous theorem gives us an Euler circuit C in G'. Then $C \setminus \{e\}$ is an Euler path





What about Königsberg?





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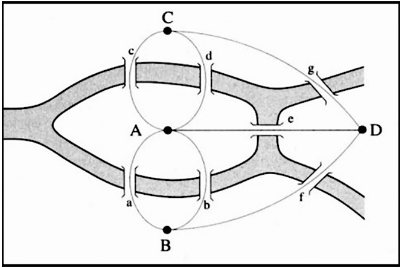

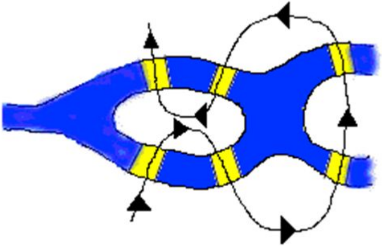

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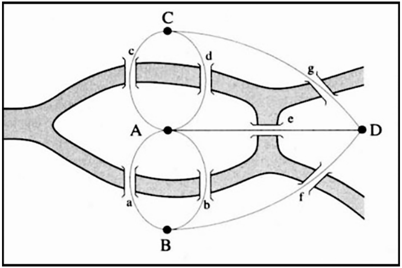

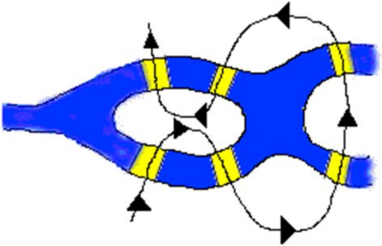

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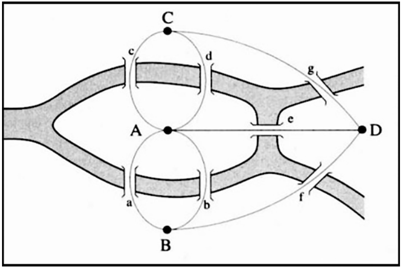

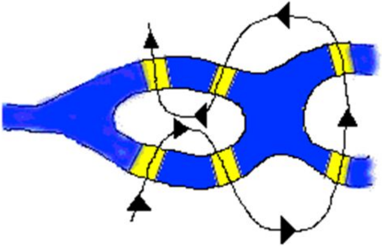

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Solution: Destroy bridge e ;-)

Topological equivalence

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, for $m, n \geq 1$

Definition

A homeomorphism $f : X \longrightarrow Y$ is a continuous map that has a continuous inverse $g : Y \longrightarrow X$. The spaces X and Y are homeomorphic if there is a homeomorphism $f : X \longrightarrow Y$

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Examples of homeomorphisms

Proposition

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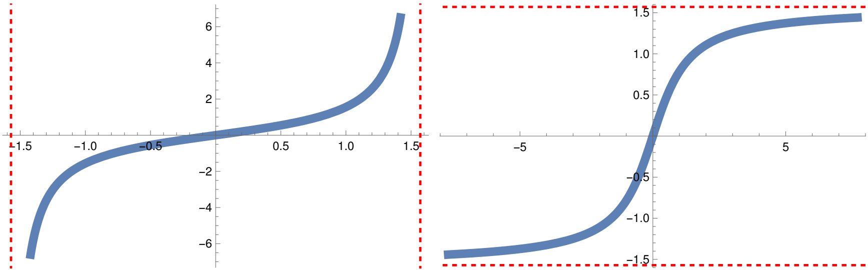
Proposition

If $a < b$, then $(a, b) \cong \mathbb{R}$

Proof It is enough to show that $(-\frac{\pi}{2}, \frac{\pi}{2}) \cong \mathbb{R}$

Proof continued

Homeomorphisms are given by $f(x) = \tan(x)$ and $g(x) = \tan^{-1}(x)$



Examples of homeomorphisms...

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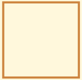
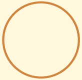
 $\cong$  $= S^1$

Examples of homeomorphisms...

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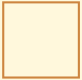
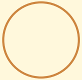
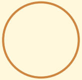
 $\cong$  $= S^1$

We show that  $\cong$ 

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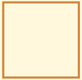
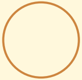
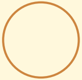
 $\cong$  $= S^1$ *We show that*  $\cong$ 

Proof

The square is $\{\, (x, y) \mid |x| + |y| = 1 \,\}$ and $S^1 = \{\, (x, y) \mid x^2 + y^2 = 1 \,\}$

Examples of homeomorphisms...

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$\square \cong \bigcirc = S^1$ *We show that* $\diamond \cong \bigcirc$

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The square is $\{\,(x,y)\,|\,|x|+|y|=1\,\}$ and $S^1=\{\,(x,y)\,|\,x^2+y^2=1\,\}$

Define: $\qquad f : \square \longrightarrow S^1; (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

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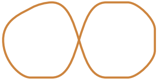
Note that $\bigcirc \ncong$ 

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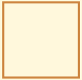
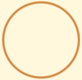
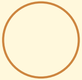
 $\cong$  $= S^1$ *We show that*  $\cong$ 

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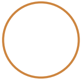
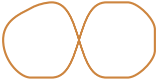
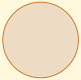
Note that  $\ncong$ 

For free we see that the square and disk are homeomorphic:

Corollary

 $\cong$  $\cong$ 

Stereographic projection in two dimensions

Think of the north pole of the circle S^1 as ∞

Stereographic projection gives a homeomorphism $\pi \colon S^1 \setminus \{\infty\} \to \mathbb{R}$:

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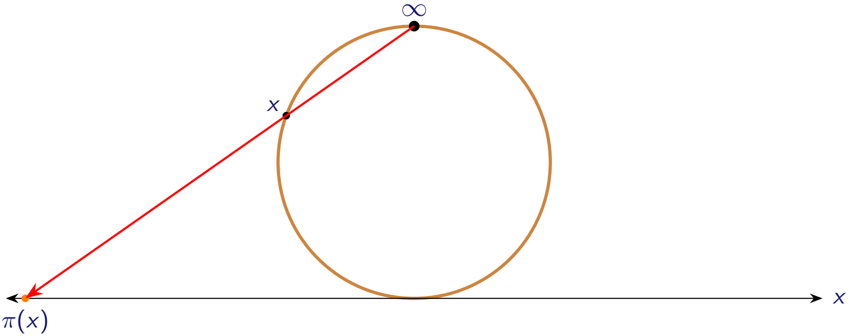

Stereographic projection in three dimensions

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Stereographic projection gives a homeomorphism $\pi\colon S^2 \setminus \{\infty\} \to \mathbb{R}^2$:

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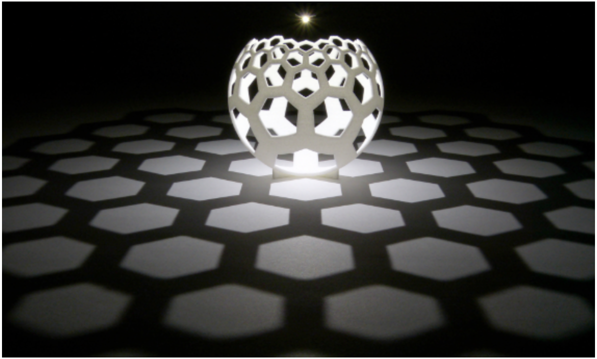
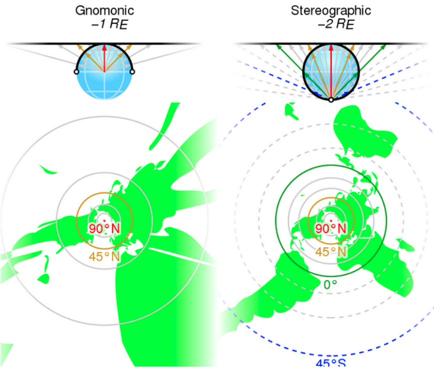


Maps

Stereographic projection is used to draw maps:



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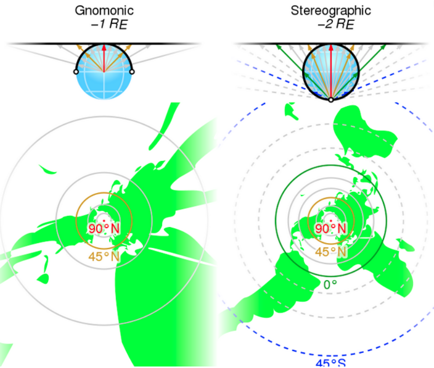

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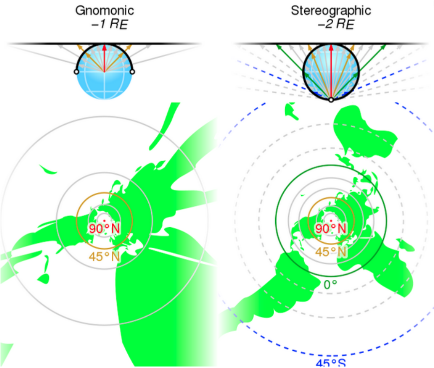

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Now that we have seen homeomorphisms we are ready to define surfaces

Surfaces — informal definition

Definition

A surface is a subset of $\mathbb{R}^n$ that, locally, is homeomorphic to the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $f(x, y) = z$ / alternatively to a disc

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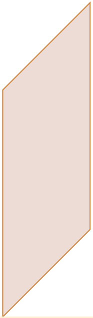
Examples

- A standard xyz-plane in $\mathbb{R}^3$



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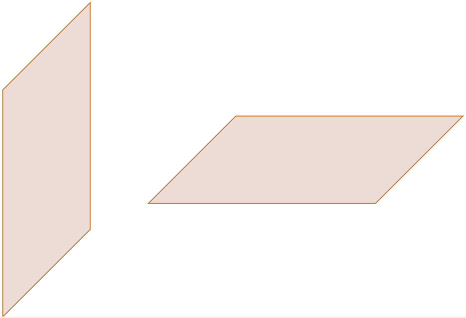

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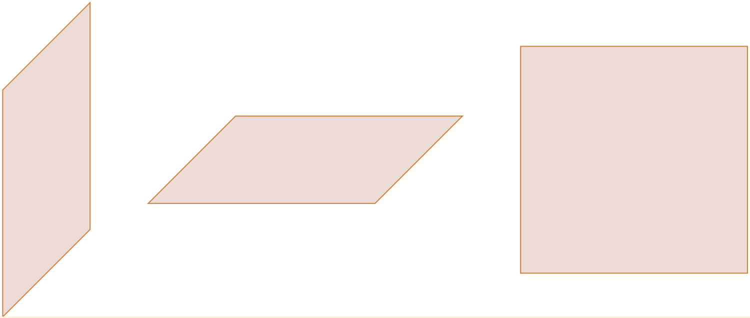
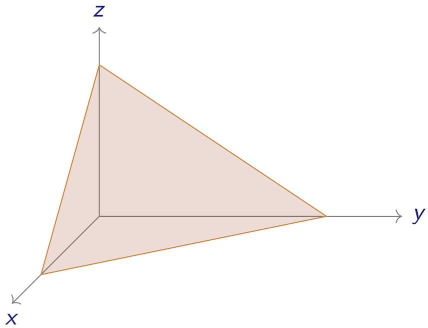


Surfaces — examples...

- Non-standard planes in $\mathbb{R}^3$



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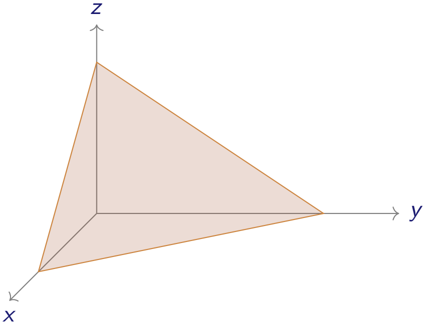
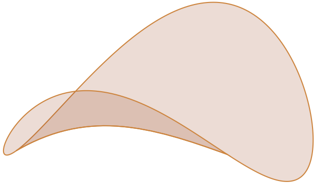
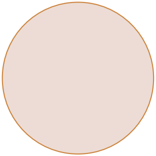


- Curved surfaces in $\mathbb{R}^3$



Surfaces — examples...

- A disk $\mathbb{D}^2$



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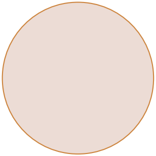
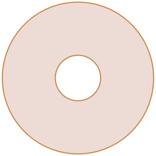


- An annulus $\mathbb{A}$



$\cong$

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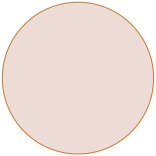
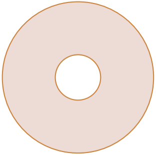


- An annulus $\mathbb{A} \cong$ cylinder



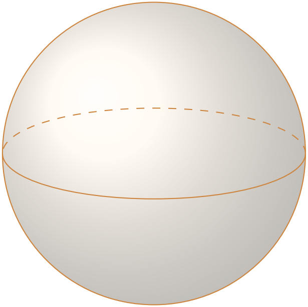
$\cong$

Strictly speaking, these are not surfaces according to our definition because they have a boundary, whereas planes in $\mathbb{R}^2$ do not have boundaries.

Our rigorous definition of a surface will allow surfaces with boundaries

Surfaces — examples...

- A sphere S^2



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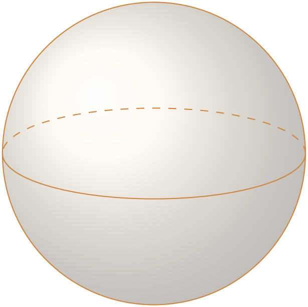
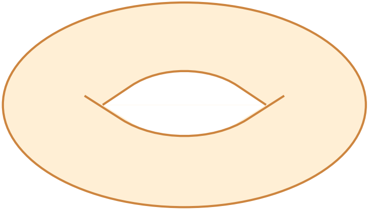


- A torus $\mathbb{T}$



Surfaces — real world examples...

- A sphere $S^2 \cong$ soccer ball

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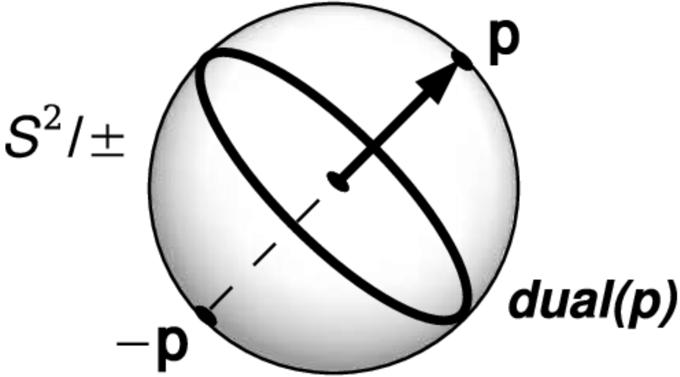
Surfaces — real world example...

- Here is a surface with boundary:

The patches are examples of neighborhoods which are discs

- The real projective plane $\mathbb{P}^2 = S^2/\text{antipode}$



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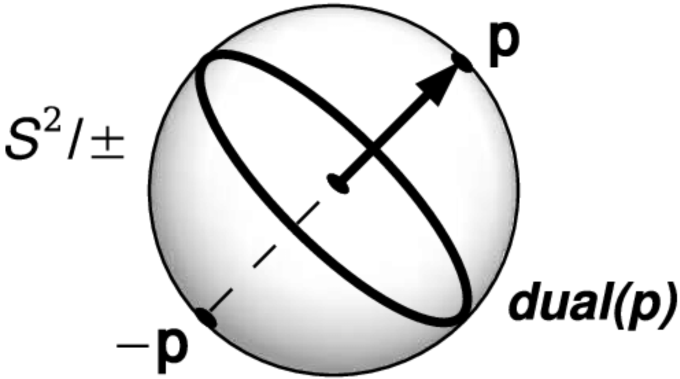
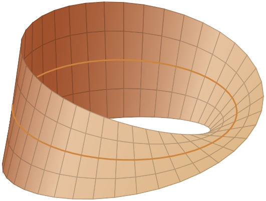


We will see other ways to describe $\mathbb{P}^2$ later

Surfaces — examples...

- A Möbius band, or Möbius strip, $\mathbb{M}$



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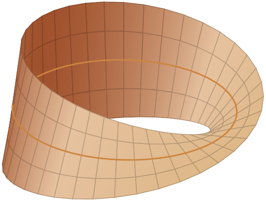
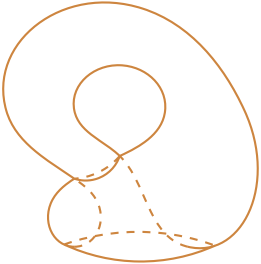


- A Klein bottle $\mathbb{K}$, also Klein surface



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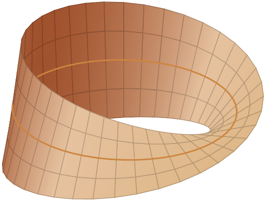

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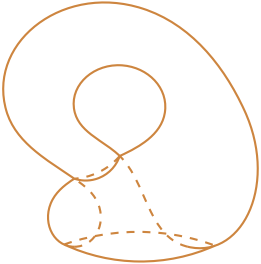
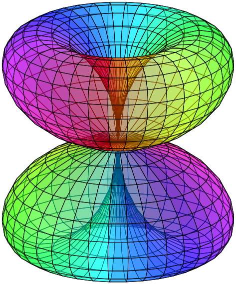


This is a three dimensional "shadow" of a four dimensional object

Surfaces — non-examples

- This is not a surface because of the cusp at the origin



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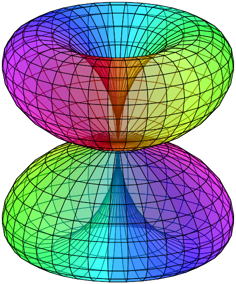
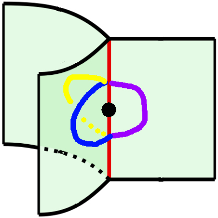


- This is not a surface because the indicated point has not a disc neighborhood



Identification spaces

A partition of a surface $S \subseteq \mathbb{R}^m$ is a collection $X_1, \ldots, X_r$ of subsets of S such that $S = X_1 \cup X_2 \cup \cdots \cup X_r$

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This makes is possible to understand Y in terms of, often, easier spaces $X_1, \ldots, X_r$, which we think of as covering Y like a patchwork quilt

Identification space for a cylinder

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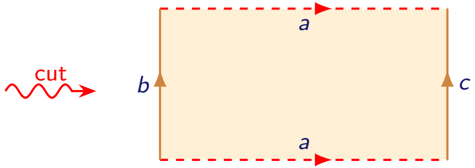

That is, the cylinder is the identification space obtained by identifying the top and bottom edges of a suitably sized rectangle

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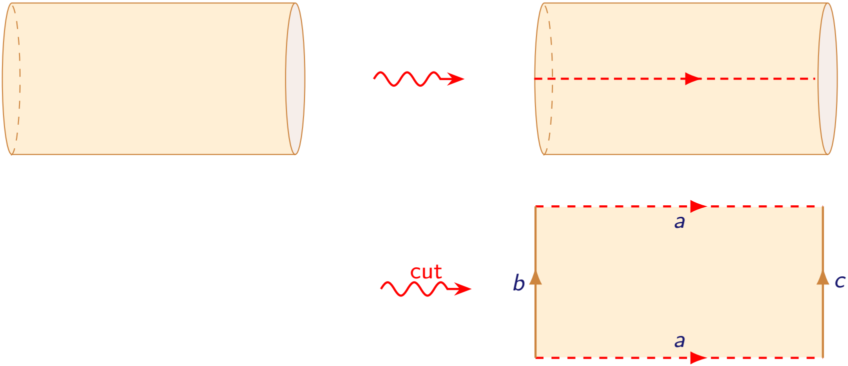

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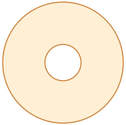
$\mathbb{A} =$ 

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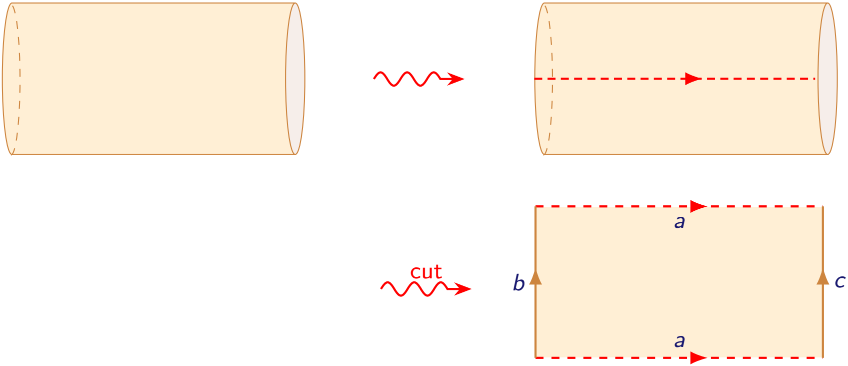

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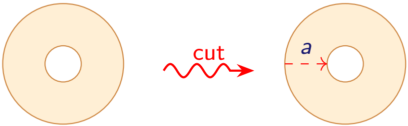
$\mathbb{A} =$ 

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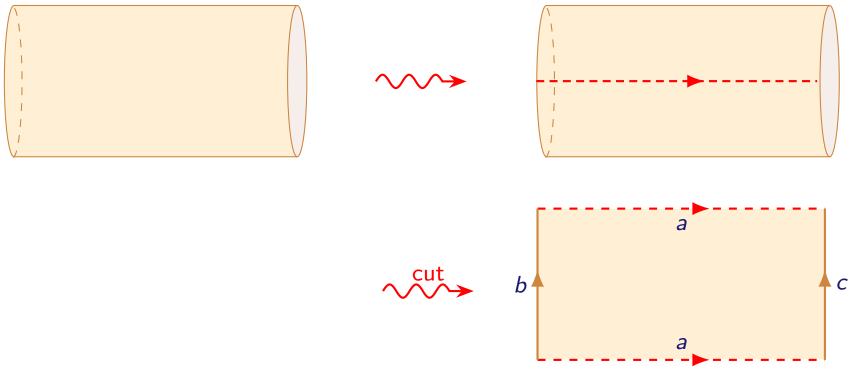

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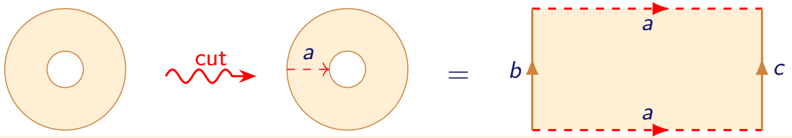
$\mathbb{A} =$ 

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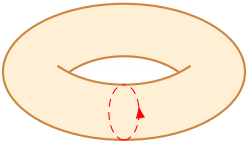

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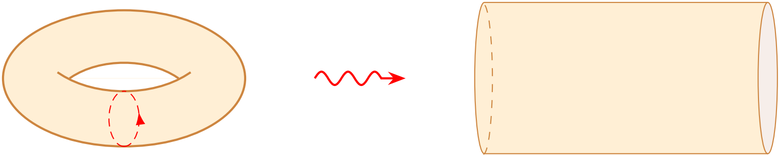

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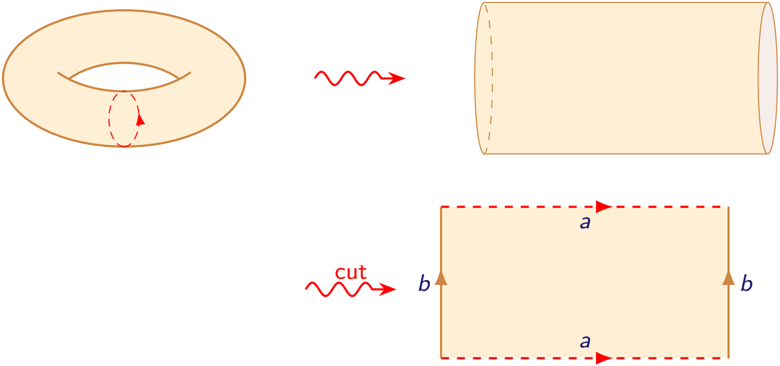
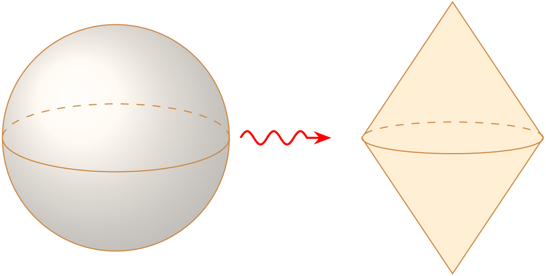


So, the torus $\mathbb{T}$ is obtained by identifying the top and bottom, and the left and right, edges of a rectangle

Identification space for a sphere



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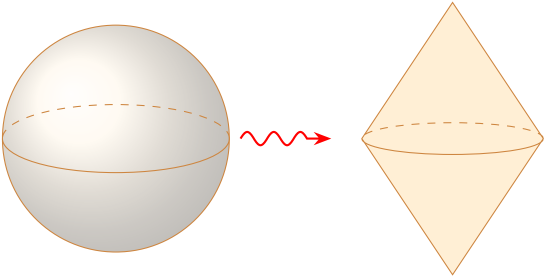

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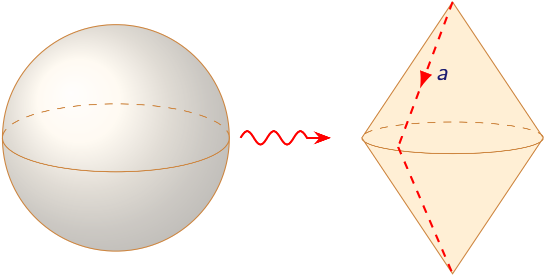

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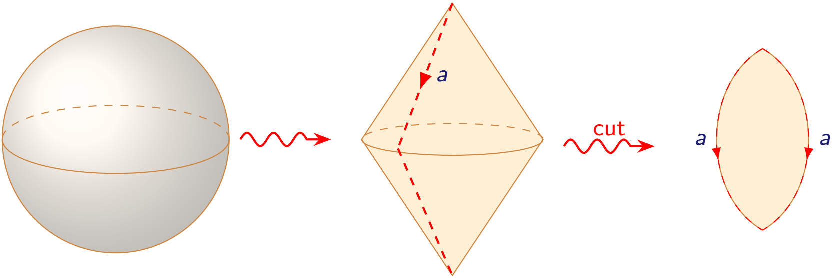

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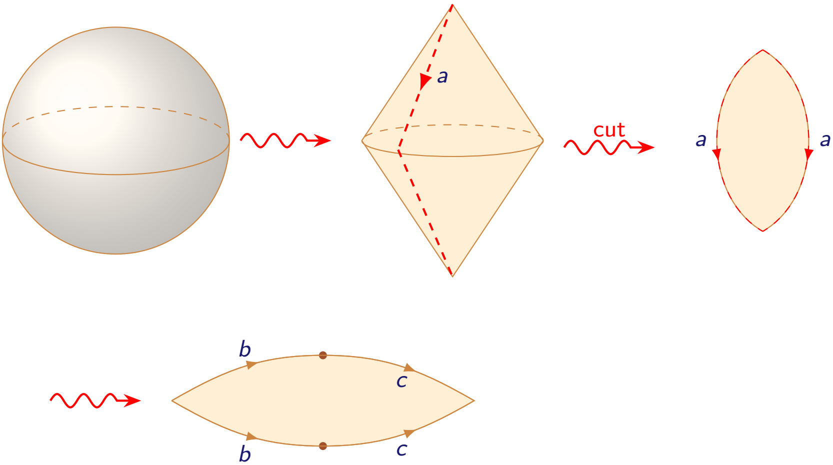

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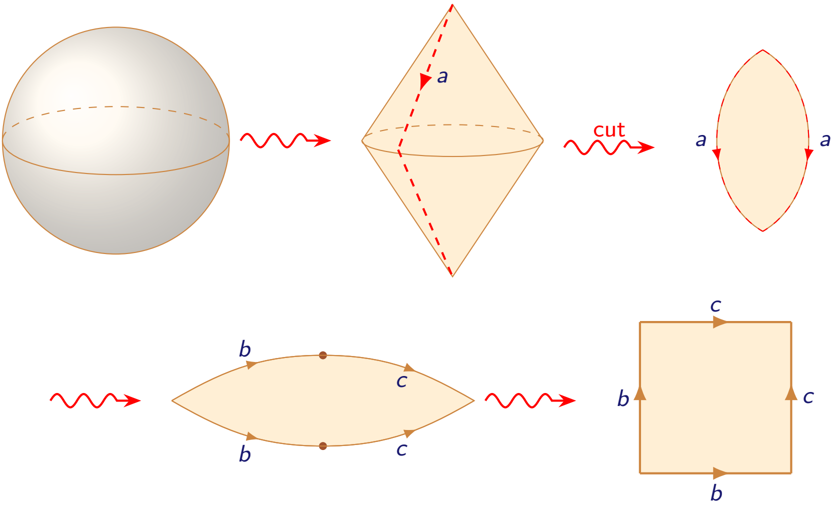

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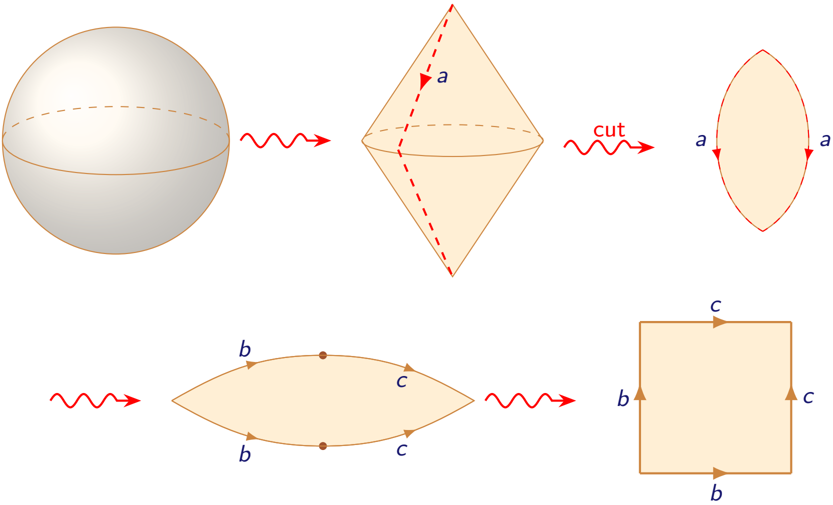
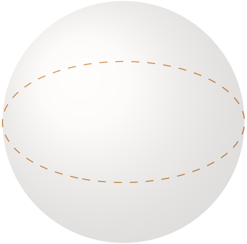


The sphere S^2 is obtained by identifying adjacent sides of a rectangle, or a 2-gon (a polygon with two sides)

Identification space for the projective plane $\mathbb{P}^2$



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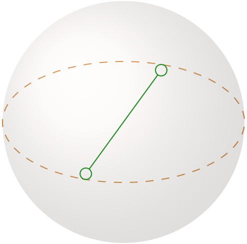

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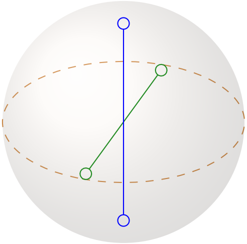

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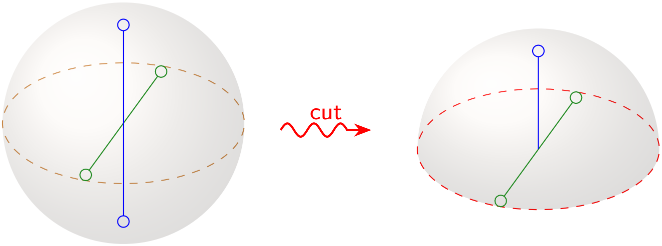

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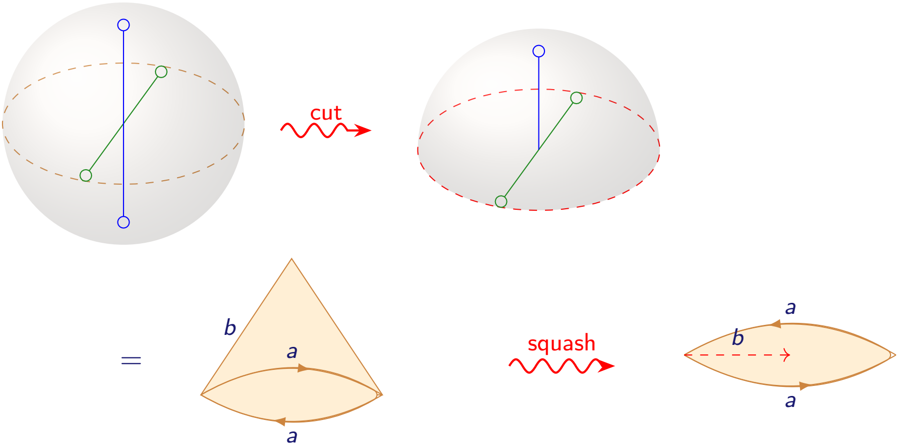

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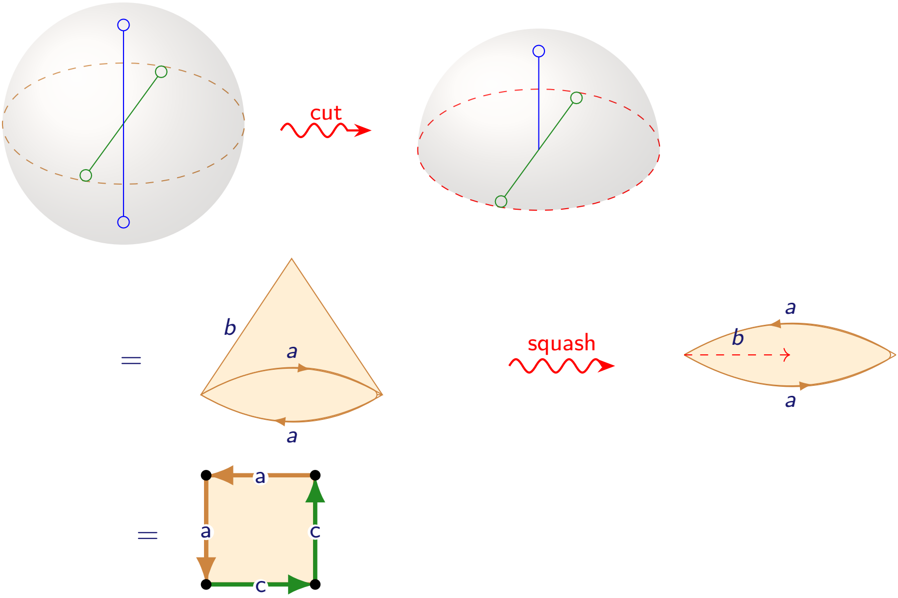
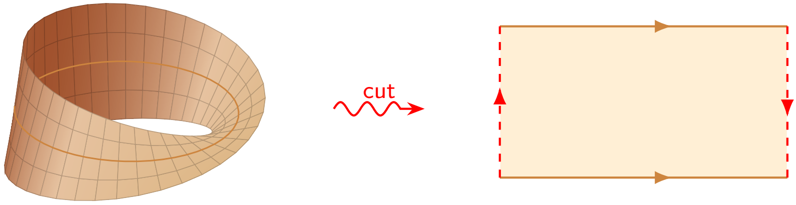


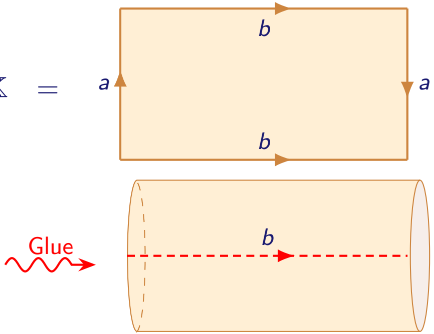
Identification space for a Möbius strip



Identification space for a Klein bottle

The Klein bottle is defined to be the identification space

$\mathbb{K} \quad =$



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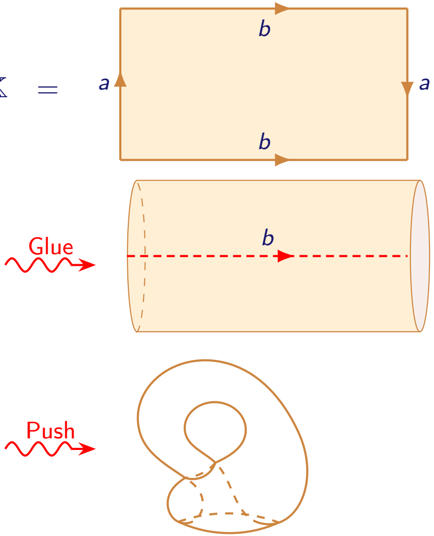

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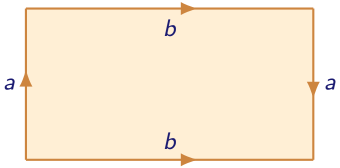
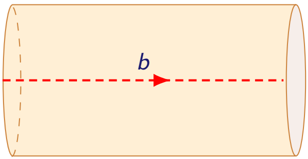
$\mathbb{K} \quad =$ 

Glue



Push

It is not clear how we to do the last step in $\mathbb{R}^3$ and, in fact, we can't!

We have seen that all of our "standard surfaces" can be viewed as identification spaces using rectangles

Polygons in $\mathbb{R}^n$

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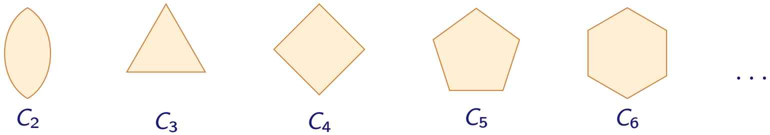

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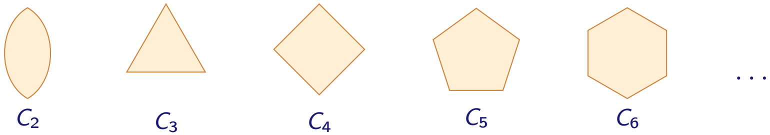

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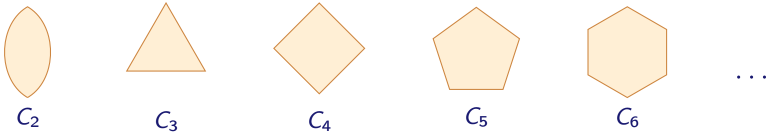

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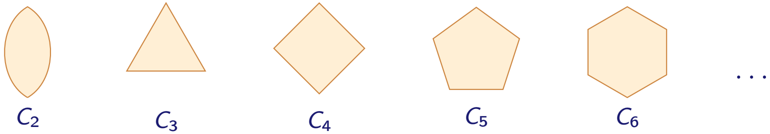

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- The graph C_2 has only one edge. When working with surfaces we think of C_2 as having two edges so that its image in $\mathbb{R}^2$ is a 2-gon

Surfaces and polygonal decompositions

Definition

A surface S is an identification space in $\mathbb{R}^n$ that is obtained by gluing together polygons along their edges in such a way that at most two edges meet along any edge

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- We sometimes write $S = (V, E, F)$, where V is the vertex set, edge set E, and face set F

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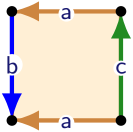
- When doing surgery always double check that you do not accidentally change the orientation of an edge

Examples of polygonal decompositions

We have already seen that:

- Annulus $\mathbb{A} \cong$ 

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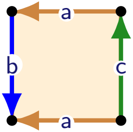
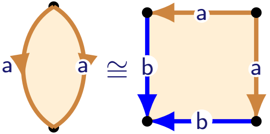
- Annulus $\mathbb{A} \cong$ 

- Sphere

$S^2 \cong$ 

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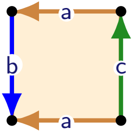

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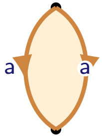

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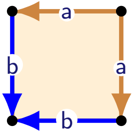

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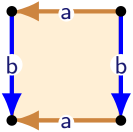

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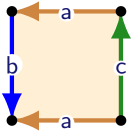
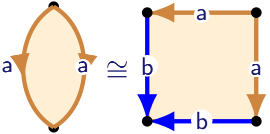
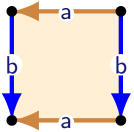
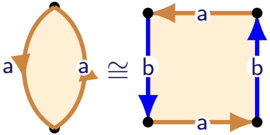
- Annulus $\mathbb{A} \cong$ 

- Sphere $S^2 \cong$ 

- Torus $\mathbb{T} \cong$ 

- Projective plane

$\mathbb{P}^2 \cong$ 

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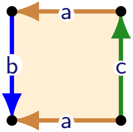
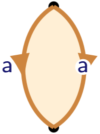
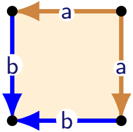
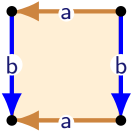
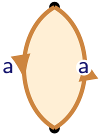
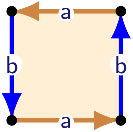
- Annulus $\mathbb{A} \cong$ 

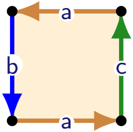
- Sphere $S^2 \cong$ 

$\cong$ 

- Torus $\mathbb{T} \cong$ 

- Projective plane $\mathbb{P}^2 \cong$ 

$\cong$ 

- Möbius strip $\mathbb{M} \cong$ 

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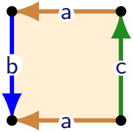
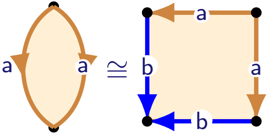
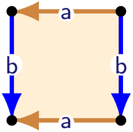
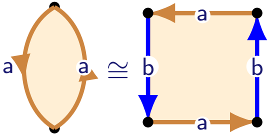
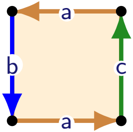
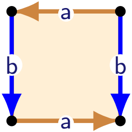
- Annulus $\mathbb{A} \cong$ 

- Sphere $S^2 \cong$ 

- Torus $\mathbb{T} \cong$ 

- Projective plane $\mathbb{P}^2 \cong$ 

- Möbius strip $\mathbb{M} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

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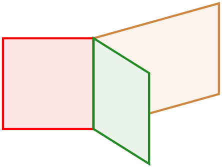

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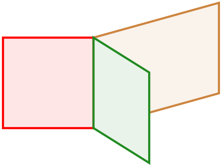

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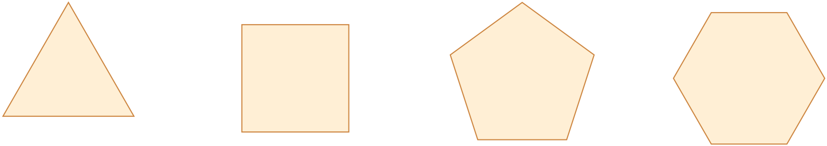

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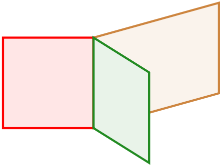

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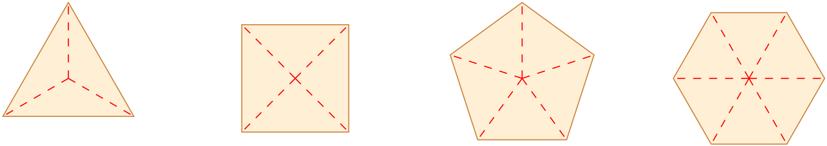

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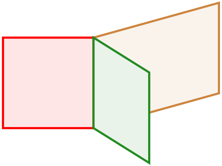

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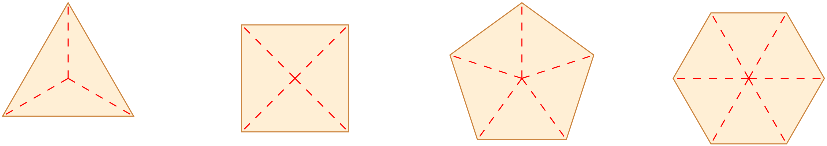

$\Longrightarrow$ Iterating this process, shows that any surface has infinitely many different polygonal decompositions!

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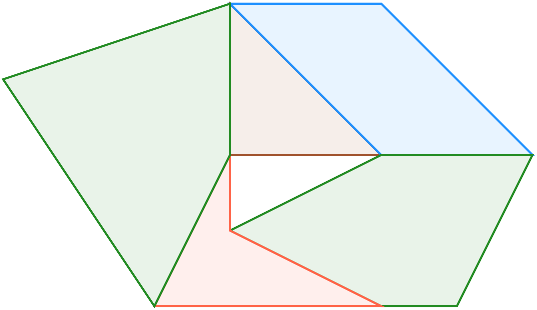
- Every connected surface has a polygonal decomposition with one polygon — with identified edges
  (A polygonal surface is connected if the underlying graph is connected)



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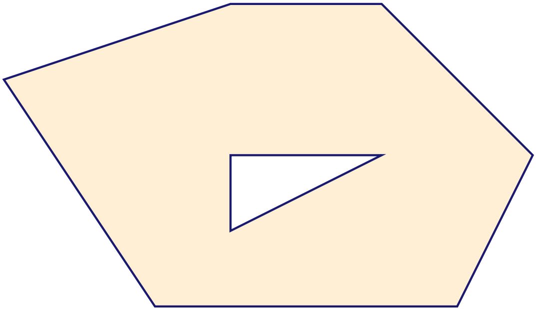

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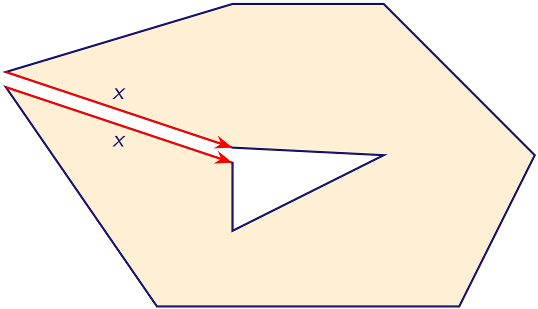

Important facts about polygonal decompositions...

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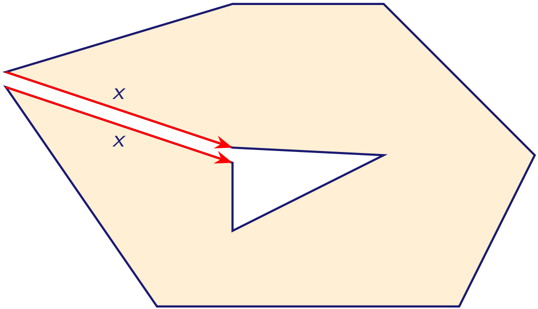

- We have to check that what we are doing does not depend on the choice of polygonal decomposition

Surgery: cutting and gluing

Surgery is our main tool for working with surfaces: it allows us to change a polygonal decomposition by cutting and gluing

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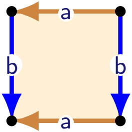
$\mathbb{T} \cong$ 

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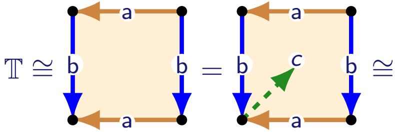

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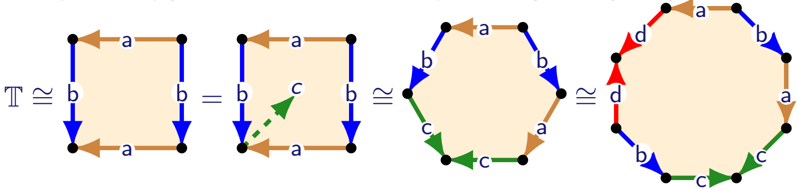

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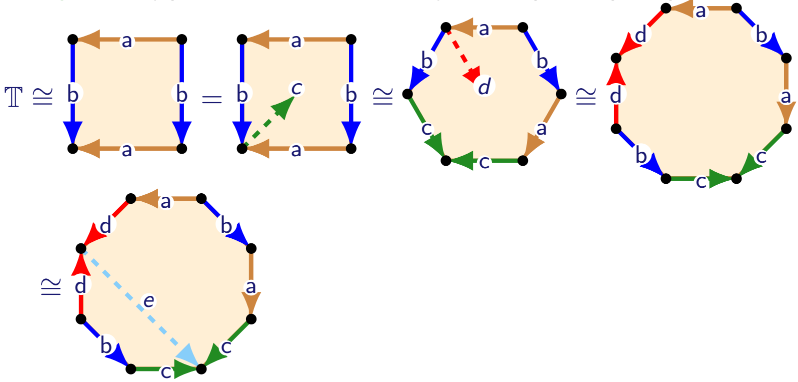

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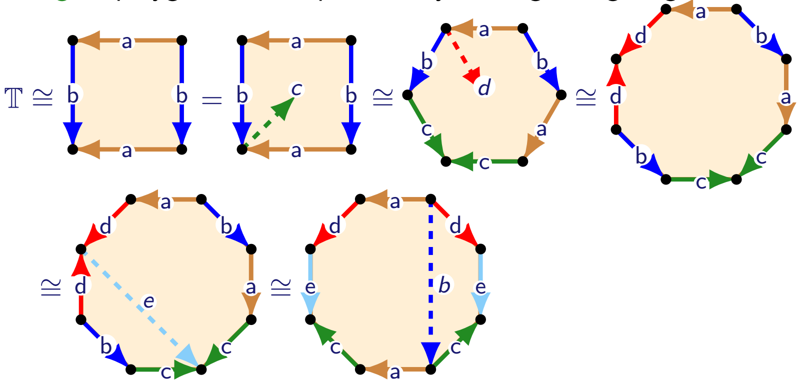

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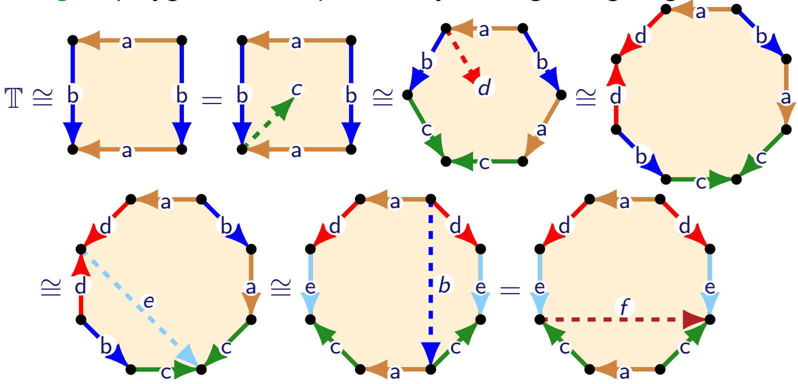

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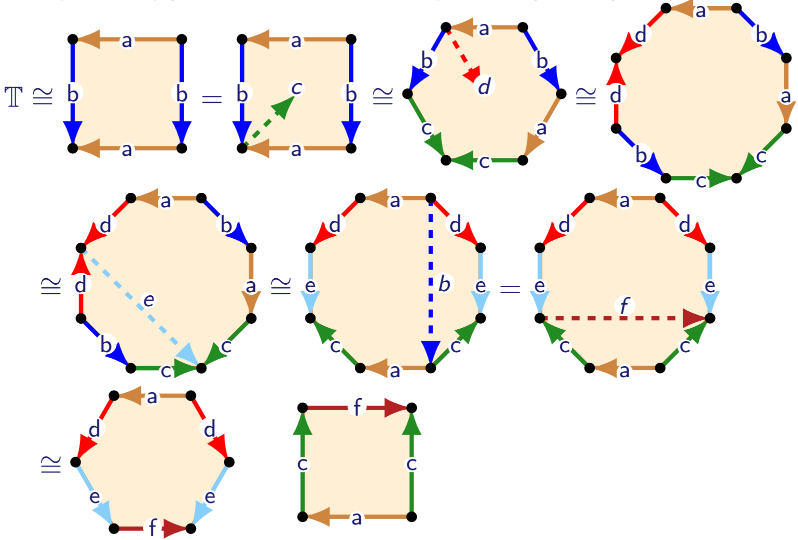

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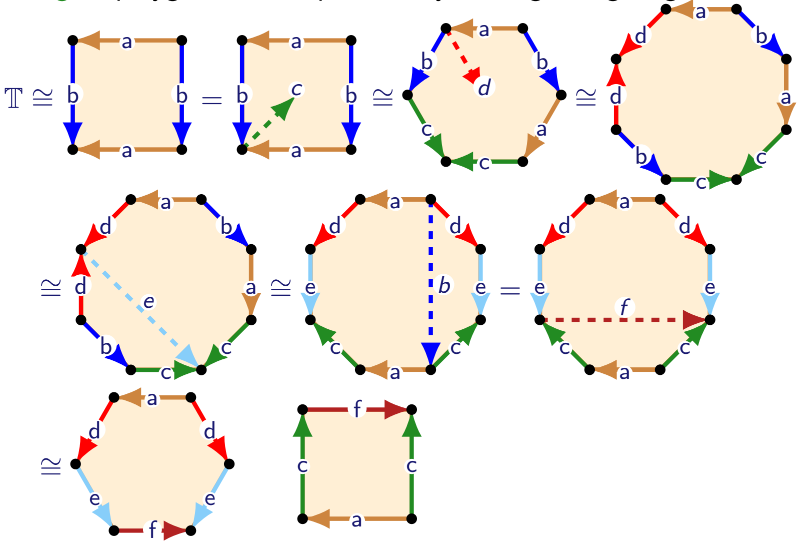
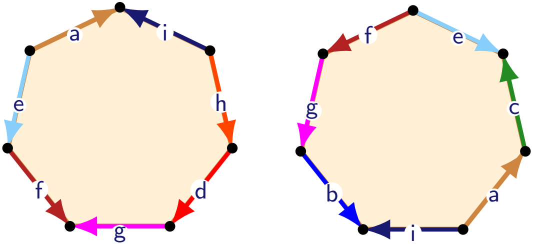


We want an easy way to identify surfaces from polygonal decompositions

Exercise Can we describe the following surface?



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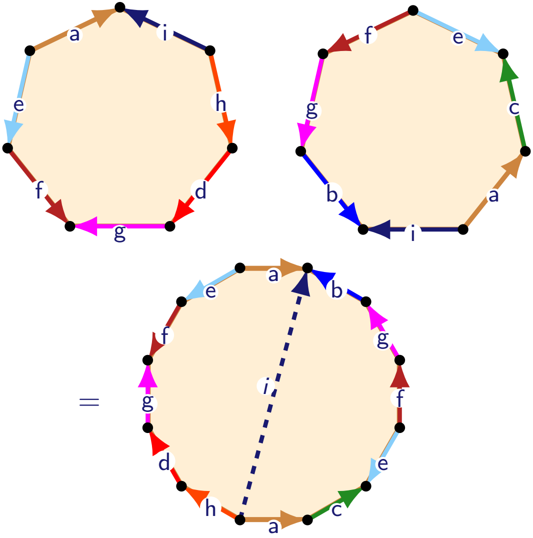

Example surface

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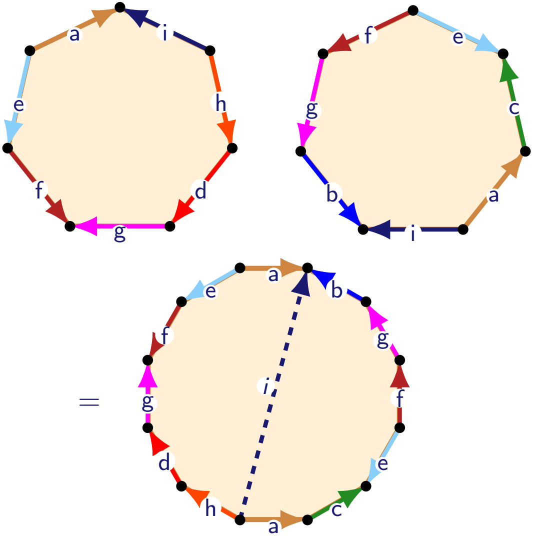

Answer Not yet! First we need more language and technology.

Free and paired edges and the boundary

Let S be a surface with a polygonal decomposition

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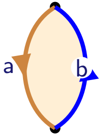
$\mathbb{D}^2 =$  $\cong$ 

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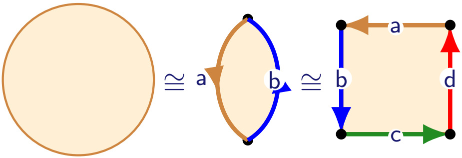
$\mathbb{D}^2 =$ 

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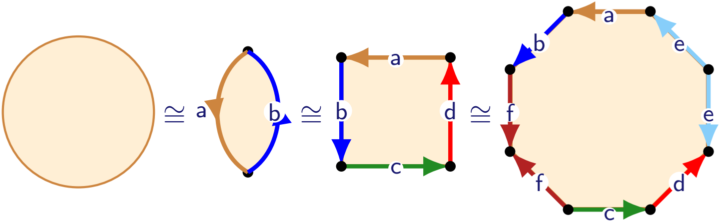
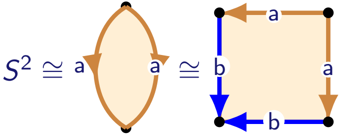
$\mathbb{D}^2 =$ 

Example boundary circles...

- Sphere



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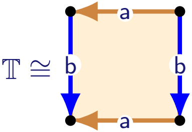
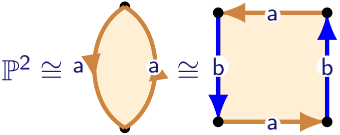
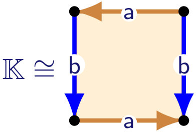


- Projective plane



- Klein bottle



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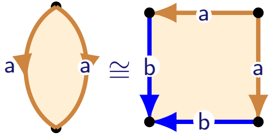
$S^2 \cong$ 

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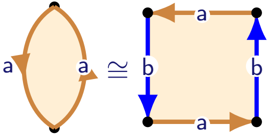
$\mathbb{P}^2 \cong$ 

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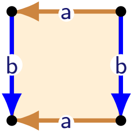
$\mathbb{T} \cong$ 

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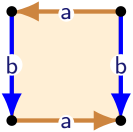
$\mathbb{K} \cong$ 

All edges paired $\implies$ no boundary

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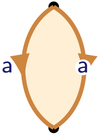
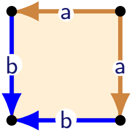
$S^2 \cong$ 

$\cong$ 

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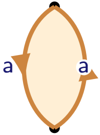
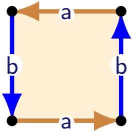
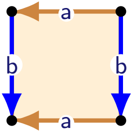
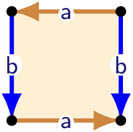
$\mathbb{P}^2 \cong$ 

$\cong$ 

- Torus $\mathbb{T} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

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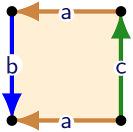
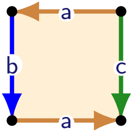
- Annulus $\mathbb{A} \cong$ 

- Möbius $\mathbb{M} \cong$ 

Example boundary circles...

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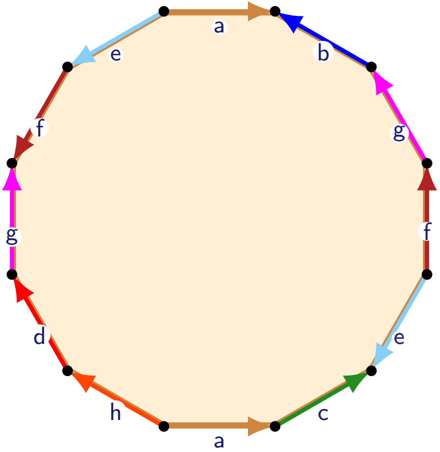

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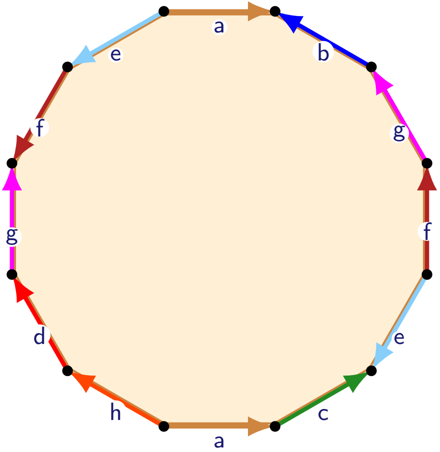

Free edges: b, c, d, h

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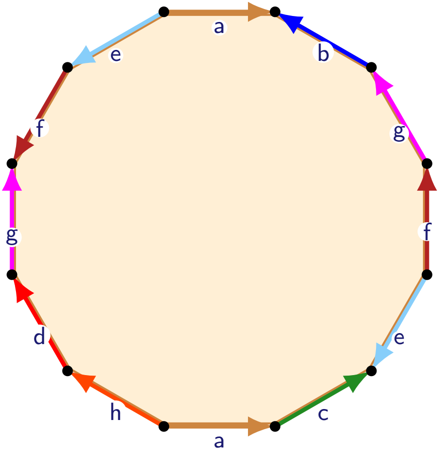

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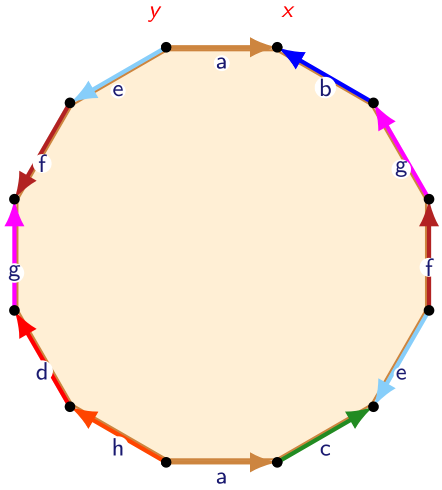

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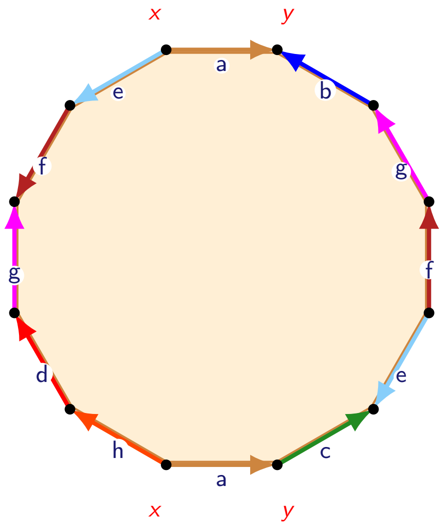

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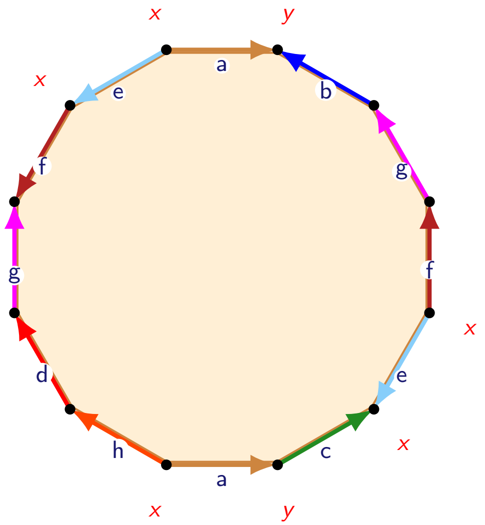

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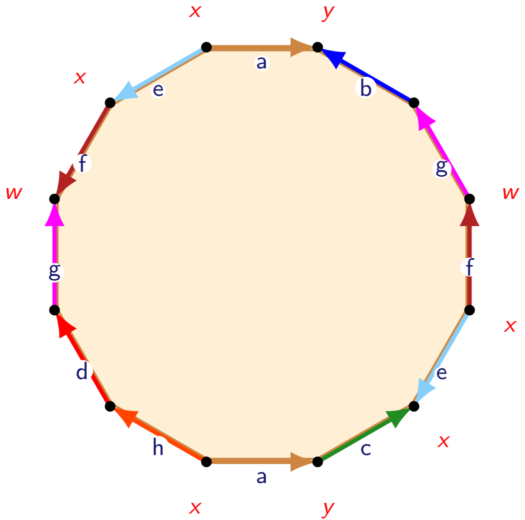

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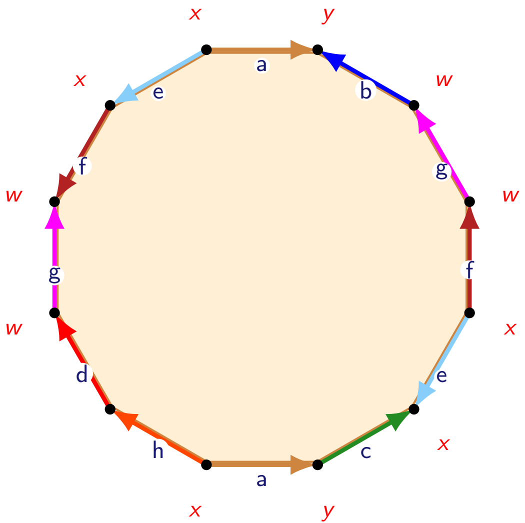

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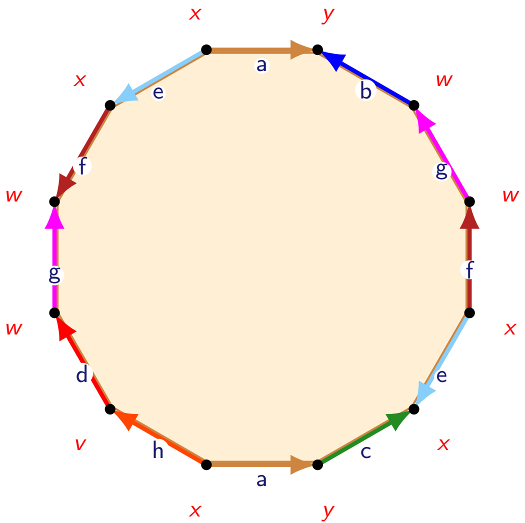

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The Euler characteristic of a surface

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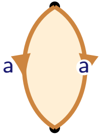
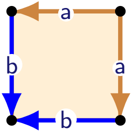
- The definition of $\chi(S)$ appears to depend on the choice of polygonal decomposition (V, E, F) of S. In fact, we will soon see that $\chi(S)$ is independent of this choice

Euler characteristic of basic surfaces.

- Sphere

$S^2 \cong$ 

$\cong$ 

$, \chi = 2$

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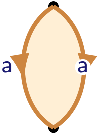
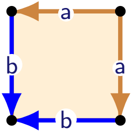
$S^2 \cong$ 

$\cong$ 

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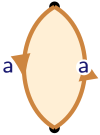
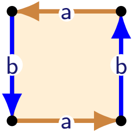
$\mathbb{P}^2 \cong$ 

$\cong$ 

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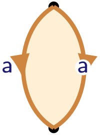
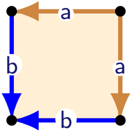
$S^2 \cong$ 

$\cong$ 

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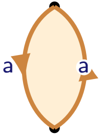
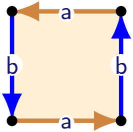
$\mathbb{P}^2 \cong$ 

$\cong$ 

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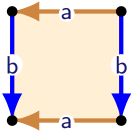
- Torus $\mathbb{T} \cong$ 

$, \chi = 0$

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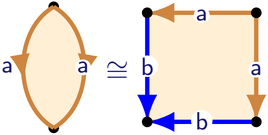
$S^2 \cong$ 

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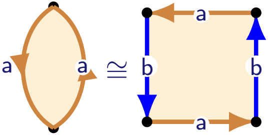
$\mathbb{P}^2 \cong$ 

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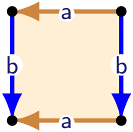
- Torus $\mathbb{T} \cong$ 

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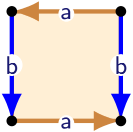
$\mathbb{K} \cong$ 

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Euler characteristic of basic surfaces.

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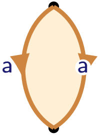
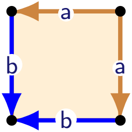
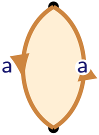
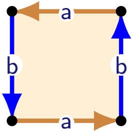
$S^2 \cong$ 

$\cong$ 

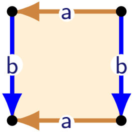
$, \chi = 2$

- Projective plane

$\mathbb{P}^2 \cong$ 

$\cong$ 

$, \chi = 1$

- Torus $\mathbb{T} \cong$ 

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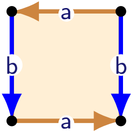
- Klein bottle

$\mathbb{K} \cong$ 

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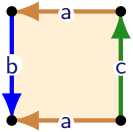
- Annulus $\mathbb{A} \cong$ 

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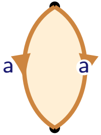
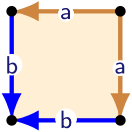
$S^2 \cong$ 

$\cong$ 

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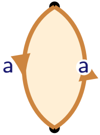
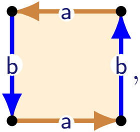
$\mathbb{P}^2 \cong$ 

$\cong$ 

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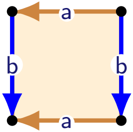
- Torus $\mathbb{T} \cong$ 

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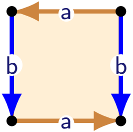
$\mathbb{K} \cong$ 

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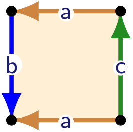
- Annulus $\mathbb{A} \cong$ 

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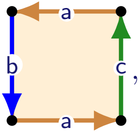
- Möbius $\mathbb{M} \cong$ 

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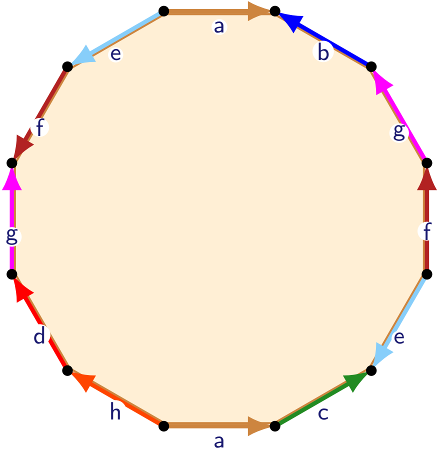
$S =$ 

Euler characteristic example

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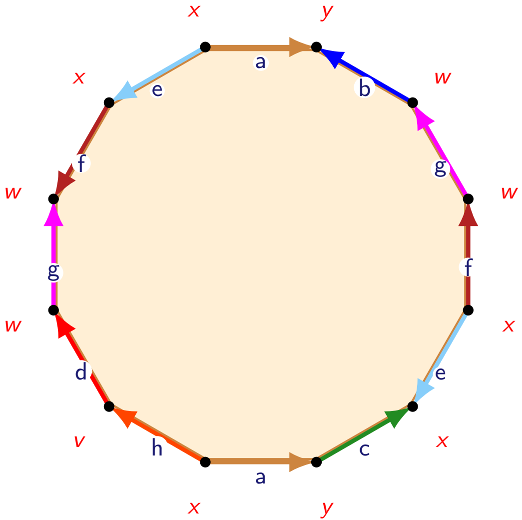

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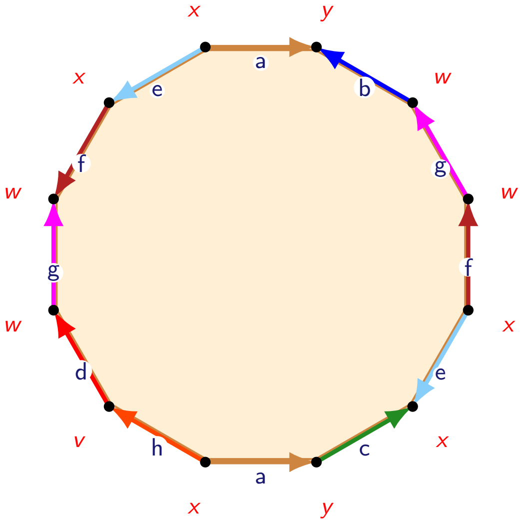

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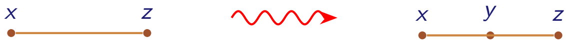

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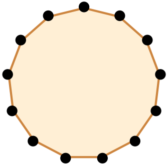

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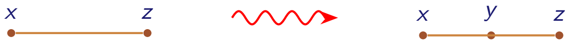

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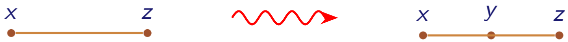

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Let $\dot{S}$ be a subdivision of S. Then $\chi(S) = \chi(\dot{S})$

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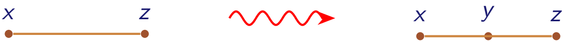

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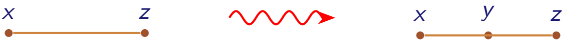

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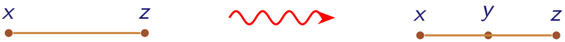

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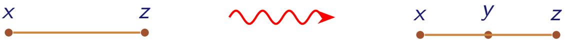

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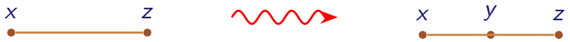

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Let S be a surface and suppose that S has polygonal decomposition $P_1 = (V_1, E_1, F_1)$ and $P_2 = (V_2, E_2, F_2)$. Then S has a polygonal decomposition (V, E, F) that is a common subdivision of P_1 and P_2

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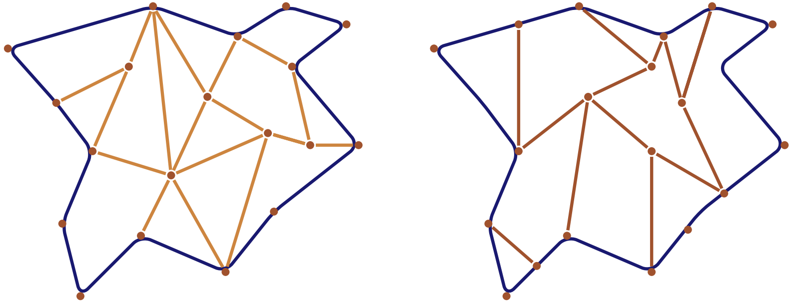

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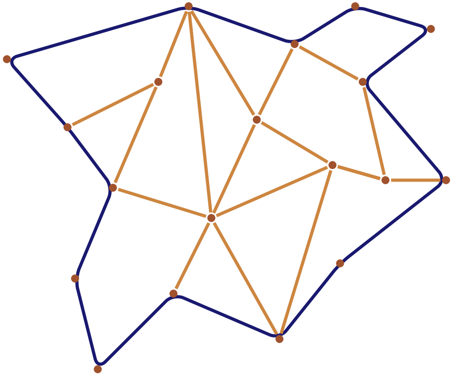

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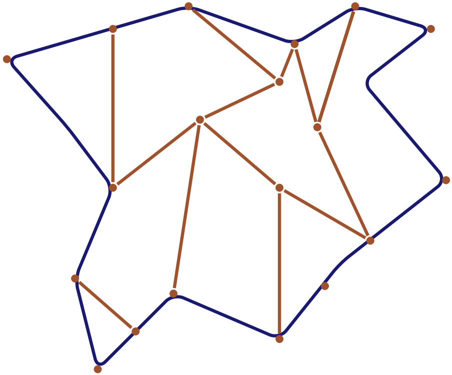

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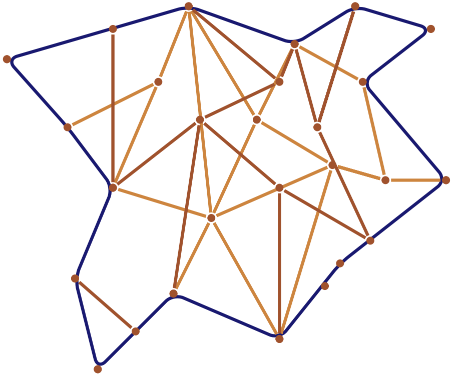

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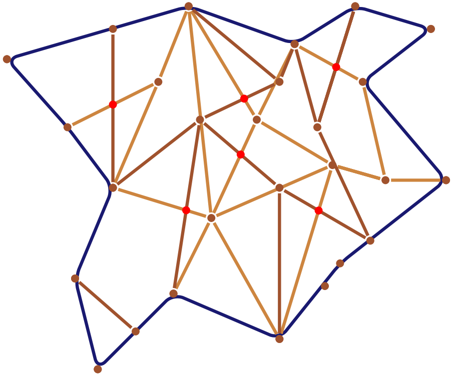
Proof Merge the two subdivisions — adding extra vertices as necessary



Two invariants

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Suppose that S *and* T *are homeomorphic surfaces that have polygonal decompositions. Then* $\chi(S) = \chi(T)$ *and* S *and* T *have the same number of boundary circles.*

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Exercise Using what we know so far, deduce that the surfaces

$$S^2,\ \mathbb{A},\ \mathbb{D}^2,\ \mathbb{K},\ \mathbb{M},\ \mathbb{P}^2$$

are pairwise non-homeomorphic (see Tutorial 9)

Topology – week 9

Math3061

Daniel Tubbenhauer, University of Sydney

Classifying surfaces using invariants

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- The same Euler characteristic
- The same number of boundary circles

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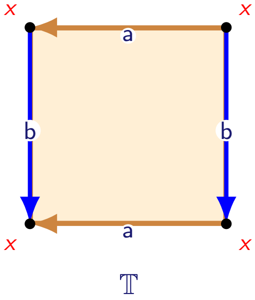

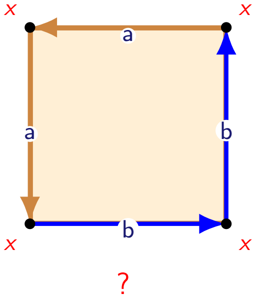

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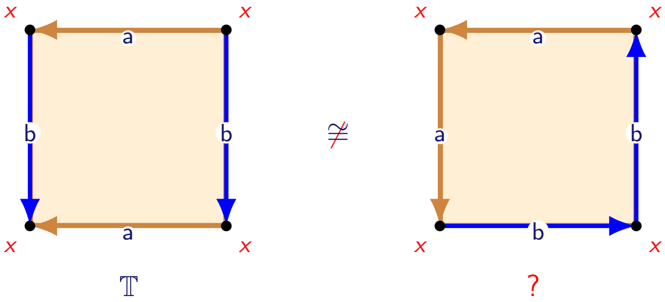

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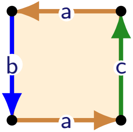
- Clearly, $\mathbb{M}$ is non-orientable, but there are no other "easy" examples

$\mathbb{M} =$ 

Orientability

Definition

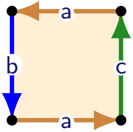
A surface S is non-orientable if it contains a Möbius strip $\mathbb{M}$

If S does not contain a Möbius strip it is orientable

Remarks

- Even though this looks hard to apply we will see it isn't
- Clearly, $\mathbb{M}$ is non-orientable, but there are no other "easy" examples

$\mathbb{M} =$



- Are S^2, $\mathbb{A}$, $\mathbb{D}^2$, $\mathbb{T}$, $\mathbb{P}^2$, $\mathbb{K}$, … orientable or non-orientable?

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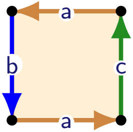
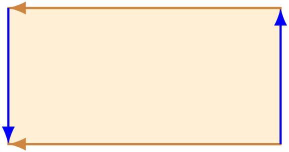
$\mathbb{M} =$ 

- Are S^2, $\mathbb{A}$, $\mathbb{D}^2$, $\mathbb{T}$, $\mathbb{P}^2$, $\mathbb{K}$, … orientable or non-orientable?
- Can a surface be orientable and non-orientable for different polygonal decompositions? (That would be bad!)

The Klein bottle $\mathbb{K}$

$\mathbb{K} \quad =$ 

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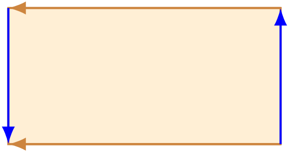

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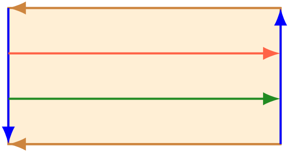

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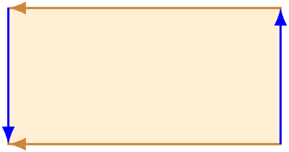
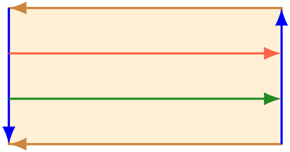
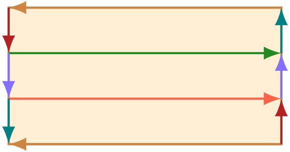
$\mathbb{K}$ =  = 

= 

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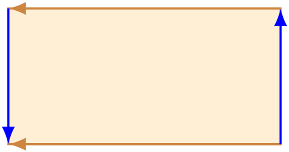

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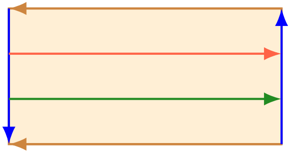

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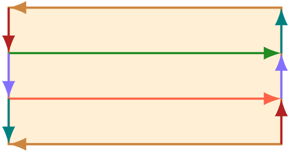

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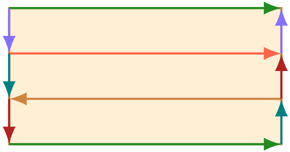

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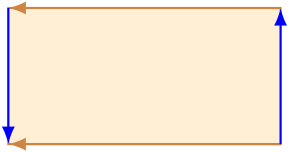
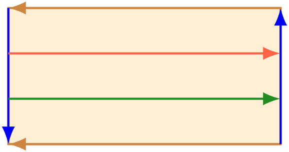
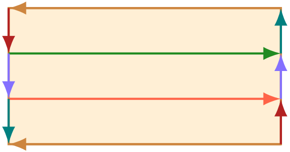
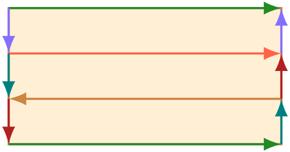
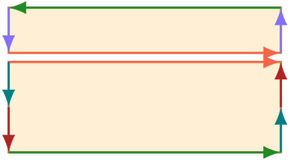
$\mathbb{K}$ $=$  $=$ 

$=$  $=$ 

$=$ 

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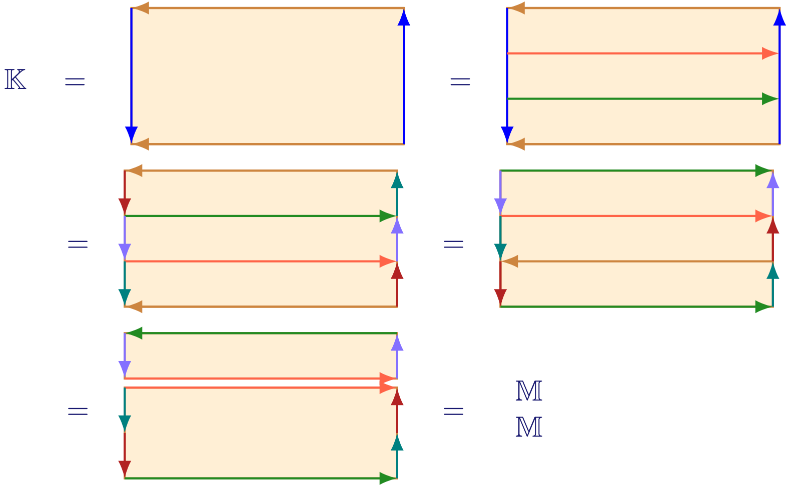

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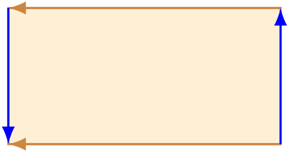

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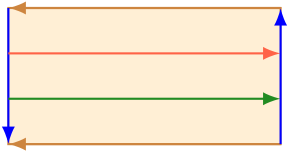

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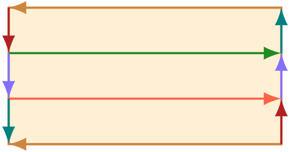

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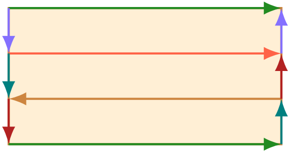

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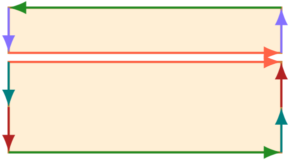
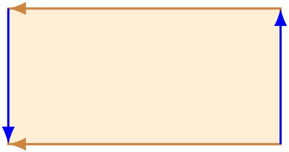
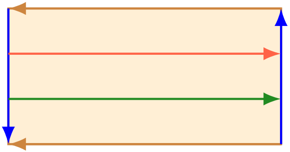
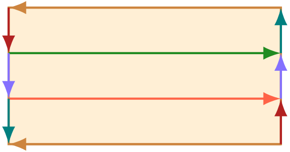
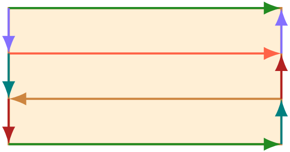


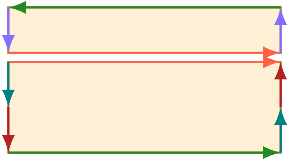
$=$ $\mathbb{M}$
$\mathbb{M}$

$\Longrightarrow$ The Klein bottle $\mathbb{K}$ is non-orientable!

The Klein bottle $\mathbb{K}$

$\mathbb{K}$ $=$  $=$ 

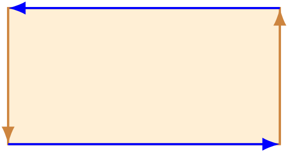
$=$  $=$ 

$=$  $=$ $\begin{matrix}\mathbb{M}\\ \mathbb{M}\end{matrix}$

$\Longrightarrow$ The Klein bottle $\mathbb{K}$ is non-orientable!

... although it might be more accurate to say that the Klein bottle is a Möbius strip without boundary

The projective plane $\mathbb{P}^2$

$\mathbb{P}^2 \quad =$ 

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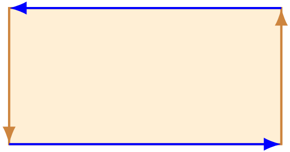
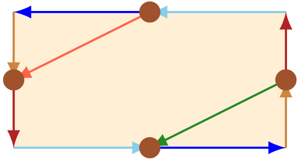
$\mathbb{P}^2 \quad =$  $=$ 

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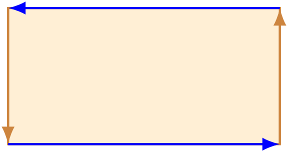
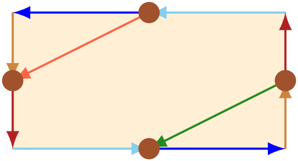
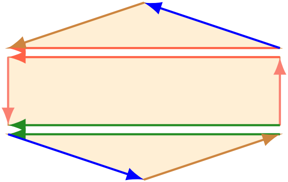
$\mathbb{P}^2 \quad =$  $=$ 

$=$ 

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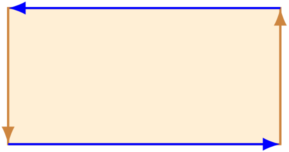
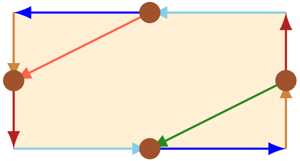
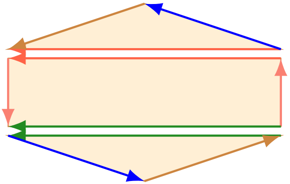
$\mathbb{P}^2$ $=$  $=$ 

$=$  $=$

$\mathbb{D}^2$
$\mathbb{M}$
$\mathbb{D}^2$

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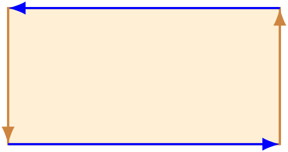
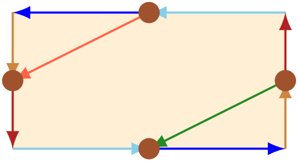
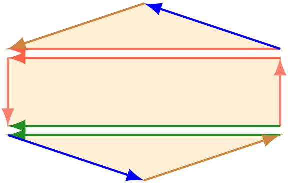
$\mathbb{P}^2 \quad =$  $=$ 

$=$  $=$ $\mathbb{D}^2$ $\mathbb{M}$ $\mathbb{D}^2$

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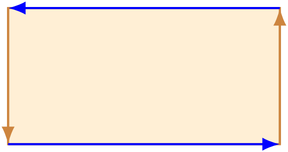
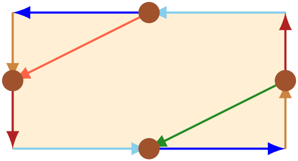
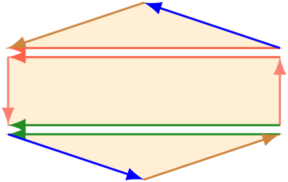
$\mathbb{P}^2 \quad =$  $=$ 

$=$  $=$ $\begin{matrix}\mathbb{D}^2 \\ \mathbb{M} \\ \mathbb{D}^2\end{matrix}$

$\Longrightarrow$ The projective plane $\mathbb{P}^2$ is non-orientable

... or maybe $\mathbb{P}^2$ and not $\mathbb{K}$
is a Möbius strip without boundary?

What does orientability mean?

Orientability is a generalisation of direction to higher dimensions

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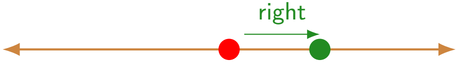
- One dimension $\mathbb{R}$



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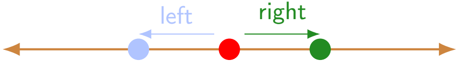

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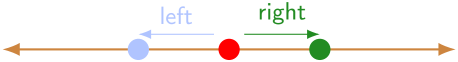
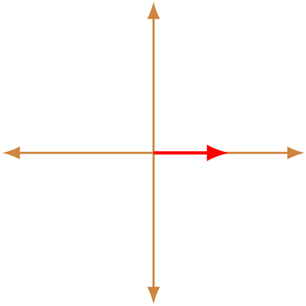


- Two dimensions $\mathbb{R}^2$



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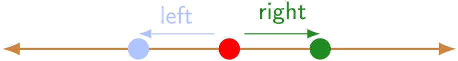

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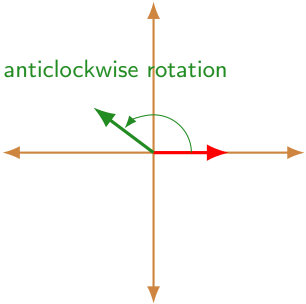

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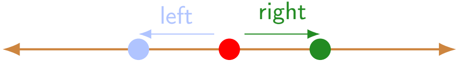

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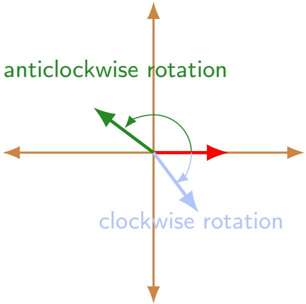

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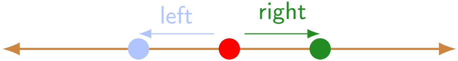

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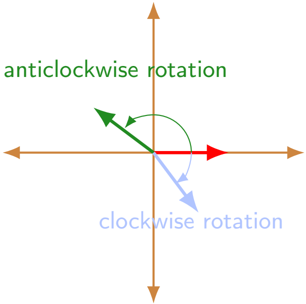

- Three dimensions $\mathbb{R}^3$???

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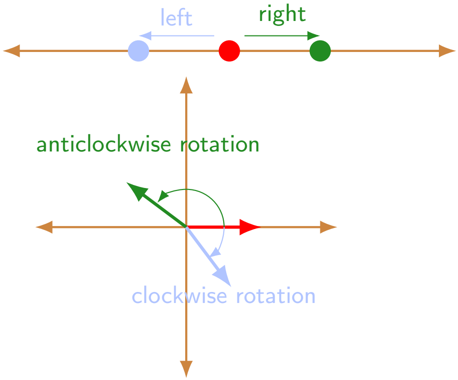
- One dimension $\mathbb{R}$
- Two dimensions $\mathbb{R}^2$



- Three dimensions $\mathbb{R}^3$???
- Higher dimensions $\mathbb{R}^n$, for $n \geq 3$???

Direction in higher dimensions

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We can compare B to the standard basis $E = \{e_1, e_2, \ldots, e_n\}$ of column vectors by computing the sign of the determinant

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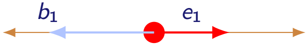

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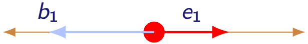
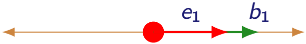


$\mathrm{sign}(B) = +1$



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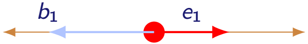
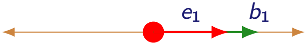


$\operatorname{sign}(B) = +1$



- Two dimensions $\mathbb{R}^2$

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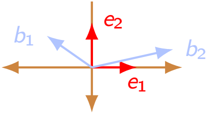

Direction in higher dimensions

To generalise direction, choose an ordered basis $B = \{b_1, b_2, \dots, b_n\}$ of $\mathbb{R}^n$

The order of the basis elements is the key to understanding direction

We can compare B to the standard basis $E = \{e_1, e_2, \dots, e_n\}$ of column vectors by computing the sign of the determinant

$$\det(B) = \det\begin{pmatrix} \vdots & \vdots & & \vdots \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & & \vdots \end{pmatrix} \rightsquigarrow \operatorname{sign}(B) = \pm 1$$

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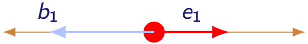

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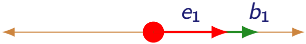

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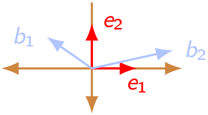

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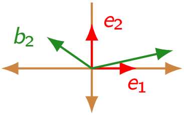

Direction on the Möbius strip

Pick a point $m \in \mathbb{M}$ on the Möbius strip and an ordered basis $B = \{b_1, b_2, b_3\}$ positioned at m with $b_3 = b_1 \times b_2$ pointing outwards

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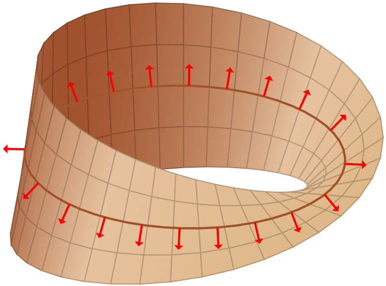

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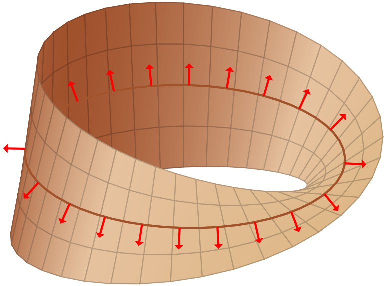

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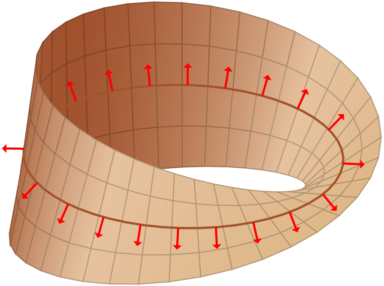

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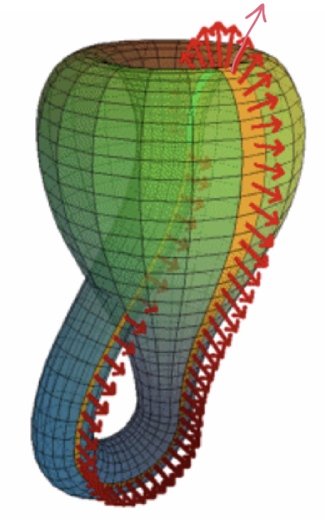
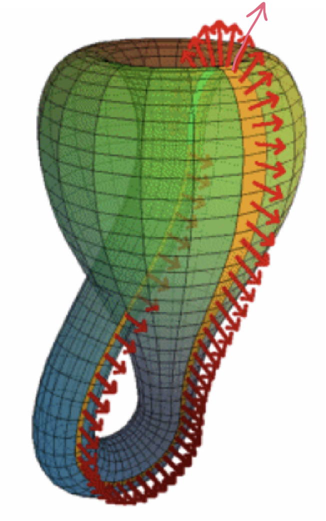
The vector b_3 is always normal to the surface of the Möbius strip. The direction of b_3 can change from pointing outside to inside because the Möbius strip is a surface with a boundary that only has one side

Direction on the Klein bottle $\mathbb{K}$



We can do the same experiment with the Klein bottle and we see the same phenomenon: the vector b_3 changes from pointing outside to pointing inside the surface



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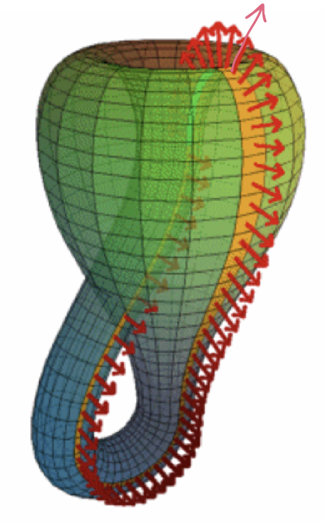

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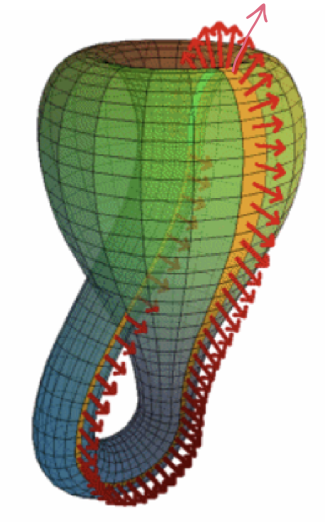

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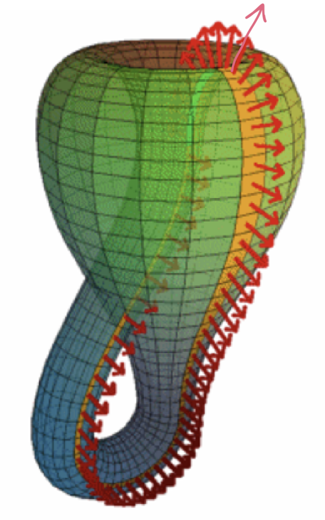

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Warning: this is a drawing of $\mathbb{K}$ in $\mathbb{R}^3$ but it is not the actual Klein bottle! Similarly, the pictures of the sphere S^2 in $\mathbb{R}^3$ are not really the sphere!

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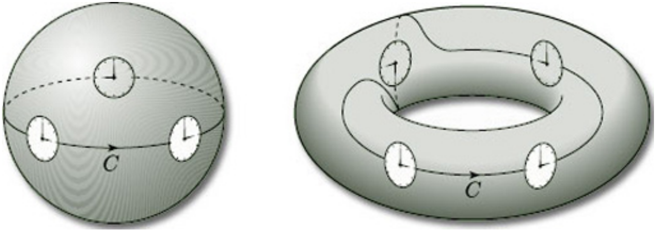

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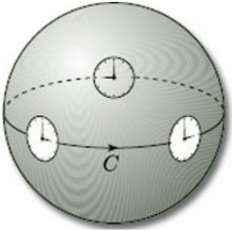

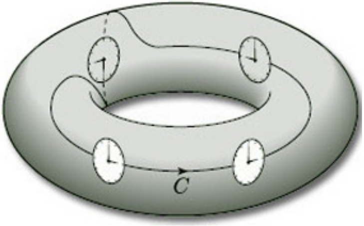

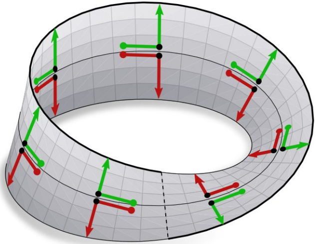

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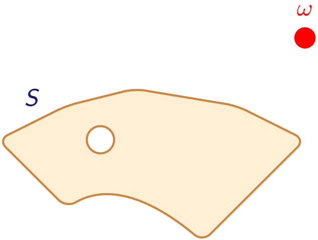

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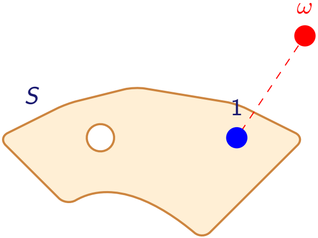

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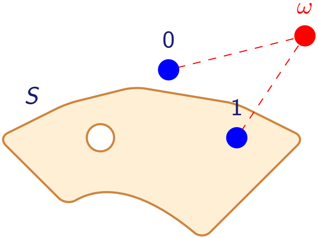

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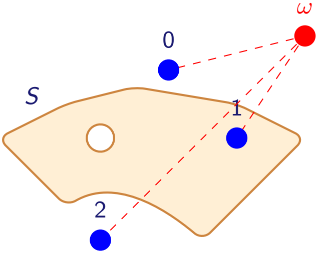

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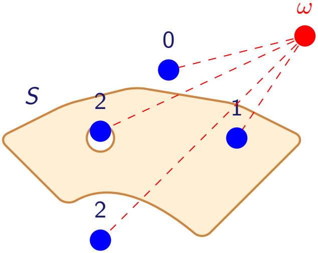

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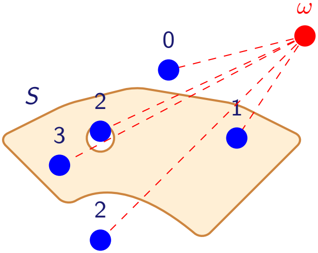

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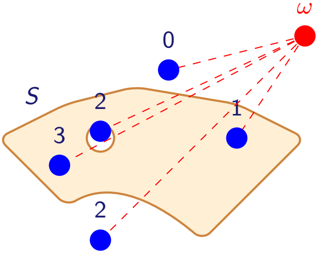

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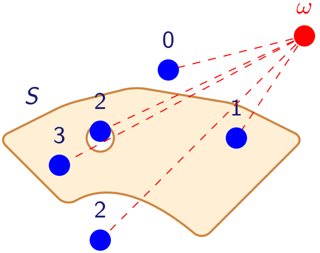

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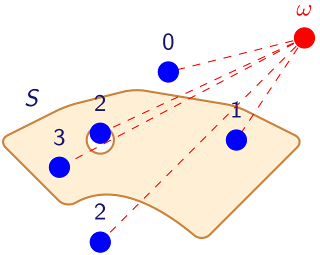

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Notice that since S is a closed surface it does not have boundary, so the "circle" in the picture, which contains a point x with $s(x) = 2$, should be interpreted as a tube through the surface

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Corollary

Let S be a non-orientable closed surface. Then S does not embed in $\mathbb{R}^3$.

You can't fill a liquid into the Klein bottle

Strictly speaking the liquid is neither in- nor outside

Jordan curve theorem

This argument used to prove theorem can be made rigorous for surfaces with finite polygonal decompositions but for "general surfaces" it is difficult to prove that $\mathbb{R}^3 = S \cup V_{\text{in}} \cup V_{\text{out}}$.

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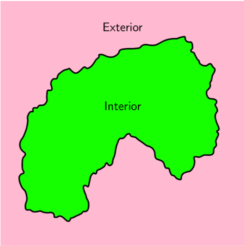

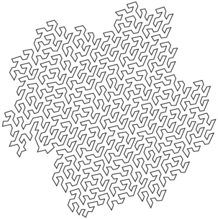

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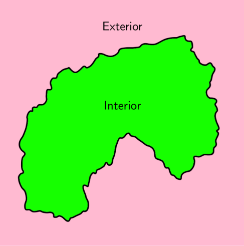
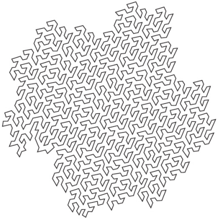
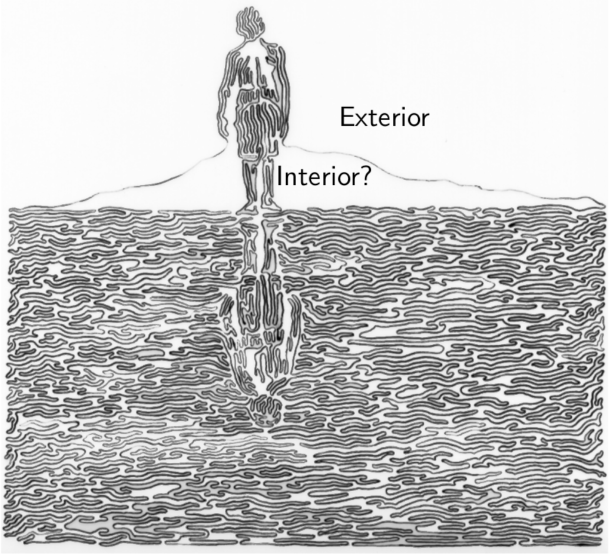




The left is easy, but can you tell for the right what is "in" or "out"?

Jordan curve theorem - 2

The main meat is that one needs to deal with “crazy” curves:



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We can embed $\mathbb{P}^2$ into $\mathbb{R}^4$ using the continuous map:

$$(x, y, z) \quad \mapsto \quad (xy, xz, yz, y^2 - z^2)$$

It is not hard to check that this is a well-defined injective function

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In contrast, every orientable surface embeds in $\mathbb{R}^3$

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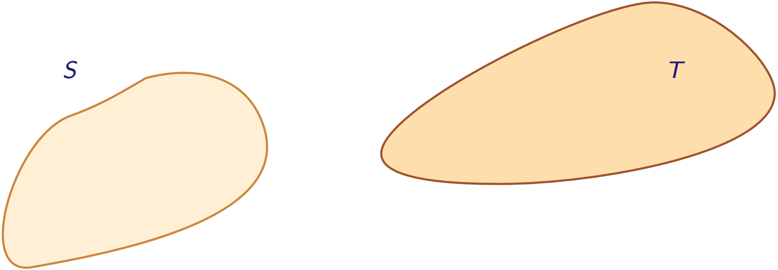

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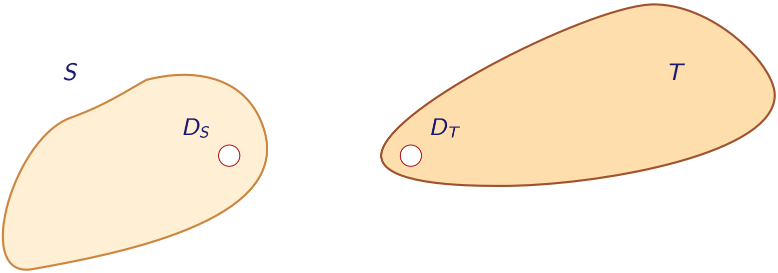

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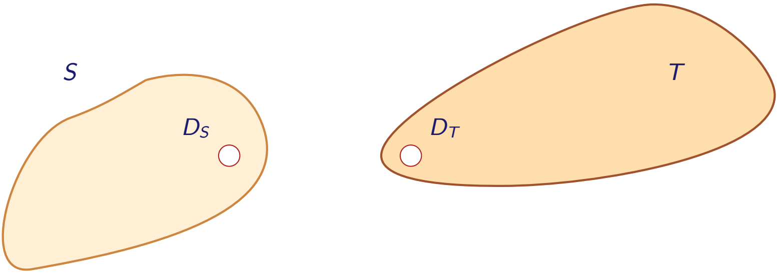

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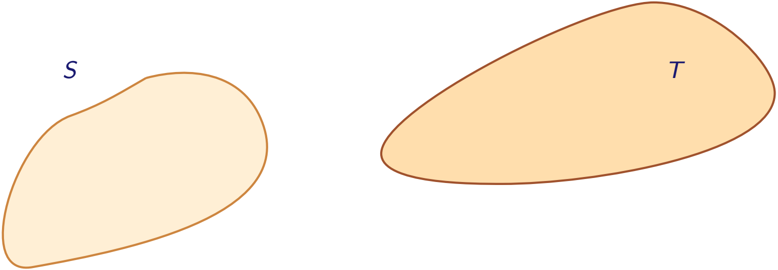

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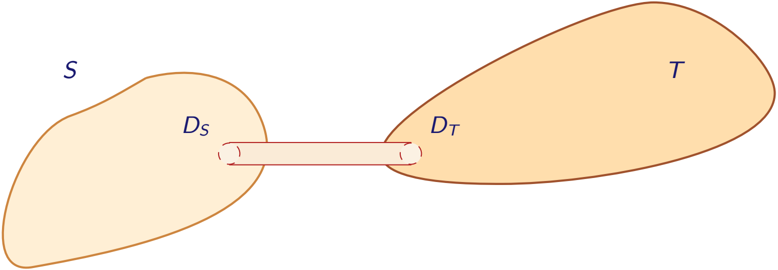

Identifying D_S and D_T is the same as connecting them with a cylinder

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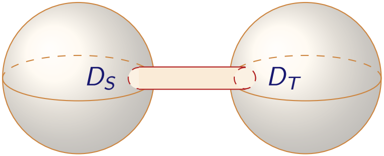

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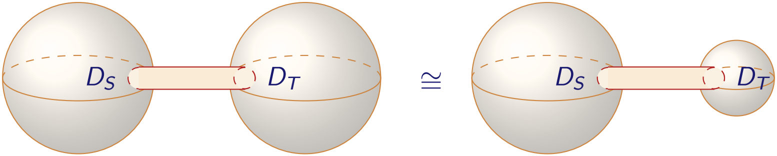

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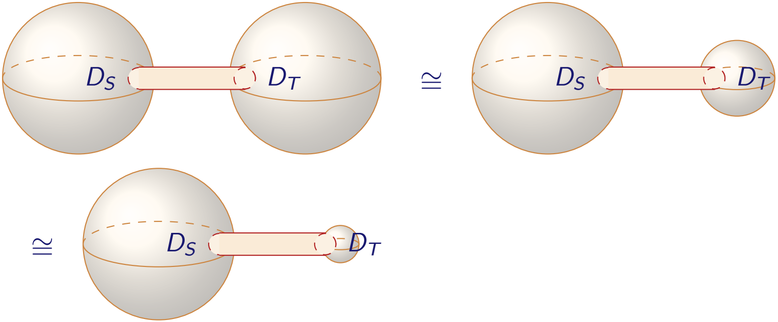

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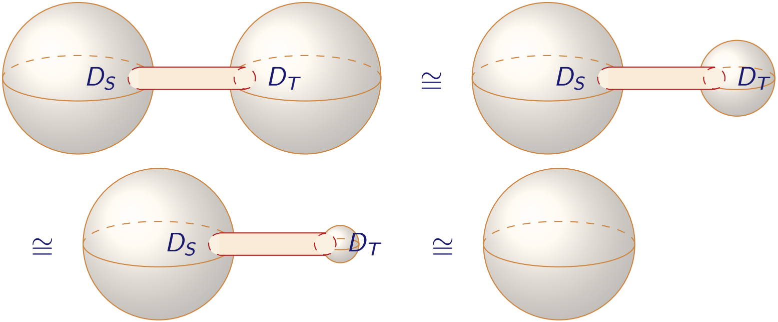

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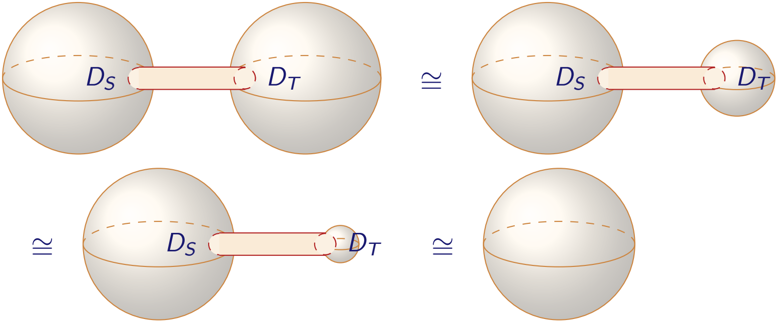

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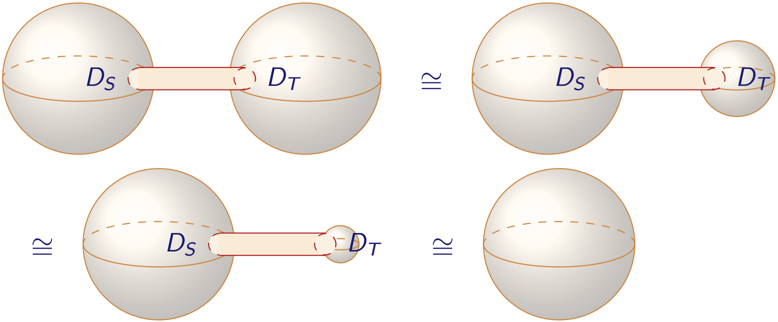

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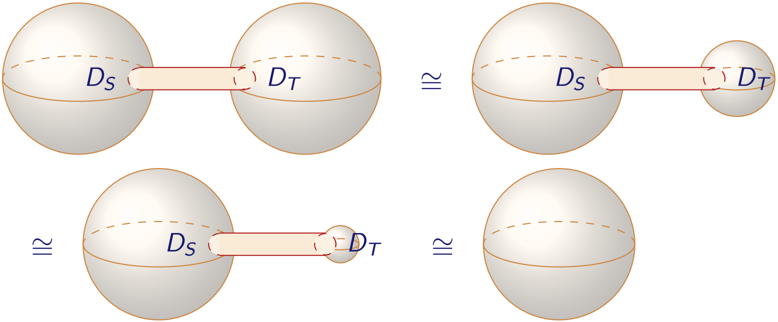

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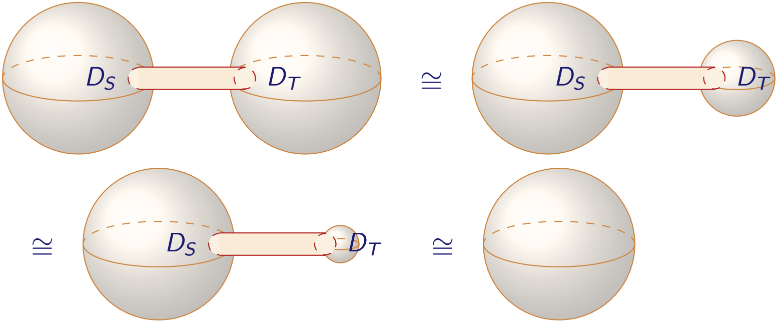

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So S^2 is the unit under the operation $\#$

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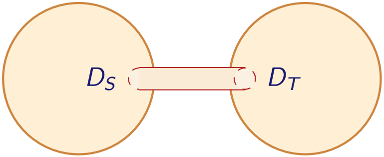

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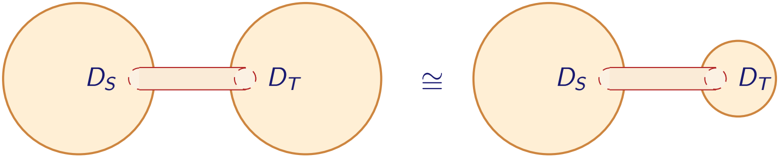

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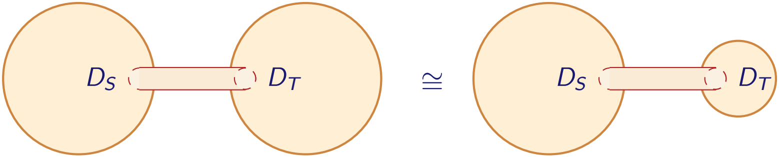

This is not the same as collapsing a sphere, which closes up the hole, because the disk has a boundary!

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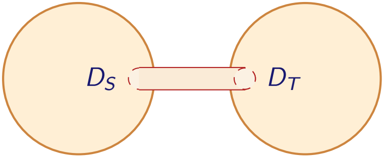

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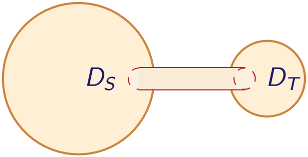

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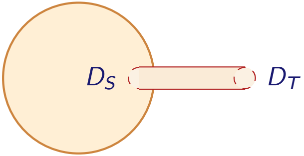

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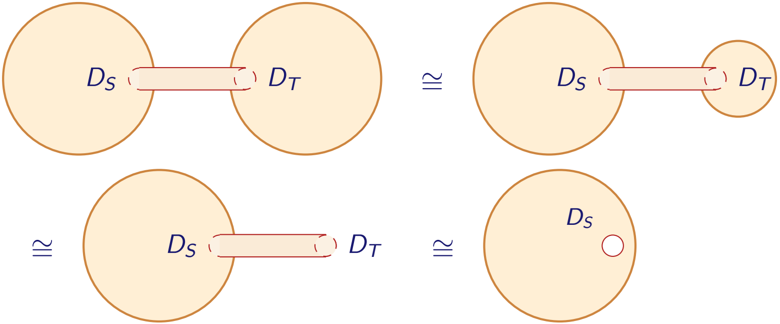

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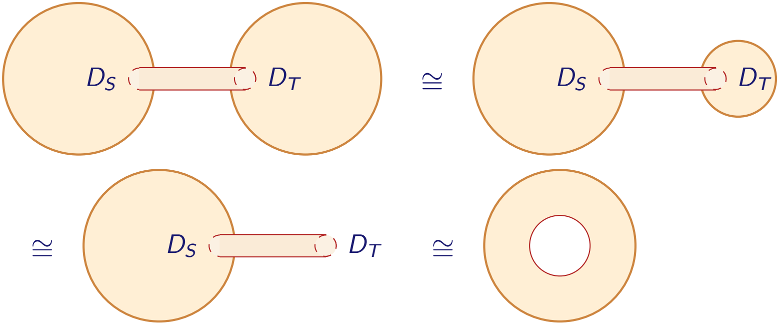

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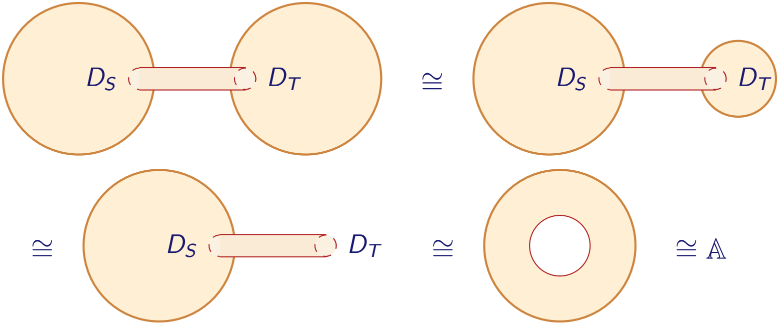

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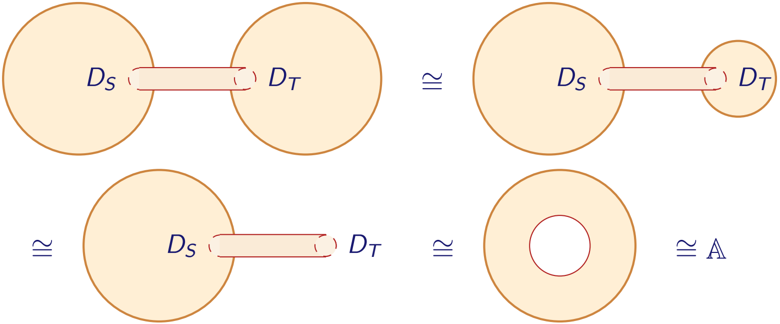

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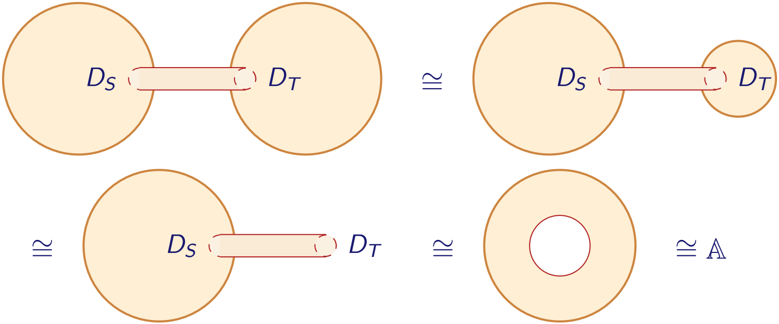

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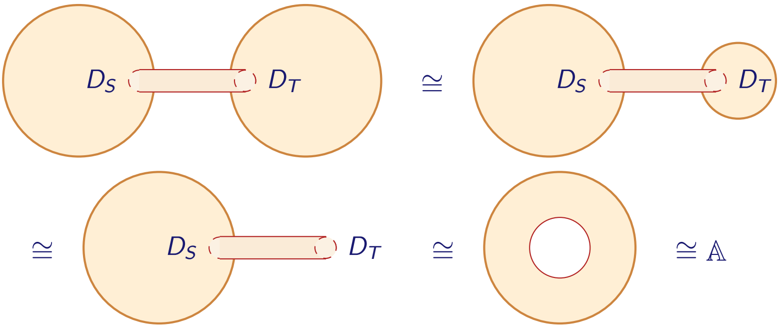

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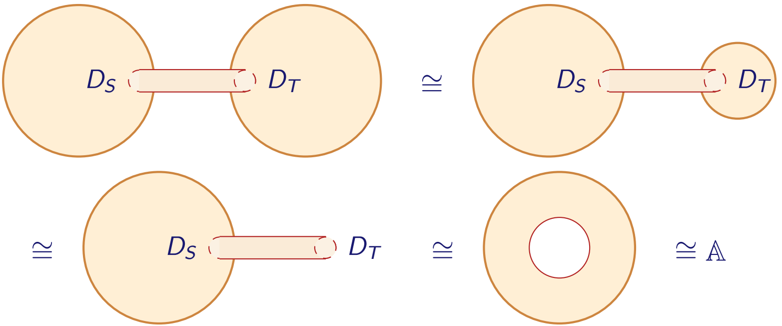

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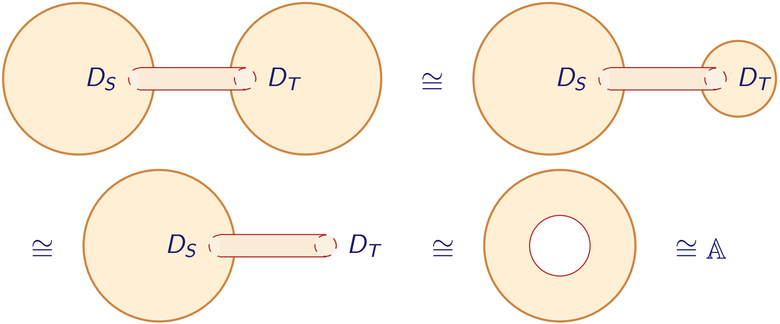

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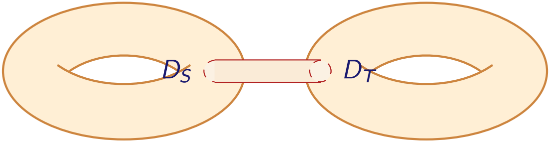

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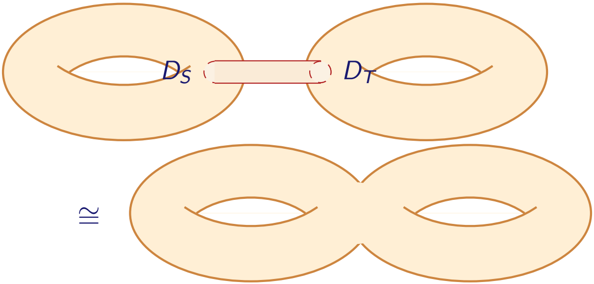

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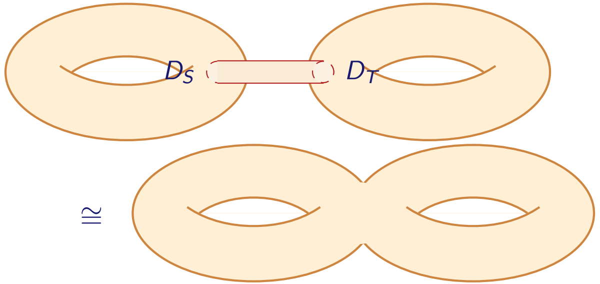

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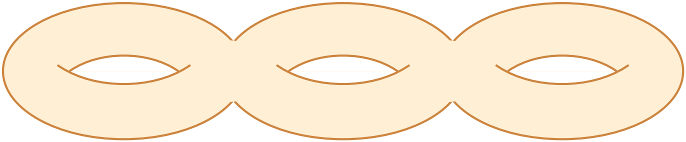

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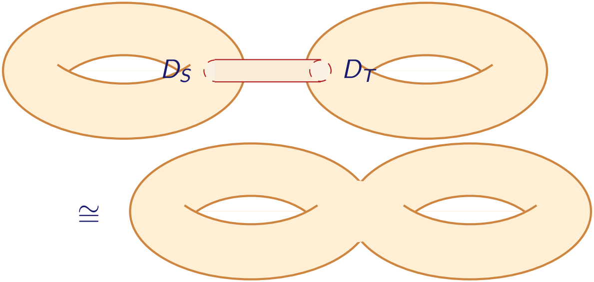

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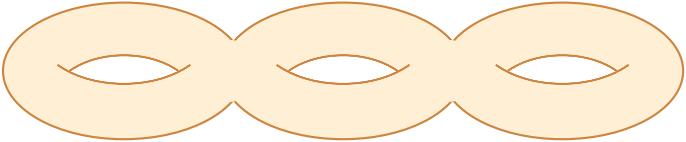
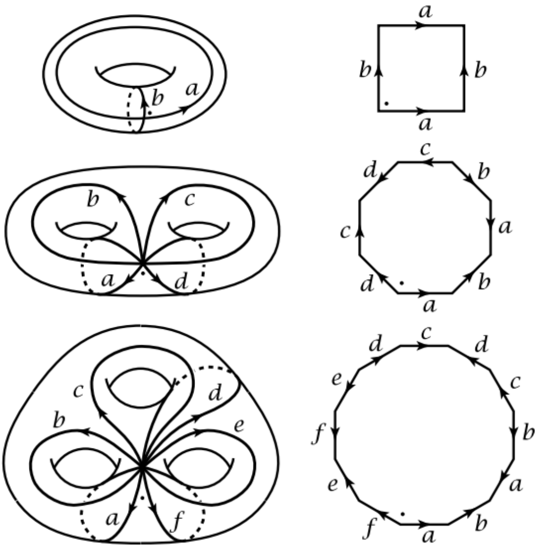


...and, more generally, t-tori $\#^t \mathbb{T}$

We already know t-tori



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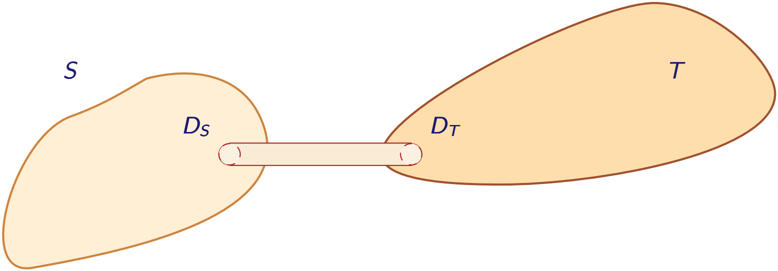

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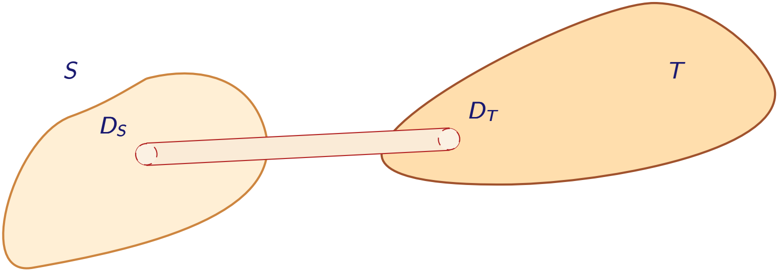

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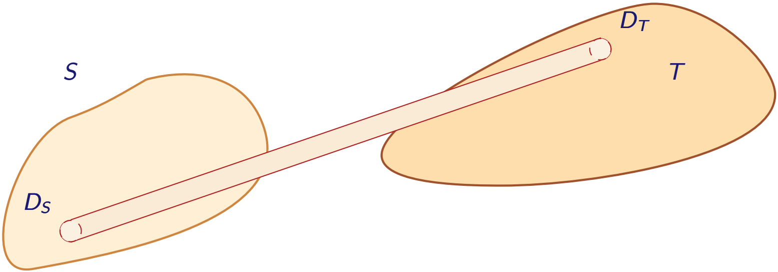

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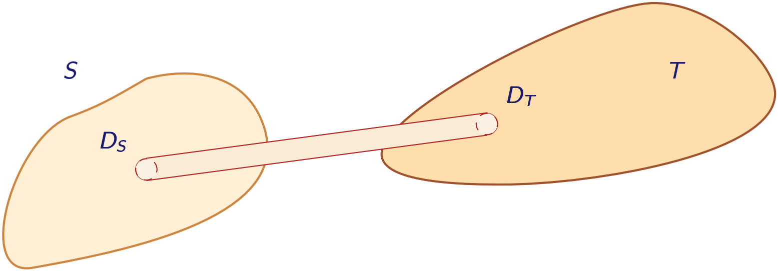

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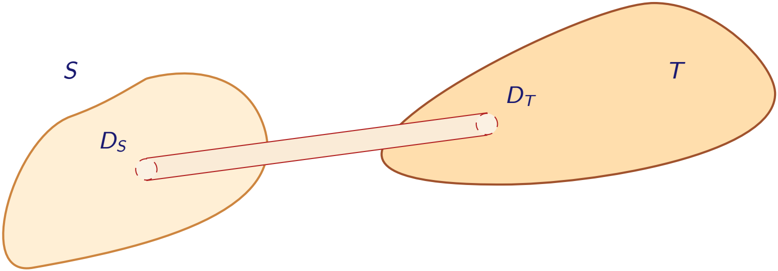

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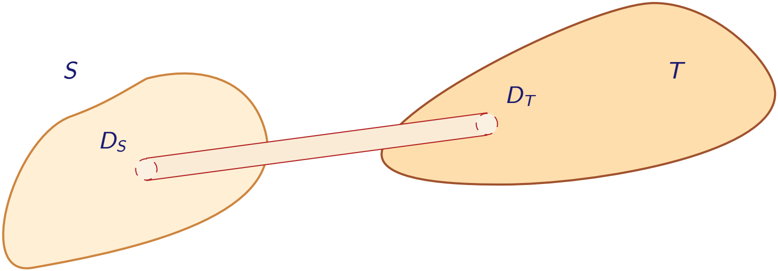

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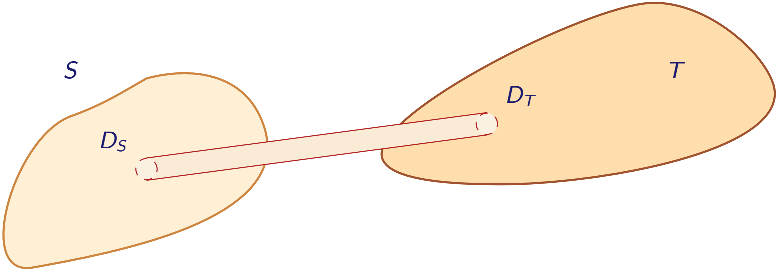

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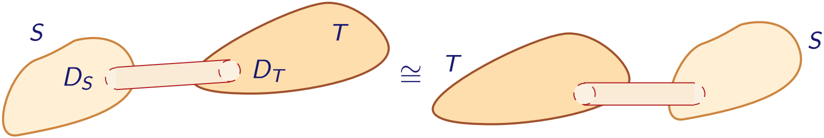

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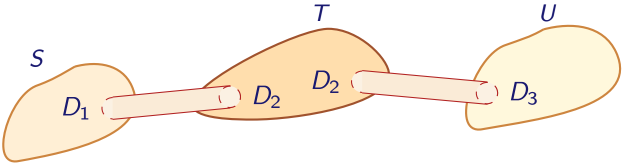

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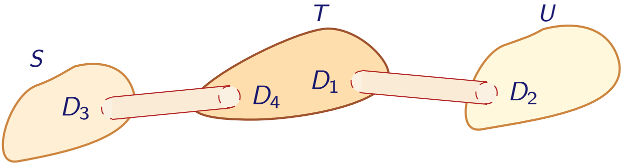
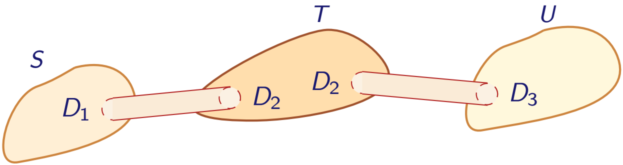


In these diagrams, D_1 and D_2 are cut first and then D_3 and D_4

Associativity of connected sums...

- $S \# (T \# U) \cong (S \# T) \# U$



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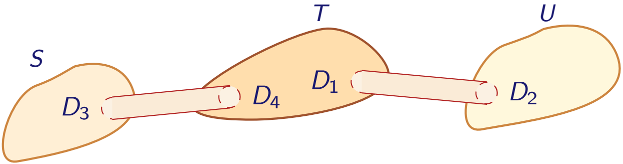

In these diagrams, D_1 and D_2 are cut first and then D_3 and D_4

$\Longrightarrow \quad \#$ is a "surface addition or multiplication"

Connected sums of Euler characteristic

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Let S and T be surfaces with polygonal decompositions. Then

$$\chi(S \# T) = \chi(S) + \chi(T) - 2$$

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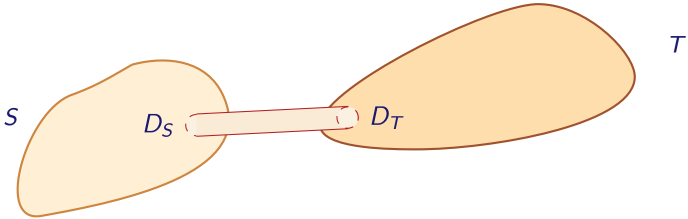

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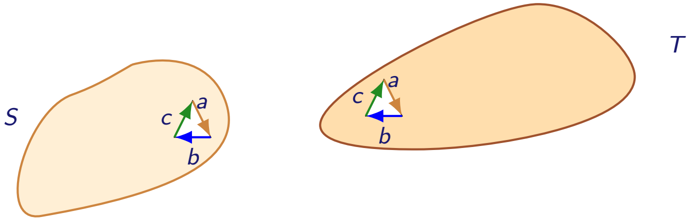

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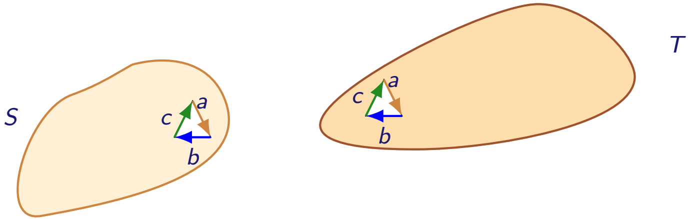

$$\implies \quad \chi(S \# T) = \big(\chi(S) - (3 - 3 + 1)\big) + \big(\chi(T) - (3 - 3 + 1)\big)$$

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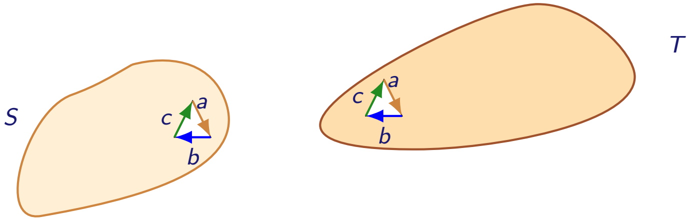

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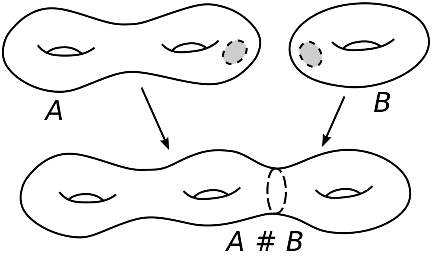
Moral The -2 comes from cutting out two disks

- If S is any surface then $S \cong S \# S^2$

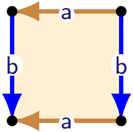
$$\implies \quad \chi(S) = \chi(S) + \underbrace{\chi(S^2)}_{=2} - 2 = \chi(S)$$

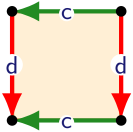
- $\mathbb{A} \cong \mathbb{D}^2 \# \mathbb{D}^2 \quad \implies \quad \chi(\mathbb{A}) = \chi(\mathbb{D}^2) + \chi(\mathbb{D}^2) - 2 = 1 + 1 - 2 = 0$

- $\chi(\mathbb{T} \# \mathbb{T} \# \mathbb{T}) = (\chi(\mathbb{T}) + \chi(\mathbb{T}) - 2) + \chi(\mathbb{T}) - 2 = -4$



Connected sums and polygonal decompositions

$\mathbb{T} \# \mathbb{T} \quad =$ 

$\#$ 

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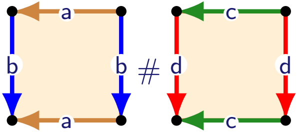
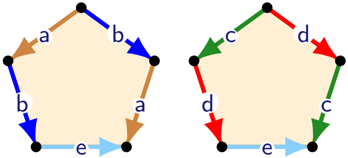
$\mathbb{T} \# \mathbb{T} =$ 

$=$ 

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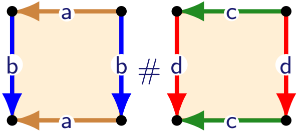

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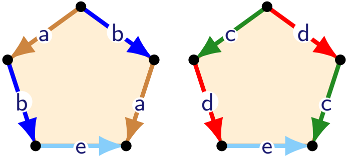

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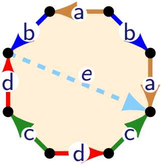

Connected sums and polygonal decompositions

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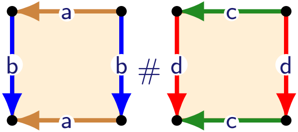

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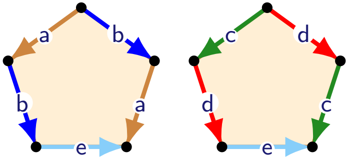

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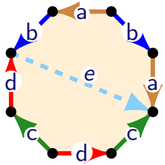
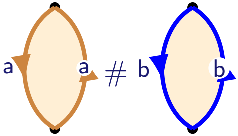


$\Longrightarrow$ For surfaces without a boundary you can cut the disks anywhere!

- What is $\mathbb{P}^2 \# \mathbb{P}^2$?

$\mathbb{P}^2 \# \mathbb{P}^2 \quad \cong$



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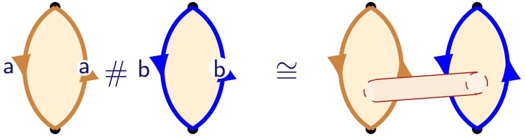

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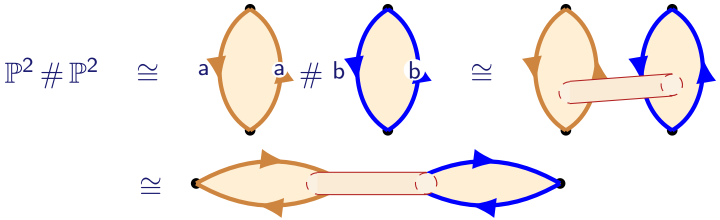

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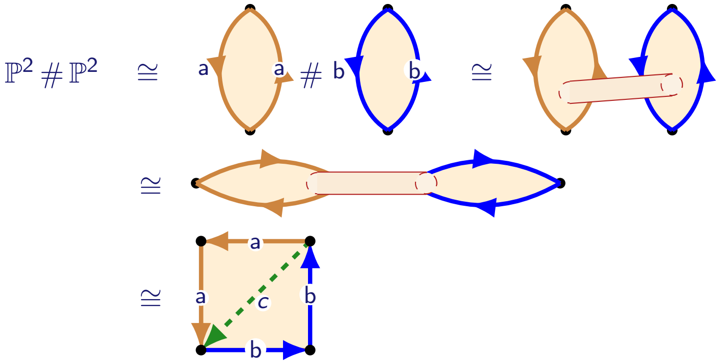

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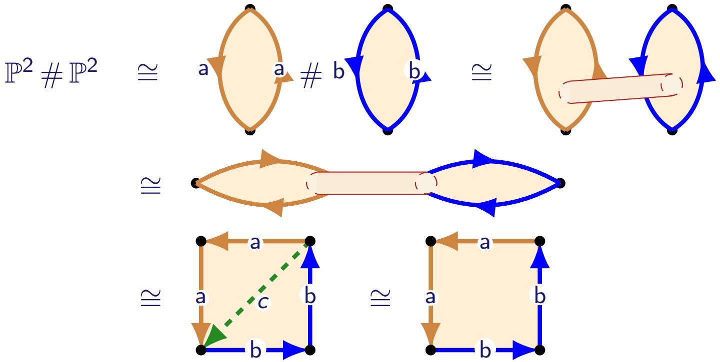

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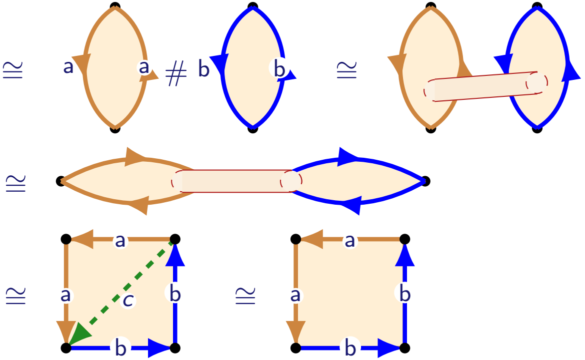
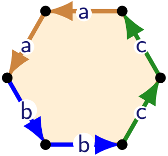


Similarly, $\#^3\mathbb{P}^2 \cong$



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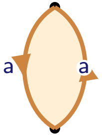
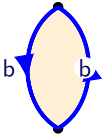
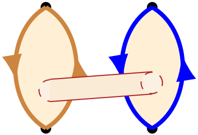
$\mathbb{P}^2 \# \mathbb{P}^2 \quad \cong$ 

$\#$ 

$\cong$ 

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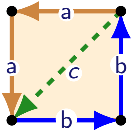
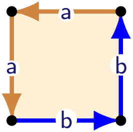
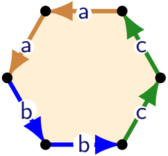
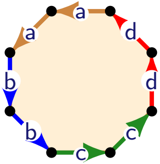
$\cong$ 

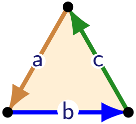
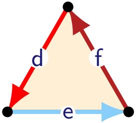
$\cong$ 

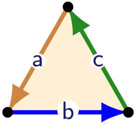
Similarly, $\#^3\mathbb{P}^2 \cong$ 

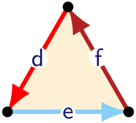
, $\#^4\mathbb{P}^2 \cong$ 

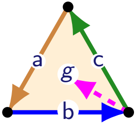
, $\dots$

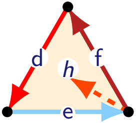
Connected sums and polygonal decompositions...

$\mathbb{D}^2 \# \mathbb{D}^2 \quad \cong$  $\#$ 

$\mathbb{D}^2 \# \mathbb{D}^2 \quad \cong$ 

$\#$ 

$\cong$ 

$\#$ 

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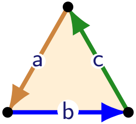

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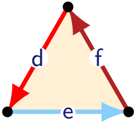

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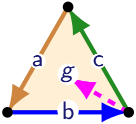

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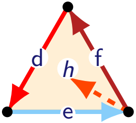

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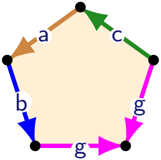

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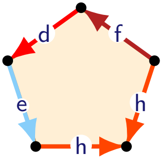

Connected sums and polygonal decompositions...

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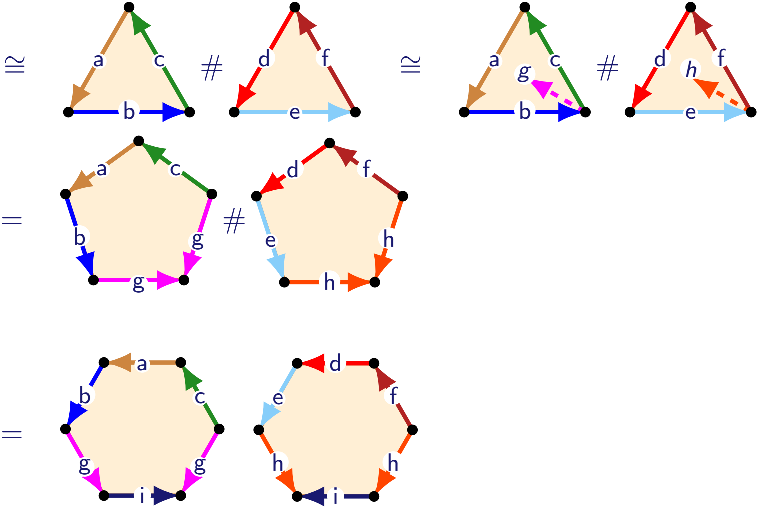

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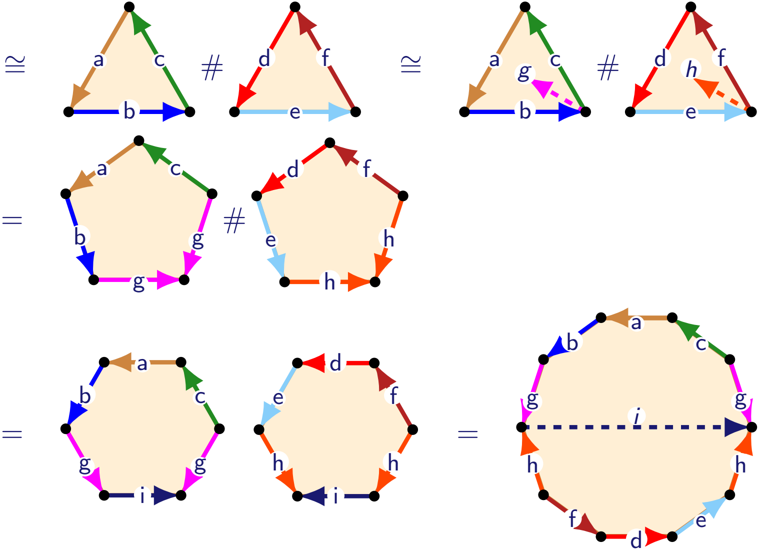

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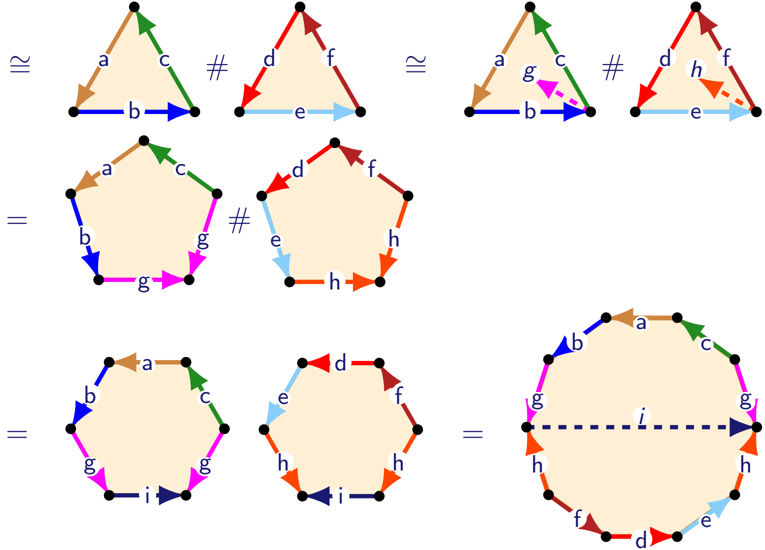

$\Longrightarrow$ For surfaces with a boundary, you can cut into the interior, if necessary, to form the connected sum

Surgery

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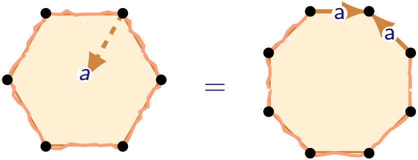

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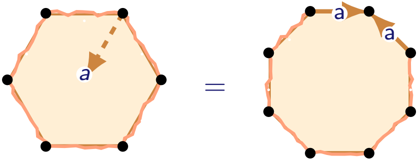

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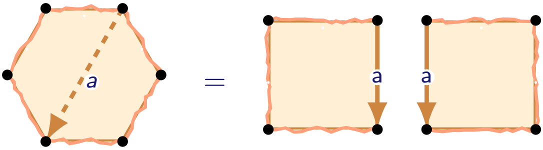

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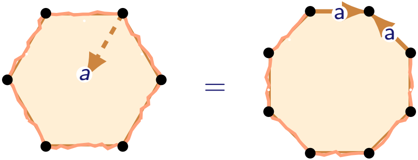

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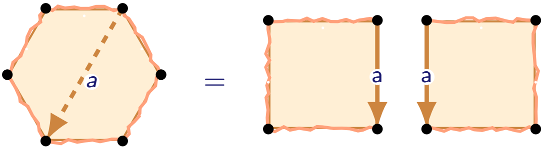

Perhaps surprisingly, these two operations and subdivision are all that we need

Lemma

$\mathbb{M} \cong \mathbb{D}^2 \# \mathbb{P}^2$ (= a *punctured projective plane*)

Proof

Surgery on the Möbius strip

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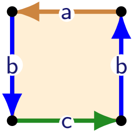
$\mathbb{M} =$ 

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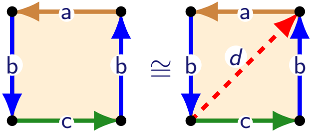
$\mathbb{M} \quad =$ 

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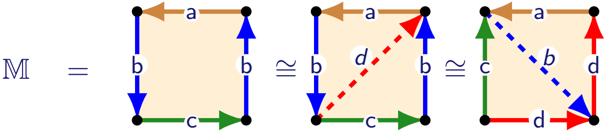

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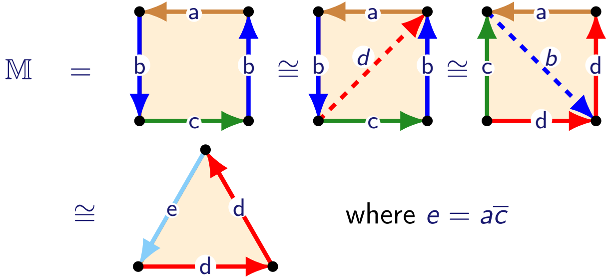

where $e = a\bar{c}$

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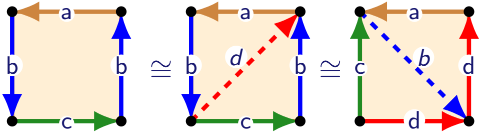

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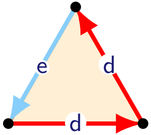

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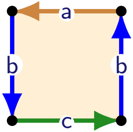
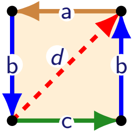
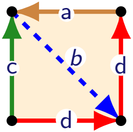
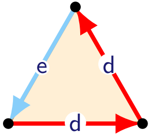
$\mathbb{M} =$ 

$\cong$ 

$\cong$ 

$\cong$ 

where $e = a\bar{c}$

$\cong \quad \mathbb{D}^2 \# \mathbb{P}^2$

$\Longrightarrow$ A Möbius strip is a punctured projective plane

Surgery on the Möbius strip

Lemma

$\mathbb{M} \cong \mathbb{D}^2 \# \mathbb{P}^2$ (= *a punctured projective plane*)

Proof

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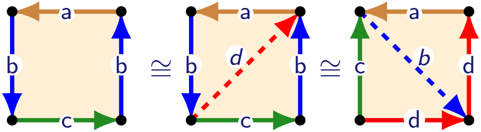

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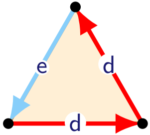

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$\Longrightarrow$ Every non-orientable surface contains the projective plane

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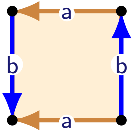
$\mathbb{K} \quad =$ 

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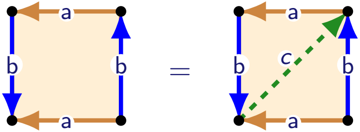

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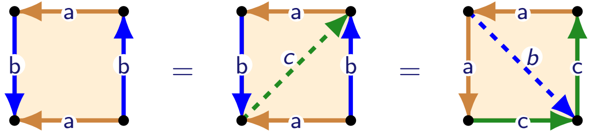

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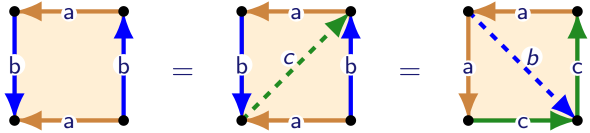
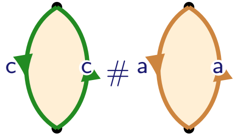
$\mathbb{K}$ $=$ 

$\cong$ 

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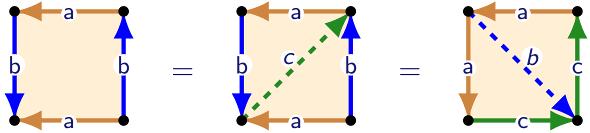

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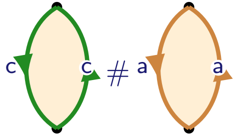

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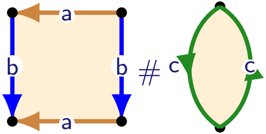
$\mathbb{T} \# \mathbb{P}^2 \quad = $ 

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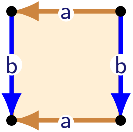

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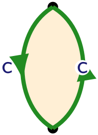

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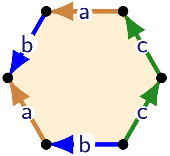

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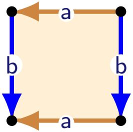

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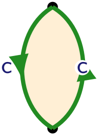

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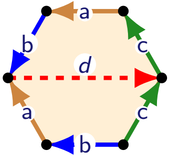

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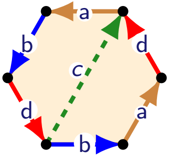

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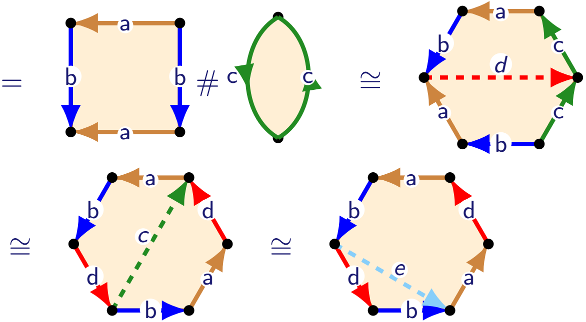

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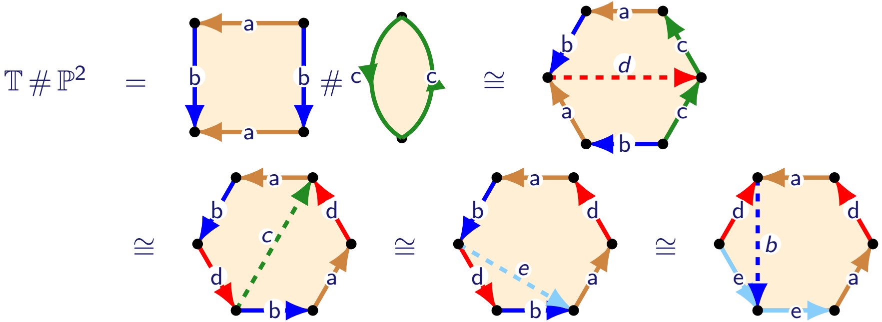

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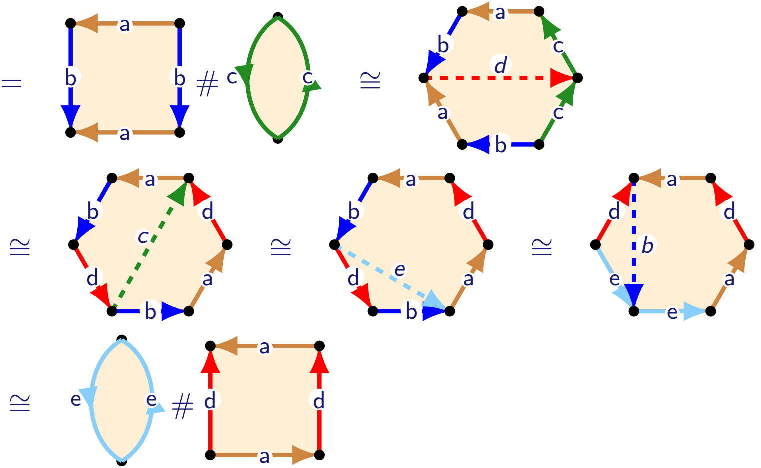

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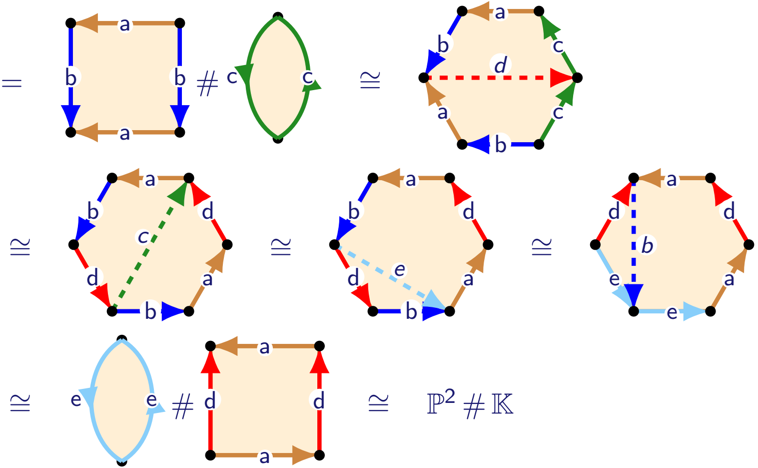

$\cong \mathbb{P}^2 \# \mathbb{K}$

Projective planes dominate

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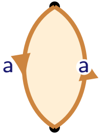
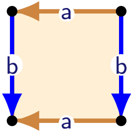
Why? $\mathbb{T}$ embeds in $\mathbb{R}^3$ but $\mathbb{K}$ does not!

Oriented and unoriented edges

Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

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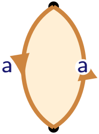
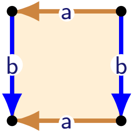
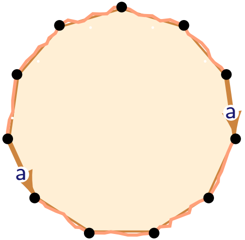
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

Paired edges on a polygon are oriented if they point in opposite directions and unoriented if they point in the same direction



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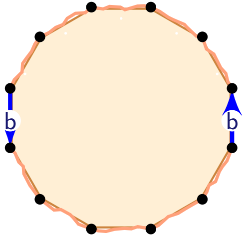

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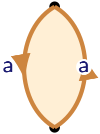
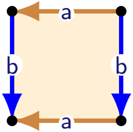
Compare: $\mathbb{P}^2 =$ 

and $\mathbb{T} =$ 

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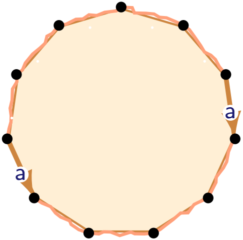

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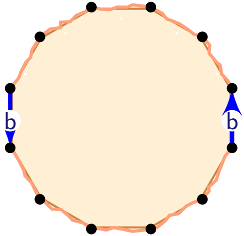

Unoriented

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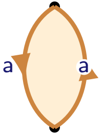
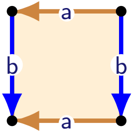
Compare: $\mathbb{P}^2 =$  and $\mathbb{T} =$ 

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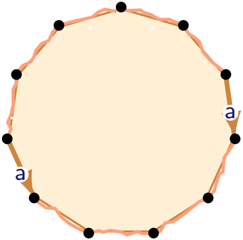

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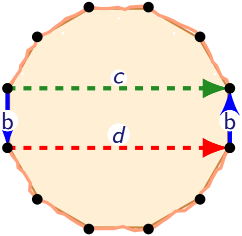

$\mathbb{M}$

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Classification of connected surfaces

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Let S be a connected surface. Then there exist non-negative integers d, p and t such that

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Moreover, we can assume that $pt = 0$, in which case S is uniquely determined up to homeomorphism by (d, p, t)

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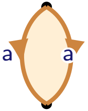
$S =$ 

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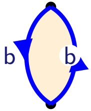
$S =$ 

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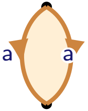
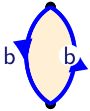
$S =$ 

$\cong S^2$ or $S =$ 

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$\Longrightarrow$ The theorem is true in this case

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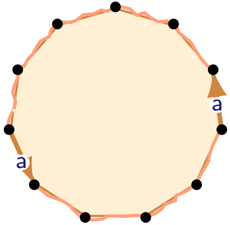

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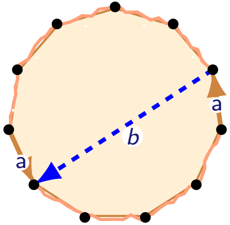

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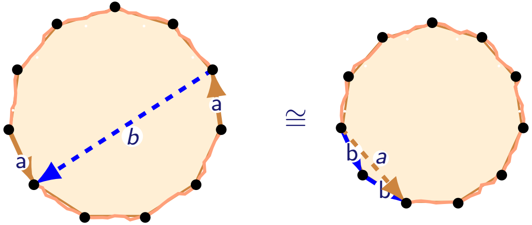

Proof of the classification theorem

Now suppose that S has at least two edges and that the theorem is true whenever all surfaces that have a polygonal decomposition with one face and fewer edges

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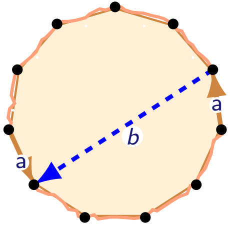
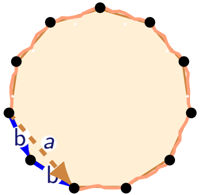
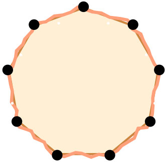


$\cong$



$\cong$ $\mathbb{P}^2 \#$



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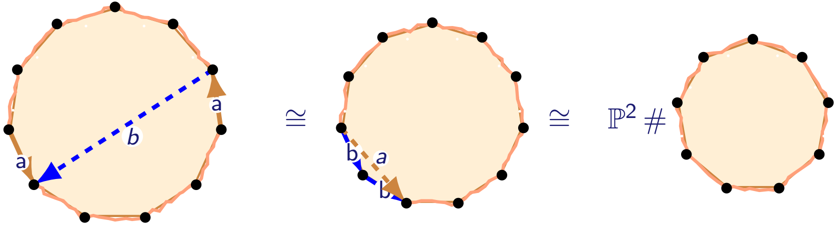

$\Longrightarrow \quad S \quad \cong \quad \mathbb{P}^2 \# T$

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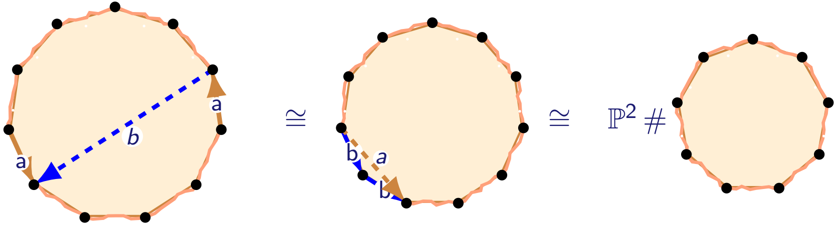

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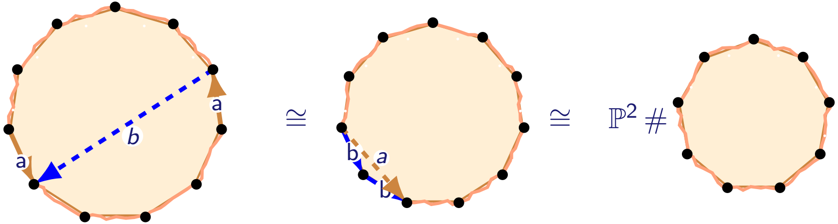

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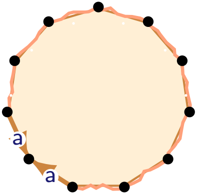
$S \cong$ 

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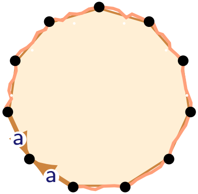
$S \cong$ 

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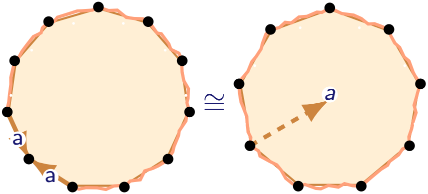
$S \cong$ 

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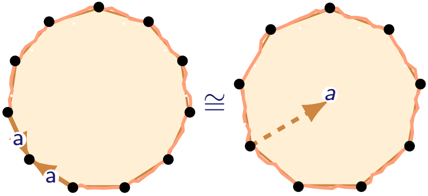
$S \cong$ 

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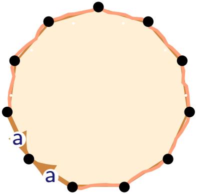
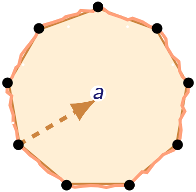
$S \cong$  $\cong$  $\cong T$

$\implies \quad S \cong T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$ by induction

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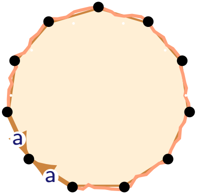
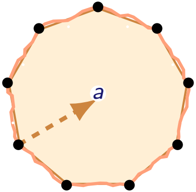
$S \cong$ 

$\cong$ 

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$$\Longrightarrow \quad S \cong T \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T} \text{ by induction}$$

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Proof of the classification theorem...

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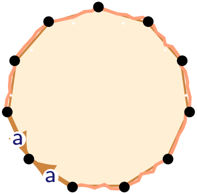
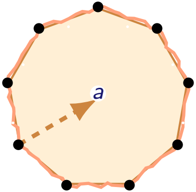
$S \cong$  $\cong$  $\cong T$

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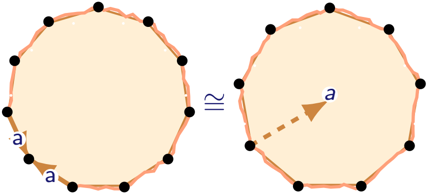
$S \cong$  $\cong T$

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Fix an (oriented) paired edge a such that the number of edges between the two copies of a is minimal

Case IIa: All edges on one side of a are free

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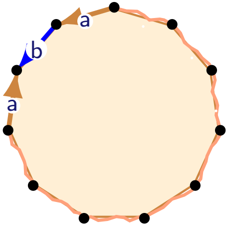
$\Longrightarrow \quad S \quad \cong$ 

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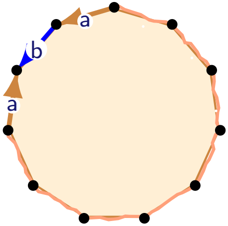
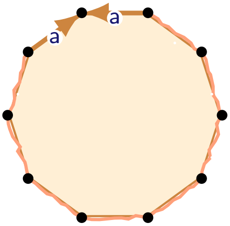
$\Longrightarrow \quad S \quad \cong$  $\cong \quad \mathbb{D}^2\,\#$ 

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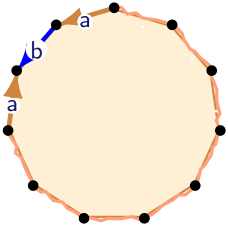
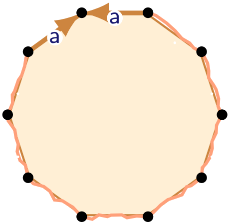


$\cong \quad \mathbb{D}^2 \#$



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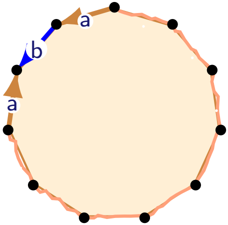

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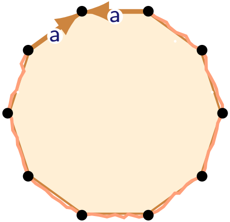

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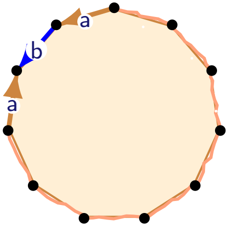
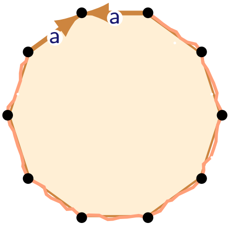
$\Longrightarrow \quad S \quad \cong$ 

$\cong \quad \mathbb{D}^2 \#$ 

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Proof of the classification theorem...

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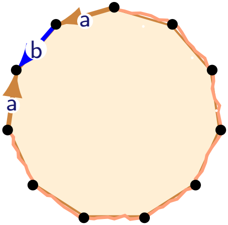

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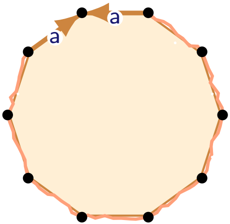

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Hence, we can assume that there are paired edges on both sides of a

Case IIb: There are paired edges on both sides of a

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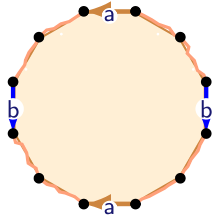
The number of edges between the ends of a is minimal, so

$\Longrightarrow \quad S \quad \cong$ 

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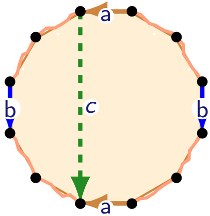
$\Longrightarrow \quad S \quad \cong$ 

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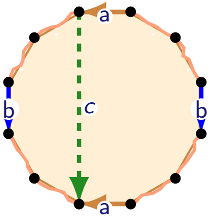
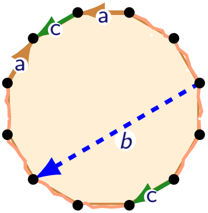
$\Longrightarrow \quad S \quad \cong$ 

$\cong$ 

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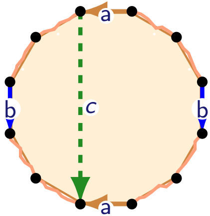
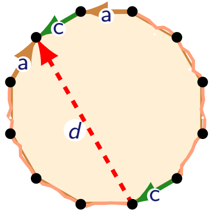
$\Longrightarrow \quad S \quad \cong$  $\cong$ 

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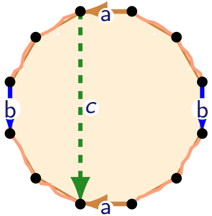
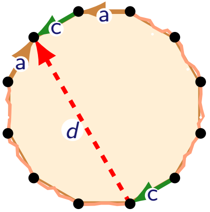
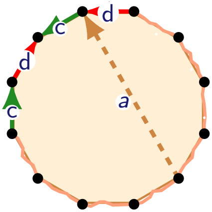
$\implies$ S $\cong$ 

$\cong$ 

$\cong$ 

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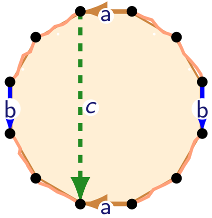
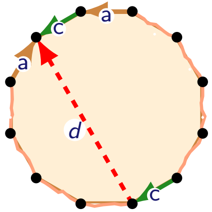
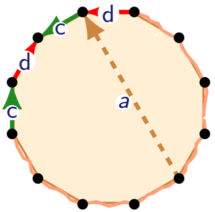
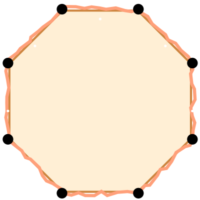
$\Longrightarrow \quad S \quad \cong$ 

$\cong$ 

$\cong$ 

$\cong \quad \mathbb{T} \,\#$  $\cong \mathbb{T} \,\# \, U$

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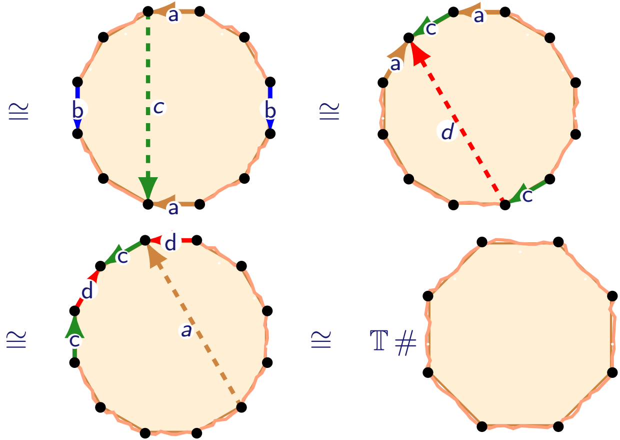

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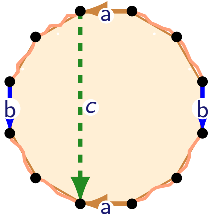

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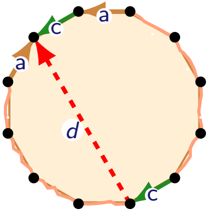

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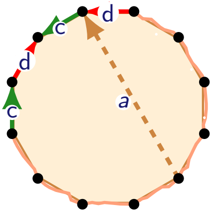

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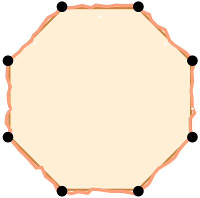

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We have now proved that every surface can be written in the form

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Hence, we now know all surfaces up to homeomorphism!

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A surface S is non-orientable if and only if its polygonal decomposition contains an unoriented edge

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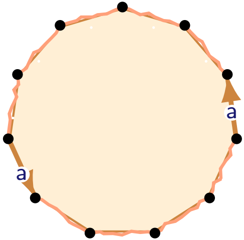
Proof Any unoriented edge gives a Möbius band inside S:



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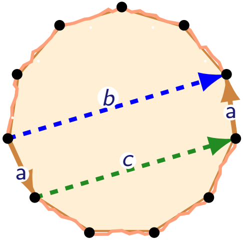

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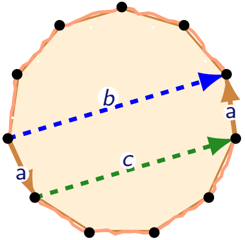

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It is now not hard to find an explicit polygonal decomposition of

$$S = S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T}$$

and check that surgery cannot create unoriented edges in S

Standard forms

Theorem

Let S be a connected surface. Then there exist non-negative integers d, p and t with $pt = 0$ such that

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The standard form of a surface that is not connected has each component in standard form

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Conversely, these three characteristics of S determine (d, p, t)

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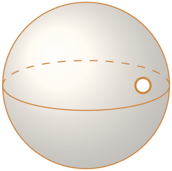
$S^2 \# \mathbb{D}^2 \quad =$ 

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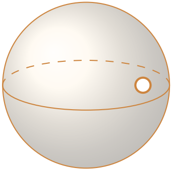
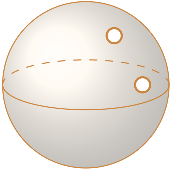
$S^2 \# \mathbb{D}^2 \quad =$ 

$S^2 \# \#^2 \mathbb{D}^2 \quad =$ 

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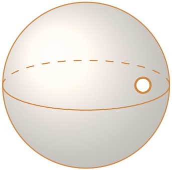

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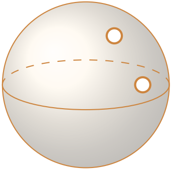

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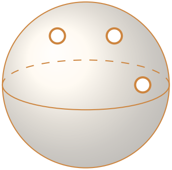

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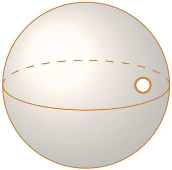

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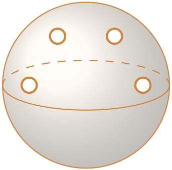

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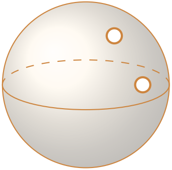

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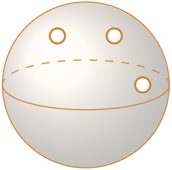

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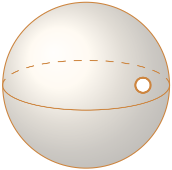
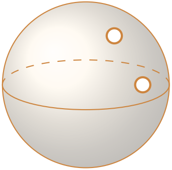
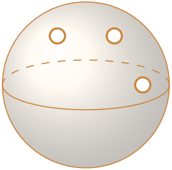
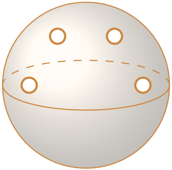
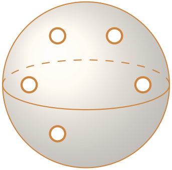
$S^2 \# \mathbb{D}^2 \quad =$ 

$S^2 \# \#^2 \mathbb{D}^2 \quad =$ 

$S^2 \# \#^3 \mathbb{D}^2 \quad =$ 

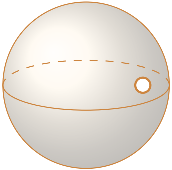
$S^2 \# \#^4 \mathbb{D}^2 \quad =$ 

$S^2 \# \#^5 \mathbb{D}^2 \quad =$ 

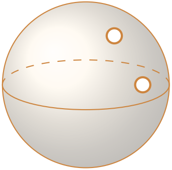
Spheres with punctures

- $S^2 \# \#^d \mathbb{D}^2$ is a sphere with d punctures

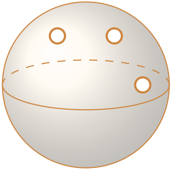
$S^2 \# \mathbb{D}^2 =$



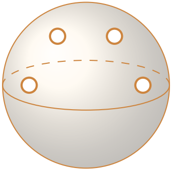
$S^2 \# \#^2 \mathbb{D}^2 =$



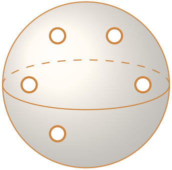
$S^2 \# \#^3 \mathbb{D}^2 =$



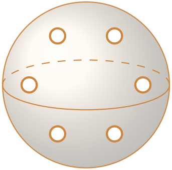
$S^2 \# \#^4 \mathbb{D}^2 =$



$S^2 \# \#^5 \mathbb{D}^2 =$

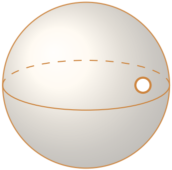


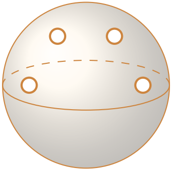
$S^2 \# \#^6 \mathbb{D}^2 =$

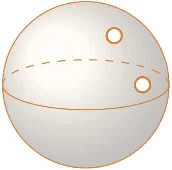


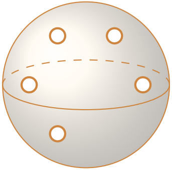
Spheres with punctures

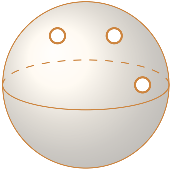
- $S^2 \# \#^d \mathbb{D}^2$ is a sphere with d punctures

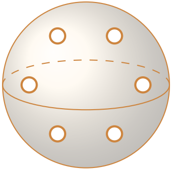
$S^2 \# \mathbb{D}^2 \quad =$ 

$S^2 \# \#^4 \mathbb{D}^2 \quad =$ 

$S^2 \# \#^2 \mathbb{D}^2 \quad =$ 

$S^2 \# \#^5 \mathbb{D}^2 \quad =$ 

$S^2 \# \#^3 \mathbb{D}^2 \quad =$ 

$S^2 \# \#^6 \mathbb{D}^2 \quad =$ 

More generally, $S \# \#^d \mathbb{D}^2$ is S with d punctures

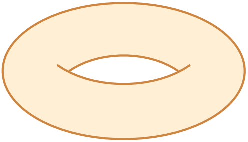
A spheres with zero and one puncture

Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

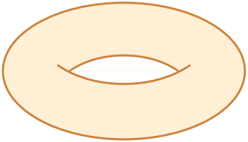
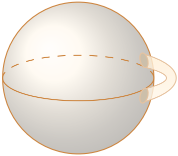
Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

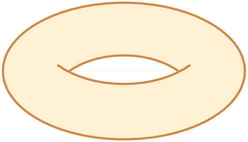
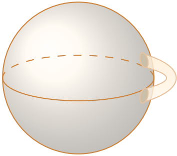
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$ 

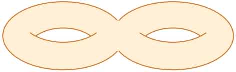
Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

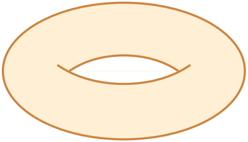
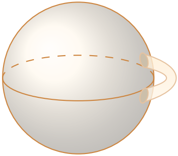
- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

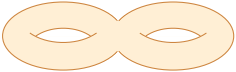
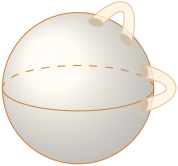
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$ 

Spheres with handles

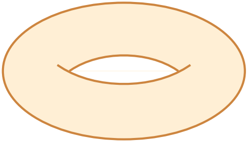
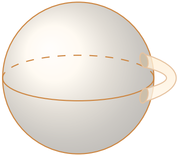
- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

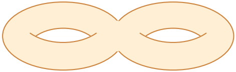
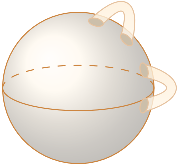
$S^2 \# \mathbb{T} \cong \mathbb{T} \cong$  $\cong$ 

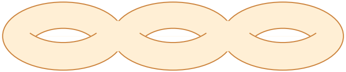
$S^2 \# \#^2 \mathbb{T} \cong \#^2 \mathbb{T} \cong$  $\cong$ 

Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

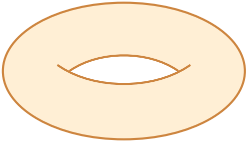
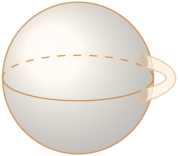
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

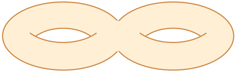
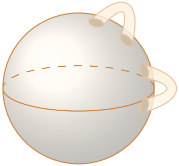
$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

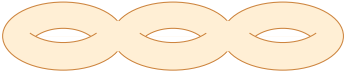
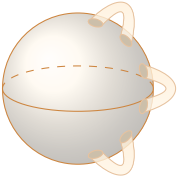
$S^2 \# \#^3 \mathbb{T} \quad \cong$ 

Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

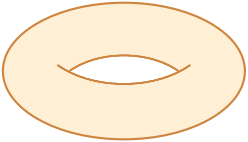
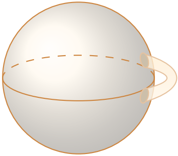
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

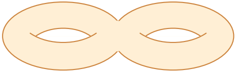
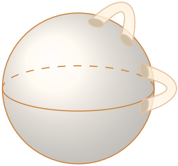
$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

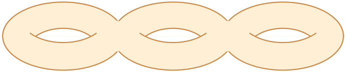
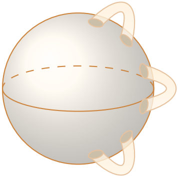
$S^2 \# \#^3 \mathbb{T} \quad \cong$  $\cong$ 

Spheres with handles

- $S^2 \# \#^t \mathbb{T}$ is a sphere with t handles

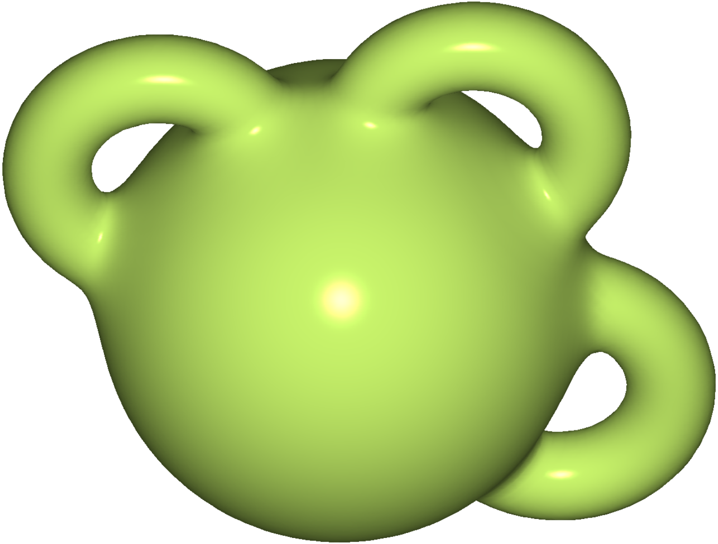
$S^2 \# \mathbb{T} \quad \cong \quad \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^2 \mathbb{T} \quad \cong \quad \#^2 \mathbb{T} \quad \cong$  $\cong$ 

$S^2 \# \#^3 \mathbb{T} \quad \cong$  $\cong$ 

Continuing like this constructs a sphere with t-handles $\#^t \mathbb{T}$

Handle decomposition



Sphere with cross-caps

- $S^2 \# \#^p \mathbb{P}^2$ is a sphere with p cross-caps

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A cross-cap is what you get when you sew a Möbius strip onto the sphere

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This shape lives in $\mathbb{R}^4$, so difficult to visualize but Wikipedia draws it as:

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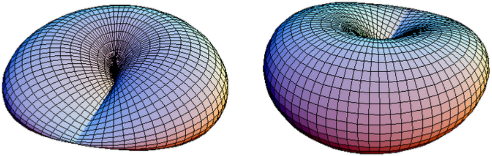

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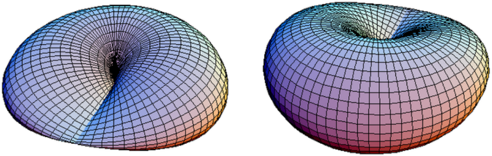
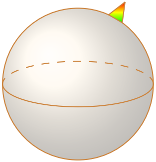


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^1 \mathbb{P}^2 \quad \cong$$



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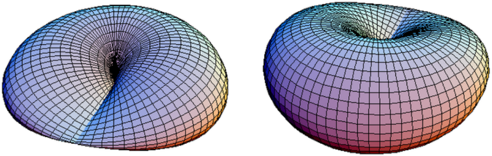
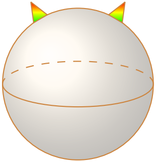


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$S^2 \# \#^2 \mathbb{P}^2 \quad \cong$



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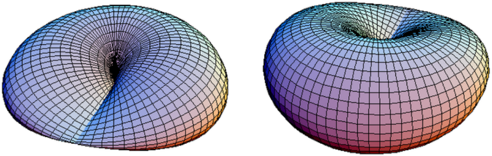
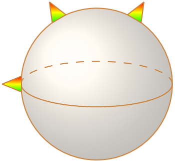


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^3 \mathbb{P}^2 \quad \cong$$



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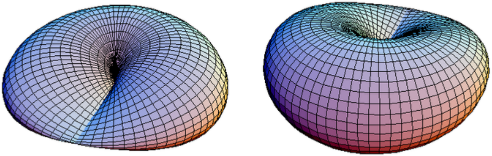
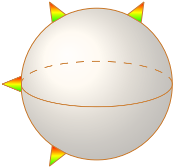


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^4 \mathbb{P}^2 \quad \cong$$



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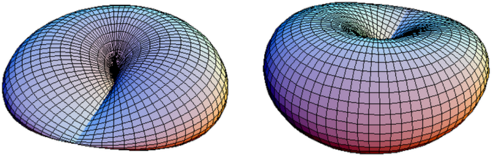
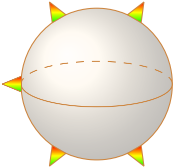


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^5 \mathbb{P}^2 \quad \cong$$



Sphere with cross-caps

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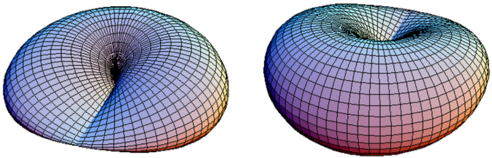
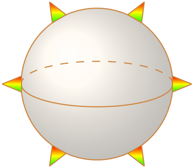


In $\mathbb{R}^3$ this surface self-intersects. We draw surfaces with cross caps as:

$$S^2 \# \#^6 \mathbb{P}^2 \quad \cong$$



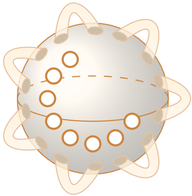
What do standard surfaces look like?

We can combine the pictures above to draw all of the standard surfaces:

What do standard surfaces look like?

We can combine the pictures above to draw all of the standard surfaces:

$$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \quad \cong$$



What do standard surfaces look like?

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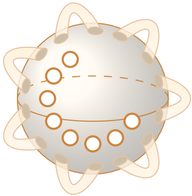
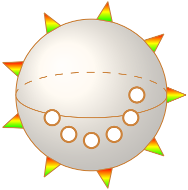
$\#^8\mathbb{D}^2 \# \#^7\mathbb{T} \quad \cong$ 

$\#^6\mathbb{D}^2 \# \#^9\mathbb{P}^2 \quad \cong$ 

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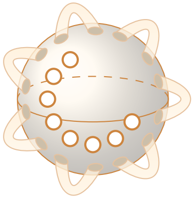

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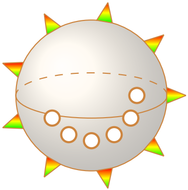
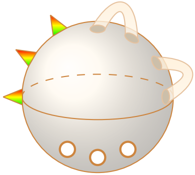


$\#^3\mathbb{D}^2 \# \#^2\mathbb{T} \# \#^3\mathbb{P}^2 \quad \cong$



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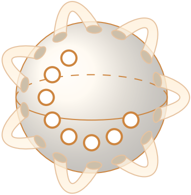

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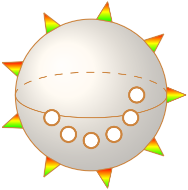
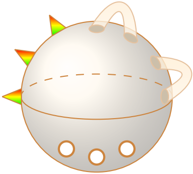
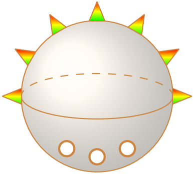


$\#^3\mathbb{D}^2 \# \#^2\mathbb{T} \# \#^3\mathbb{P}^2 \quad \cong$



$\cong$



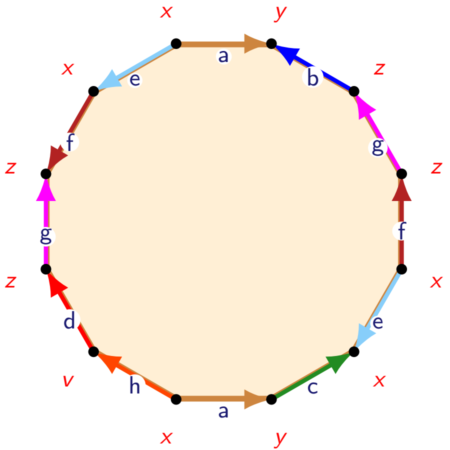
Putting a surface in standard form

Given a polygonal decomposition for a surface we can put it in standard form by:

- Find all of the vertices (identified edges implicitly identify vertices)
- Count the number d of boundary circles
- S is orientable $(p = 0)$ if all edges are oriented otherwise it is non-orientable $(t = 0)$
- Compute $\chi(S) = 2 - d - p - 2t$ to determine the missing variable, which is t if S is orientable and or p if non-orientable

Example 1

What is the surface with the below polygonal decomposition?

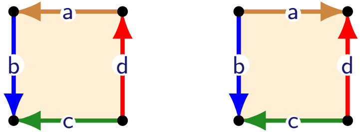


$a\, c\, \overline{e}\, f\, g\, b\, \overline{a}\, e\, f\, \overline{g}\, \overline{d h}$ (overline=opposite direction)

$\Longrightarrow$ This is $\#^1\mathbb{D}^2 \# \#^0\mathbb{T} \# \#^4\mathbb{P}^2$

Example 2

What is the standard form of the surface with polygonal decomposition?



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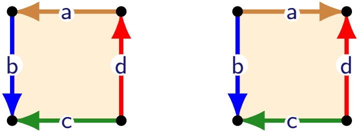

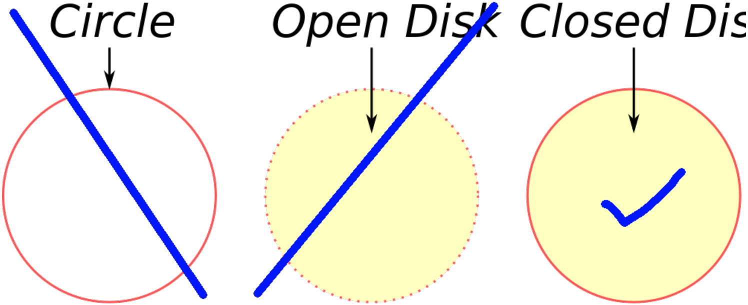

Topology – week 10

Math3061

Daniel Tubbenhauer, University of Sydney

Words for surfaces

A polygonal decomposition for a surface that has one face can be encoded in a word

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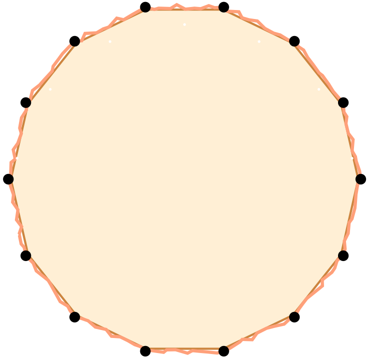

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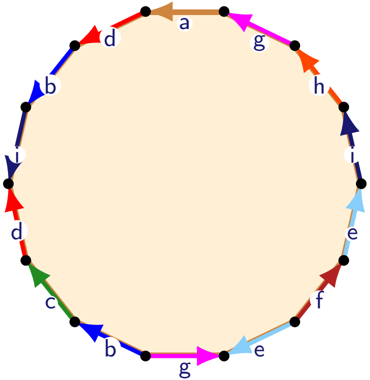

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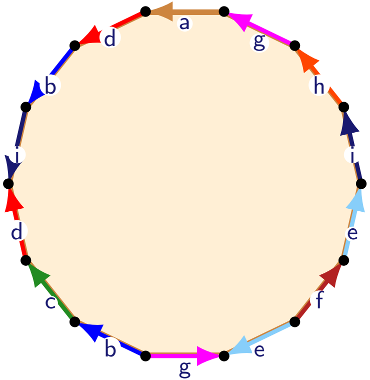

- write x for an edge pointing anticlockwise
- write $\overline{x}$ for an edge pointing clockwise

Words for surfaces

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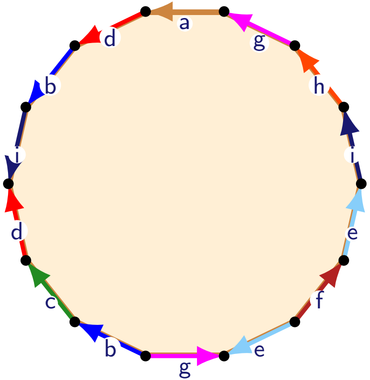
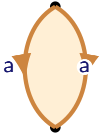


- write x for an edge pointing anticlockwise
- write $\overline{x}$ for an edge pointing clockwise
- We always read the word in anticlockwise order

Words for basic surfaces

- $S^2 =$ 

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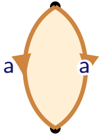
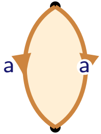
- $S^2 =$ 

  $= a\,\overline{a}$

Words for basic surfaces

- $S^2 =$ 

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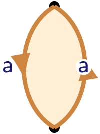
- $\mathbb{P}^2 =$ 

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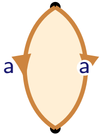
- $S^2 =$ 

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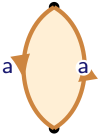
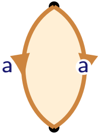
- $\mathbb{P}^2 =$ 

$= a \ a$

- $S^2 =$ 

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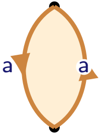
- $\mathbb{P}^2 =$ 

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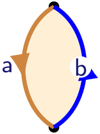
- $\mathbb{D}^2 =$ 

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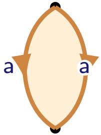
- $S^2 =$ 

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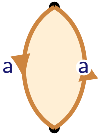
- $\mathbb{P}^2 =$ 

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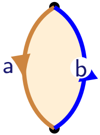
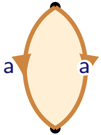
- $\mathbb{D}^2 =$ 

  $= a\ b$

- $S^2 =$ 

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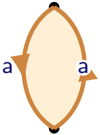
- $\mathbb{P}^2 =$ 

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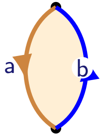
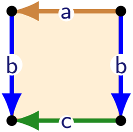
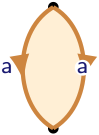
- $\mathbb{D}^2 =$ 

  $= a \; b$

- $\mathbb{A} =$ 

- $S^2 =$ 

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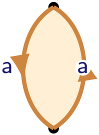
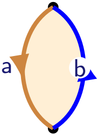
- $\mathbb{P}^2 =$ 

$= a\ a$

- $\mathbb{D}^2 =$ 

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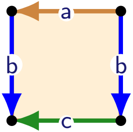
- $\mathbb{A} =$ 

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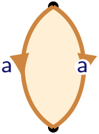
- $S^2 =$ 

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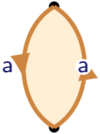
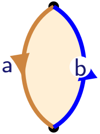
- $\mathbb{P}^2 =$ 

$= a \ a$
- $\mathbb{D}^2 =$ 

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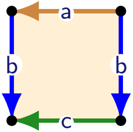
- $\mathbb{A} =$ 

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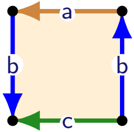
- $\mathbb{M} =$ 

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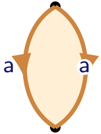
- $S^2 =$ 

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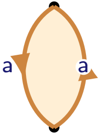
- $\mathbb{P}^2 =$ 

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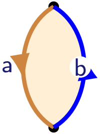
- $\mathbb{D}^2 =$ 

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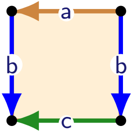
- $\mathbb{A} =$ 

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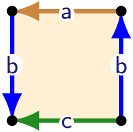
- $\mathbb{M} =$ 

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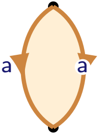
- $S^2 =$ 

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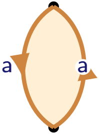
- $\mathbb{P}^2 =$ 

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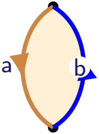
- $\mathbb{D}^2 =$ 

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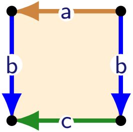
- $\mathbb{A} =$ 

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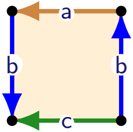
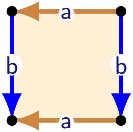
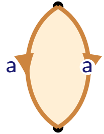
- $\mathbb{M} =$ 

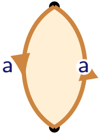
  $= a\ b\ \overline{c}\ b$

- $\mathbb{T} =$ 

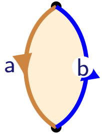
Words for basic surfaces

- $S^2 =$ 

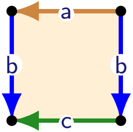
  $= a\ \overline{a}$

- $\mathbb{P}^2 =$ 

  $= a\ a$

- $\mathbb{D}^2 =$ 

  $= a\ b$

- $\mathbb{A} =$ 

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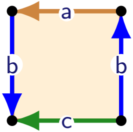
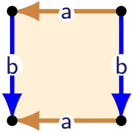
- $\mathbb{M} =$ 

  $= a\ b\ \overline{c}\ b$

- $\mathbb{T} =$ 

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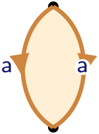
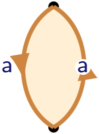
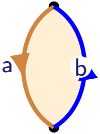
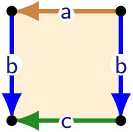
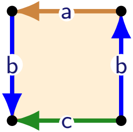
Words for basic surfaces

- $S^2 =$ 

$= a\ \overline{a}$
- $\mathbb{P}^2 =$ 

$= a\ a$
- $\mathbb{D}^2 =$ 

$= a\ b$
- $\mathbb{A} =$ 

$= a\ b\ \overline{c}\ \overline{b}$
- $\mathbb{M} =$ 

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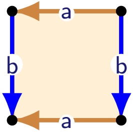
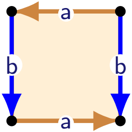
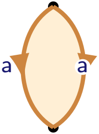
- $\mathbb{T} =$ 

$= a\ b\ \overline{a}\ \overline{b}$
- $\mathbb{K} =$ 

Words for basic surfaces

- $S^2 =$ 

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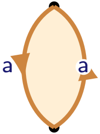
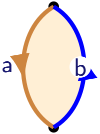
- $\mathbb{P}^2 =$ 

$= a\ a$

- $\mathbb{D}^2 =$ 

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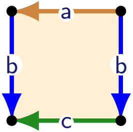
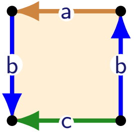
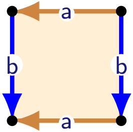
- $\mathbb{A} =$ 

$= a\ b\ \overline{c}\ \overline{b}$

- $\mathbb{M} =$ 

$= a\ b\ \overline{c}\ b$

- $\mathbb{T} =$ 

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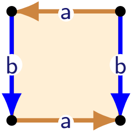
- $\mathbb{K} =$ 

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Properties of words

- Words encode orientability
  - Orientable: $\ldots a \ldots \overline{a} \ldots$ or $\ldots \overline{a} \ldots a \ldots$
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Example The following words are all words for the torus $\mathbb{T}$:

$$\begin{array}{cccc} a\,b\,\overline{a}\,\overline{b} & b\,\overline{a}\,\overline{b}\,a & \overline{a}\,\overline{b}\,a\,b & \overline{b}\,a\,b\,\overline{a} \\ a\,\overline{b}\,\overline{a}\,b & \overline{b}\,\overline{a}\,b\,a & \overline{a}\,b\,a\,\overline{b} & b\,a\,\overline{b}\,\overline{a} \end{array}$$

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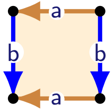
- The word of a surface can be used to give generators and relations for the first homotopy group of the surface — this generalises independent cycles and are beyond the scope of this unit

Standard words for closed orientable surfaces

- Connected sums of tori: $\#^t \mathbb{T}$
  - $\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}$

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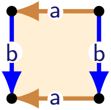
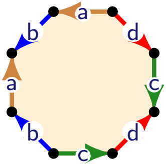
  - $\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}$

  - $\#^2\mathbb{T} =$  $= a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d}$

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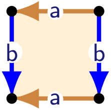
  - $\mathbb{T} =$ 

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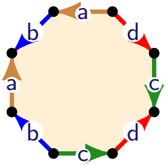
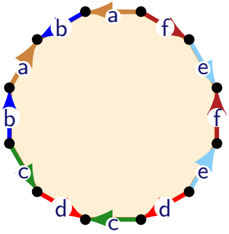
  - $\#^2 \mathbb{T} =$ 

$= a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d}$
  - $\#^3 \mathbb{T} =$ 

$= a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d}\ e\ f\ \overline{e}\ \overline{f}$

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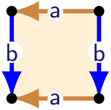
  - $\mathbb{T} =$ 

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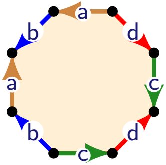
  - $\#^2\mathbb{T} =$ 

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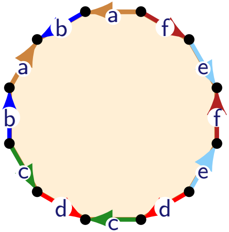
  - $\#^3\mathbb{T} =$ 

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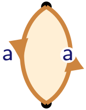
  - ... $\#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \ldots\ a_t\ b_t \overline{a_t}\ \overline{b_t}$

Words for closed non-orientable surfaces

- Connected sums of projective planes $\#^p \mathbb{P}^2$

  - $\mathbb{P}^2 =$  $= a\ a$

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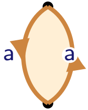
  - $\mathbb{P}^2 =$ 

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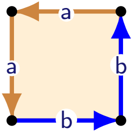
  - $\#^2\mathbb{P}^2 =$ 

$= a\ a\ b\ b$

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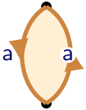
  - $\mathbb{P}^2 =$ 

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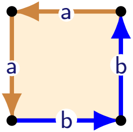
  - $\#^2\mathbb{P}^2 =$ 

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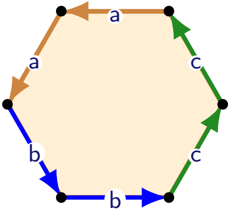
  - $\#^3\mathbb{P}^2 =$ 

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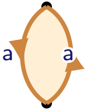
  - $\mathbb{P}^2 =$ 

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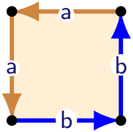
  - $\#^2\mathbb{P}^2 =$ 

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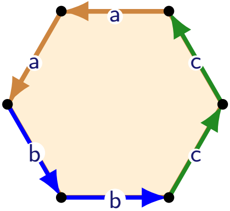
  - $\#^3\mathbb{P}^2 =$ 

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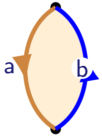
  - $\mathbb{D}^2 =$ 

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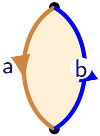
  - $\mathbb{D}^2 =$ 

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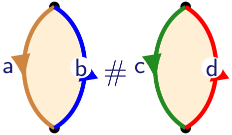
  - $\#^2 \mathbb{D}^2 =$ 

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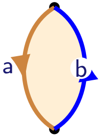
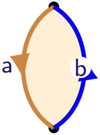
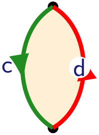
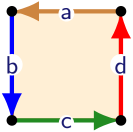
  - $\mathbb{D}^2 =$  $= a\ b$
  - $\#^2\mathbb{D}^2 =$  $\#$  $\overset{?}{\cong}$ 

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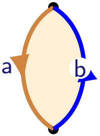
  - $\mathbb{D}^2 =$ 

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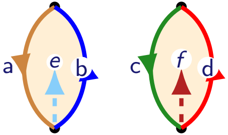
  - $\#^2 \mathbb{D}^2 =$ 

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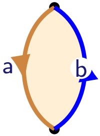
  - $\mathbb{D}^2 =$ 

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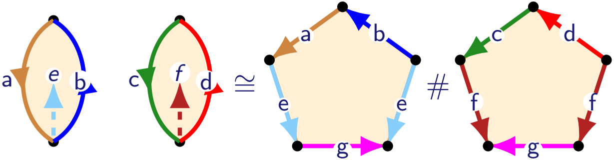
  - $\#^2 \mathbb{D}^2 =$ 

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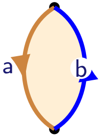
  - $\mathbb{D}^2 =$ 

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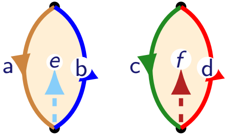
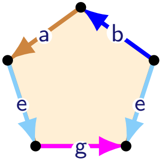
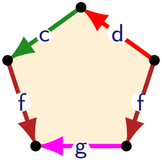
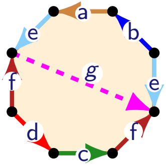
  - $\#^2 \mathbb{D}^2 =$ 

$\cong$ 

$\#$ 

$\cong$ 

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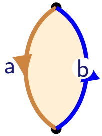
  - $\mathbb{D}^2 =$ 

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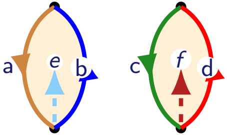
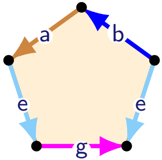
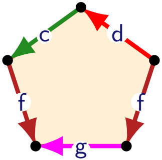
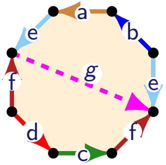
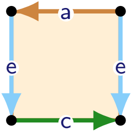
  - $\#^2\mathbb{D}^2 =$ 

$\cong$ 

$\#$ 

$\cong$ 

$\cong$ 

$= a\ e\ c\ \overline{e}$

Standard words for surfaces with boundary

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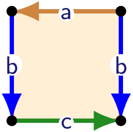
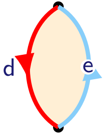
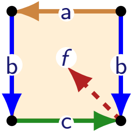
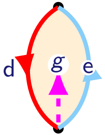
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \, \# \, \mathbb{D}^2 \cong$ 

$\#$ 

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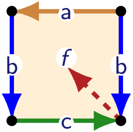
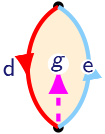
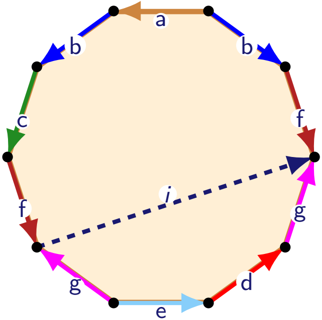
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$  $\#$ 

$\cong$ 

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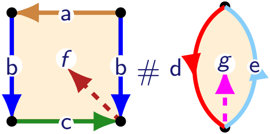
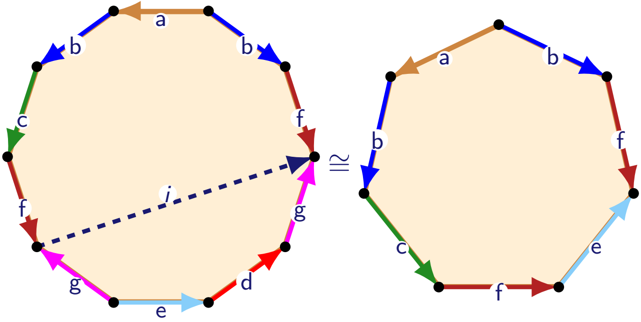
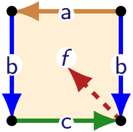
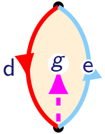
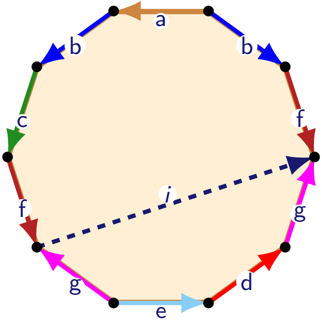
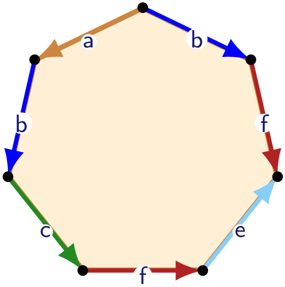
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\cong$ 

- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

$\cong$ 

$\cong$ 

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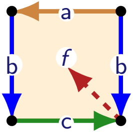
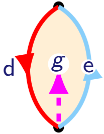
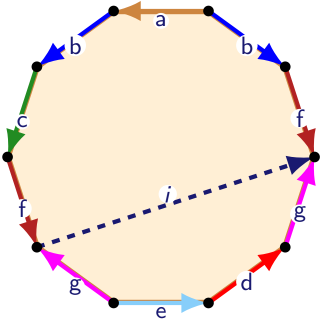
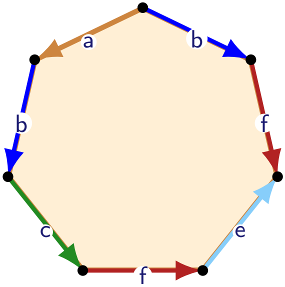
- $\#^3\mathbb{D}^2 = \#^2\mathbb{D}^2 \# \mathbb{D}^2 \cong$ 

$\#$ 

$\cong$ 

$\cong$ 

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- $\#^d\mathbb{D}^2 = a_1\ b_1\ a_2\ b_2\ \ldots\ b_{d-1} a_d\ \overline{b}_{d-1}\ \ldots \overline{b}_2\ \overline{b}_1$

Words to surfaces

What standard surface is given by the word $a\ d\ b\ \overline{c}\ e\ \overline{c}\ \overline{f}\ d\ h\ g\ e\ \overline{a}\ h$?

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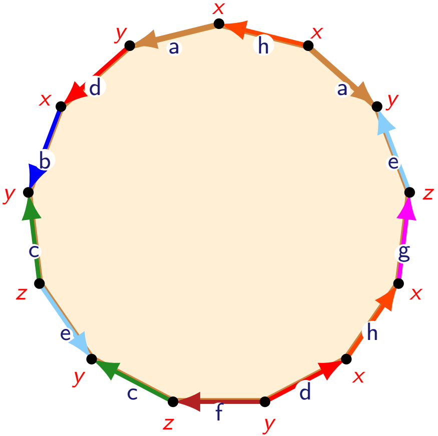

Free edges
$x \xrightarrow{b} y$
$x \xrightarrow{g} z$
$y \xrightarrow{f} z$

$\implies$ $d = 1$ and $\chi(S) = 3 - 8 + 1 = -4$

$\implies$ $S \cong \mathbb{D}^2 \# \#^5 \mathbb{P}^2$

$\implies$ $S = a\ b\ b\ c\ c\ d\ d\ e\ e\ f\ f$

The vertex-degree equation revisited

When we looked at graphs we proved the vertex-degree equation:

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Answer Yes and no!

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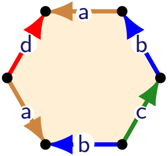

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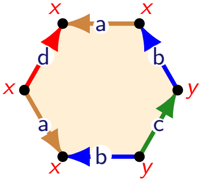

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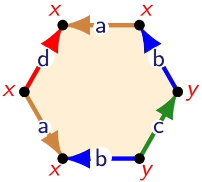

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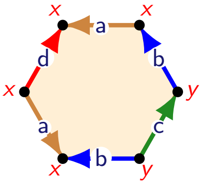

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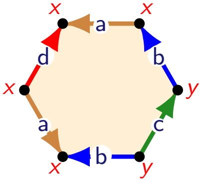

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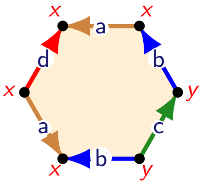

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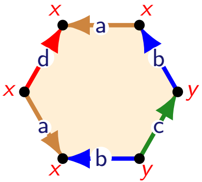

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The vertex-degree equation holds using either identified or non-identified edges and vertices because in both cases the degree of a vertex is defined to be the number of incident edges to the vertex

The surface degree-vertex equation

Proposition

Let $S = (V, E, F)$ *be a surface with polygonal decomposition. Then*

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Therefore, we have two degree-vertex equations:

- The graph degree-vertex equation where we do not identify edges and vertices in S
- The surface degree-vertex equation where we do identify edges and vertices in S

The degree of a face

Let $S = (V, E, F)$ be a surface with polygonal decomposition

Let $f \in F$ be a face of S. The degree of f is

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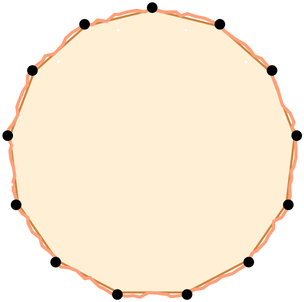

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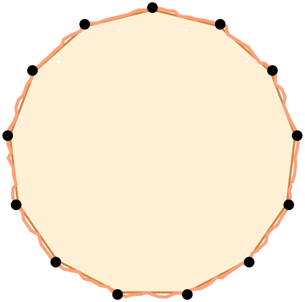

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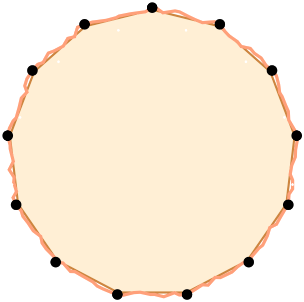

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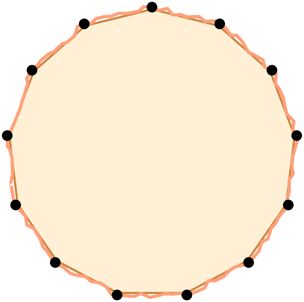

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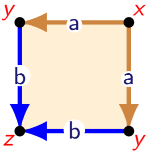
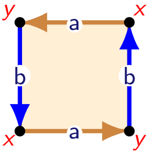
Question How are $\sum \deg(f)$ and $2|E|$ related?

In all cases $\deg(\text{face}) = 4$ as there are 4 non-identified edges

- Sphere

$S^2 \cong$ 

- Projective plane

$\mathbb{P}^2 \cong$ 

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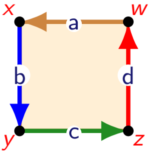
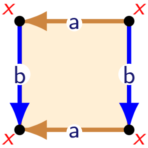
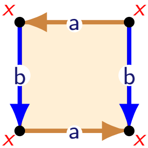
$\mathbb{D}^2 \cong$ 

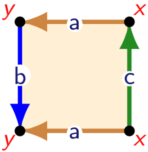
- Torus

$\mathbb{T} \cong$ 

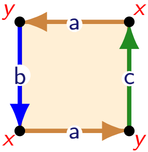
- Klein bottle

$\mathbb{K} \cong$ 

- Annulus

$\mathbb{A} \cong$ 

- Möbius band

$\mathbb{M} \cong$ 

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Remark To use this formula we need to know the number of identified edges in the polygonal decomposition

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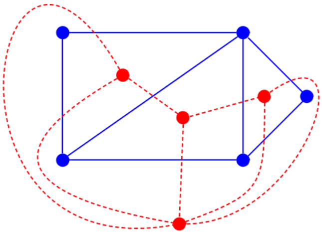
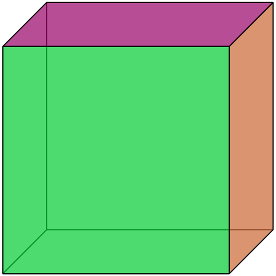


The dual of the cube



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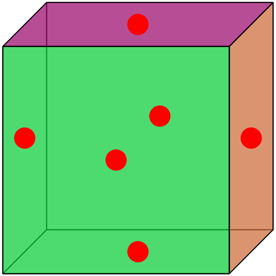

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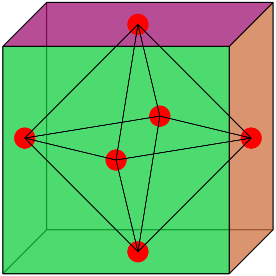

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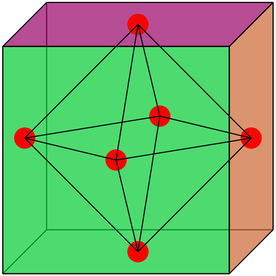

$\Longrightarrow$ the dual surface to the cube is the octahedron

Dual surfaces and the degree equations

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Example

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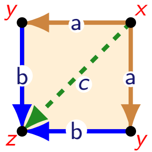

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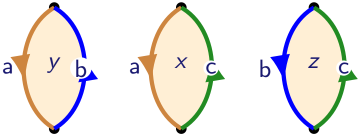
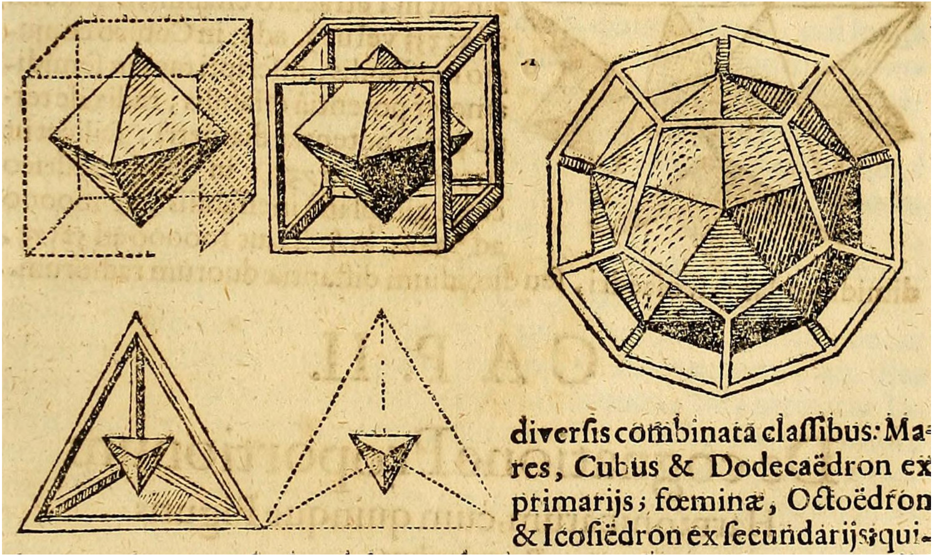


We will see better examples when we look at Platonic solids

Kepler's Harmonices Mundi



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Recall that a graph G is planar if it can be drawn in $\mathbb{R}^2$ without edge crossings

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- If $e, e' \in E$ then the paths $F(e)$ and $F(e')$ can intersect only at the images of their endpoints

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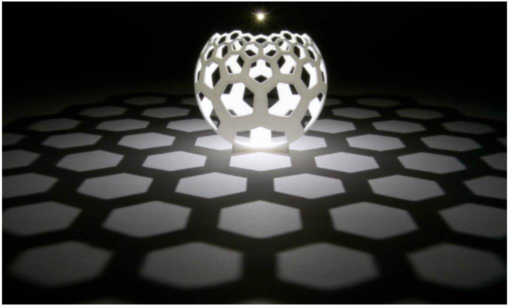
Proof Stereographic projection! (Move G away from ∞.)



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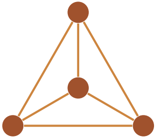
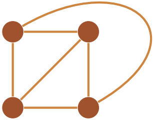
$K_4 \quad =$  $=$ 

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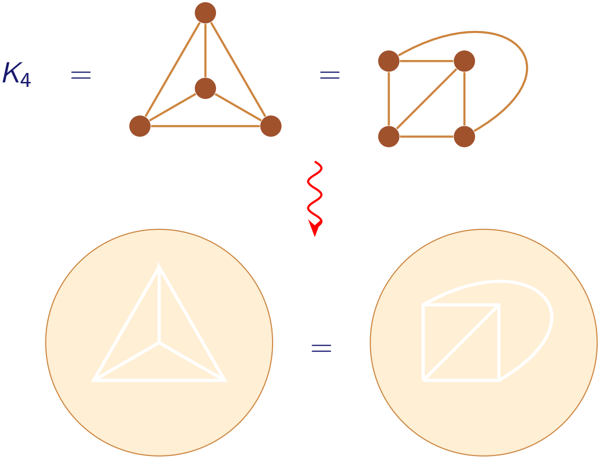

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Remark The argument cheats slightly because we are implicitly assuming that the edges are "nice" curves. This allows us to side-step issues connected with the Jordan curve theorem

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Case 1 G is a tree

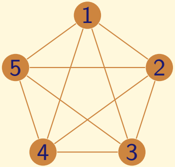
Combine $|V| - |E| = 1$ (previous lectures) and that there is only one face

Case 2 G is not a tree

By $\chi(S^2) = 2$ and the previous theorem

Planarity of K_5

Proposition

The complete graph $K_5 =$ 

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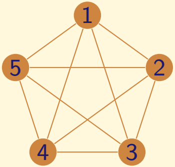
The complete graph $K_5 =$  *is not planar*

Proof Assume that K_5 is planar with $|F|$ faces

We have $|V| = 5$ and $|E| = 10$, so $2 = |V| - |E| + |F| \quad \Longrightarrow \quad |F| = 7$

Let's count the number of faces in this polygonal decomposition differently

- The faces correspond to cycles in K_5
- Every face has at least 3 edges, so by the degree-face equation

$$\Longrightarrow \quad 2|E| = \sum_{f \in F} \deg(f) \geq 3|F|$$

$$\Longrightarrow \quad 2|E| = 20 \geq 21 = 3|F| \qquad \lightning\lightning\lightning$$

Hence, the complete graph K_5 is not planar

Planarity of complete graphs

Corollary

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Proof

K_5 sits in K_n for $n \geq 5$, and the previous theorem applies

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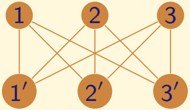
The bipartite graph $K_{3,3} =$ 

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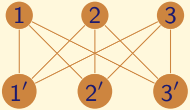
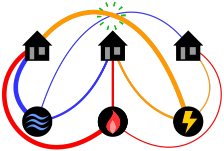
The bipartite graph $K_{3,3} =$  *is not planar*

Proof Tutorials



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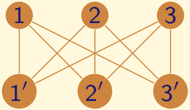
The bipartite graph $K_{3,3} =$  *is not planar*

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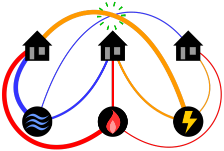

Theorem (Kuratowski)

Let G *be a graph. Then* G *if planar if and only if it has no subgraph isomorphic to a subdivision of* K_5 *or* $K_{3,3}$

The proof is out of the scope of this unit!

Platonic solids

A Platonic solid is a surface that has a polygonal decomposition that is constructed using regular n-gons of the same shape and size such that the same number of polygons meet at every vertex

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	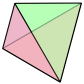	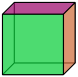	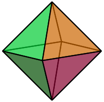	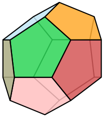	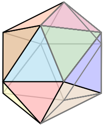
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	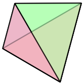	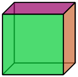	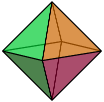	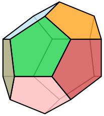	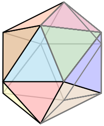
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	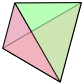	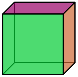	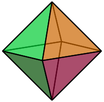	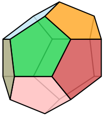	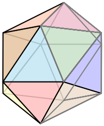
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- Can we understand them as polygonal decompositions of the sphere?

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The equations above give:

$$|E| = \left(\frac{1}{p} + \frac{1}{n} - \frac{1}{2}\right)^{-1}, \; |V| = \frac{2|E|}{p} \text{ and } |F| = \frac{2|E|}{n}$$

Classification of Platonic solids

Theorem

The complete list of Platonic solids is:

p	n	$\frac{1}{p}+\frac{1}{n}$	$e=\left(\frac{1}{p}+\frac{1}{n}-\frac{1}{2}\right)^{-1}$	$v=\frac{2e}{p}$	$f=\frac{2e}{n}$	*Platonic solid*
3	3	$\frac{2}{3}$	6	4	4	*Tetrahedron*
3	4	$\frac{7}{12}$	12	8	6	*Cube*
3	5	$\frac{8}{15}$	30	20	12	*Dodecahedron*
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Proof Since $\frac{1}{p}+\frac{1}{n}>\frac{1}{2}$ and $p, n \geq 3$ we get $n<6$ since $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$
Case-by-case we then get the above values for p, n as the only possible values for Platonic solids.

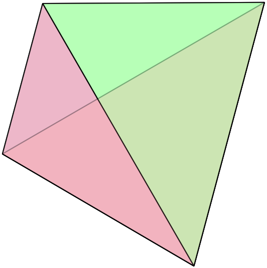
To prove existence we need to actually construct them

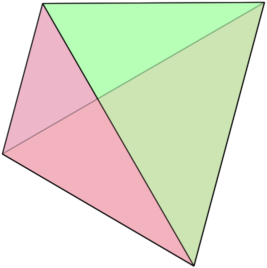
Classification of Platonic solids

Proof Continued Their construction is well-known:

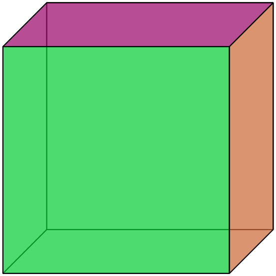
Dual tetrahedron = tetrahedron

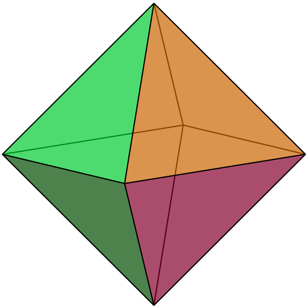
There is a symmetry in the Platonic solids given by $(p, n) \leftrightarrow (n, p)$. This corresponds to taking the dual surface



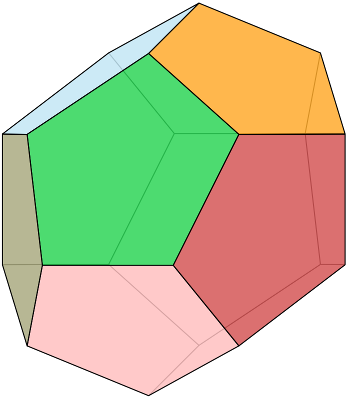


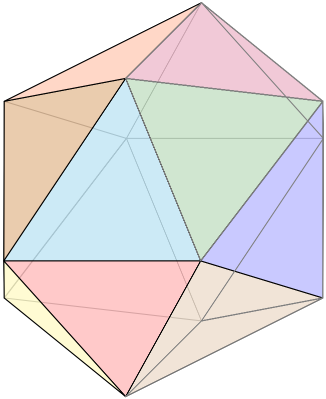
Cube and octahedron





Dodecahedron and icosahedron





Platonic soccer balls

Here are two dodecahedral decompositions of S^2

Soccer ball

Example A ball is made by gluing together triangles and octagons so that each octagon is connected to four non-touching triangles. Determine the number of octagons and triangles used

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Write $|F| = o + t$, where $o = \#$octagons and $t = \#$triangles

$$\implies \quad 2 = |V| - |E| + o + t$$

We have:

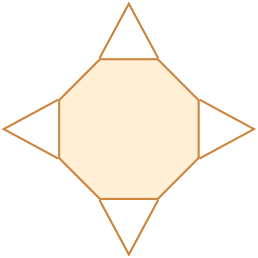
- vertex-degree equation: $3|V| = 2|E|$
- face-degree equation: $2|E| = 3t + 8o$
- Every octagon meets 4 triangles,

$\implies \quad 3t = 4o \quad \implies \quad 2|E| = 12o$

$\implies \quad 2 = o\big(4 - 6 + 1 + \frac{4}{3}\big) = \frac{o}{3}$

$\implies \quad o = 6$ and $t = 8$

$\implies \quad |E| = 36$ and $|V| = 24$



The octacube

As with the Platonic solids, we have only shown that if such a surfaces exists then there are 6 octagons, 8 triangles, 24 vertices and 36 edges but we have not shown that such a surface exists!

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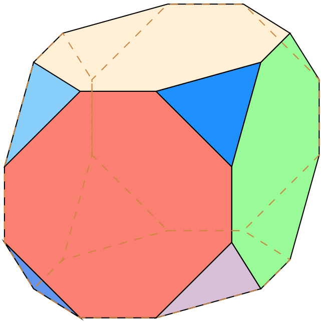
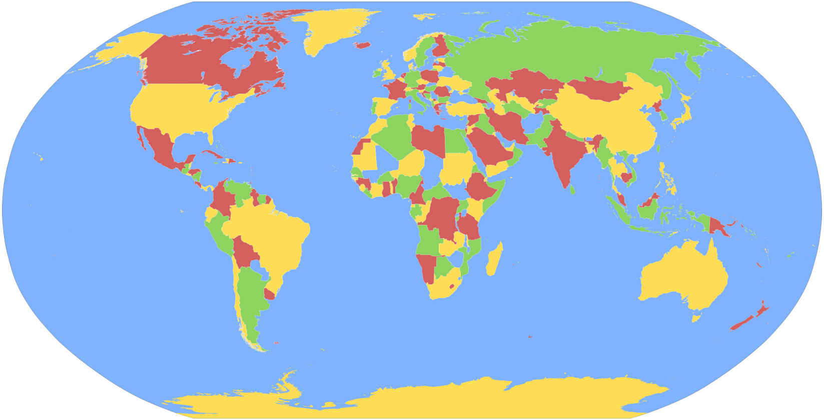
In fact, this surface does exist and it can be constructed by cutting triangular corners off a cube



Coloring maps

Question

How many different colors do you need to color a map so that adjacent countries have different colors?



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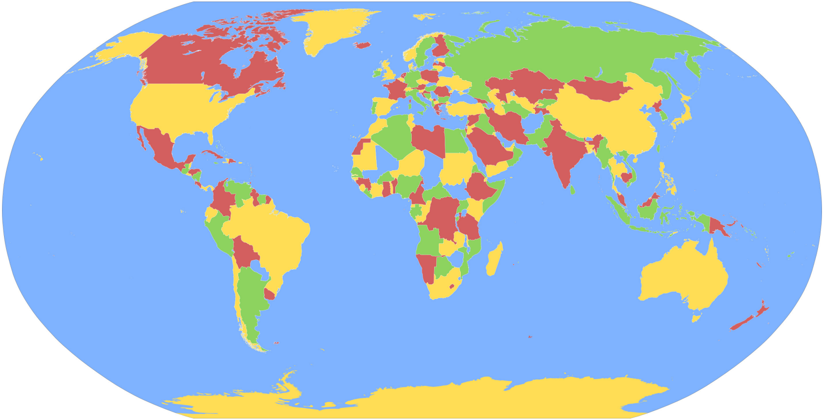

A map is a polygonal decomposition. The answer to this question involves the same ideas we used to understand Platonic solids

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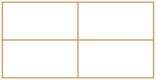

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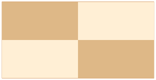

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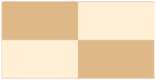

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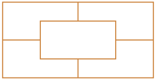

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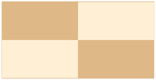

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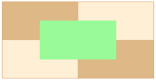

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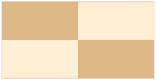

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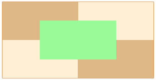

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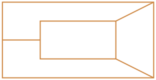

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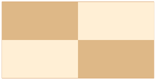

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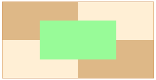

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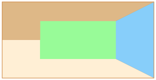

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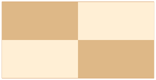

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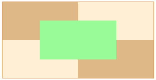

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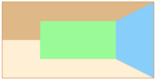

$C_P(\mathbb{D}^2) = 4$

$\implies \quad C(\mathbb{D}^2) \geq 4$

For maps of the world we are most interested in $C(\mathbb{D}^2) = C(S^2)$

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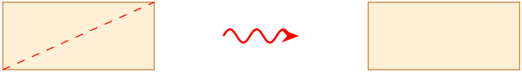

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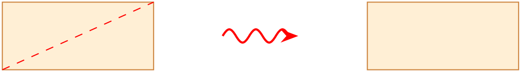
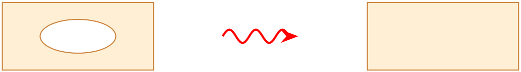


- No region contains a hole



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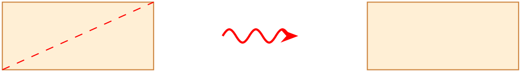
- All vertices have degree at least 3
- No region (i.e. face or polygon) has a border with itself



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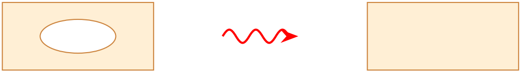

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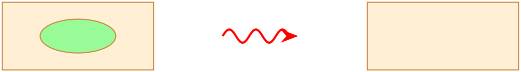

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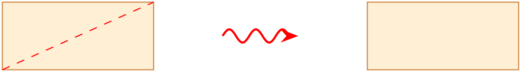

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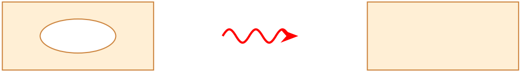

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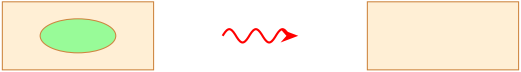

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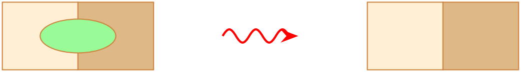

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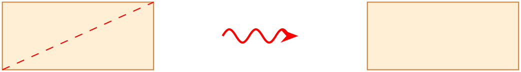

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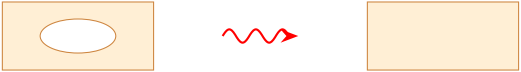

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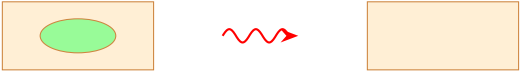

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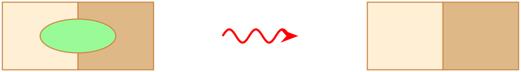

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Understanding map colourings

The basic idea is to use the Euler characteristic and the degree-vertex and degree-face equations to understand colourings

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Remark For a Platonic solid that is made from n-gons with p polygons meeting at each vertex we have $\partial_V = p$ and $\partial_F = n$

Bounding the face degree

Lemma

Suppose that M is a map on a closed surface S. Then

$$\partial_F = \left(1 - \frac{\chi(S)}{|F|}\right) \Big/ \left(\frac{1}{2} - \frac{1}{\partial_V}\right)$$

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Let M *be a map on* S^2 *or* $\mathbb{P}^2$*. Then* $\partial_F < 6$

Maps on sphere and projective planes

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2. If the average face degree $\partial_F < 6$ then there must be at least one face f with $\deg(f) \leq 5$
   This observation will be important when we prove the Five color theorem (not quite the four color theorem)

Topology – week 11

Math3061

Daniel Tubbenhauer, University of Sydney

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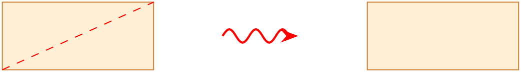

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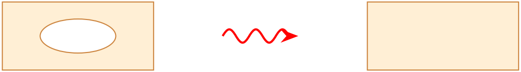

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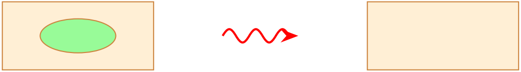

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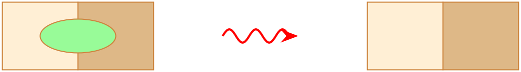

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- If M is a map on a closed surface S, then we proved that
$$\partial_F \leq 6\Big(1 - \frac{\chi(S)}{|F|}\Big)$$

Maps on surfaces with $\chi(S) \leq 0$

Lemma

Let M be a map on a closed surface S with $\chi(S) \leq 0$. Then

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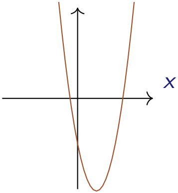

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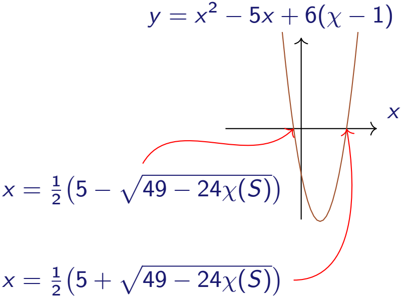

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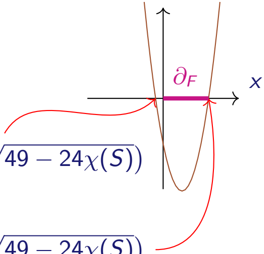

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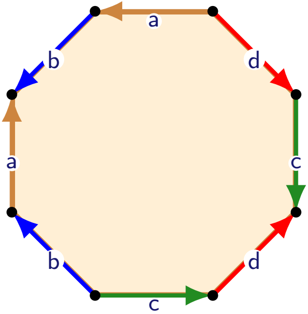

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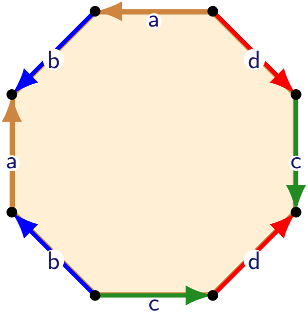

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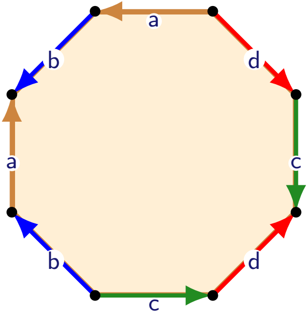

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This is not a contradiction because we are assuming that no region has a border with itself, which is never true for a polygonal decomposition that has only one face

Heawood's theorem

Theorem

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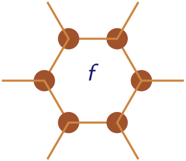

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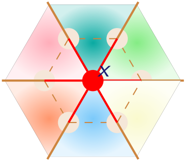

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3. If $S = S^2$ then $\chi(S^2) = 2$ so $\frac{7+\sqrt{49-24\chi(S)}}{2} = 4$!?

Why is $C(S^2) \geq 4$ easy to see? Well:

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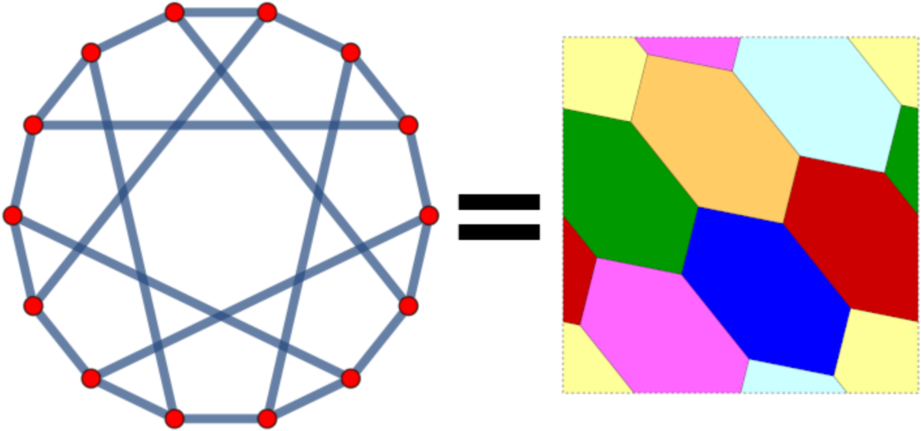
Hence, $C(\mathbb{T}) = 7$ (see the tutorials)

Hexagons on the torus



Coloring the projective plane

Heawood's estimate for the projective plane $\mathbb{P}^2$ is

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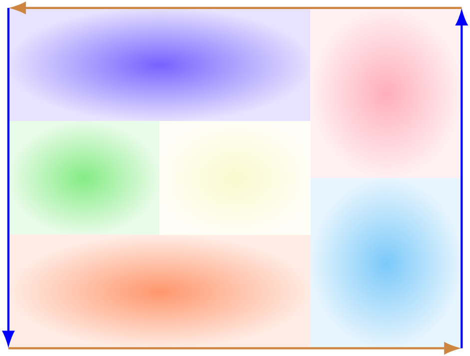

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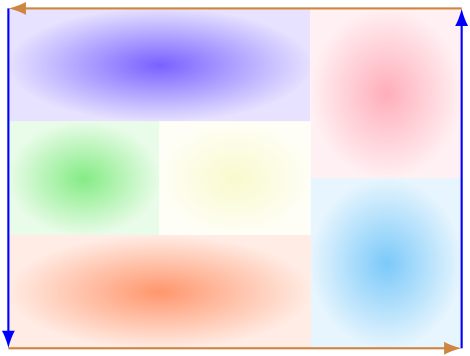

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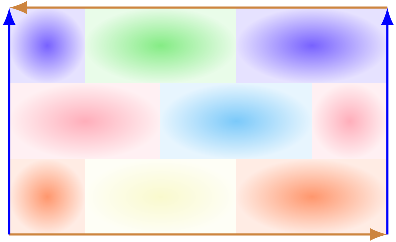
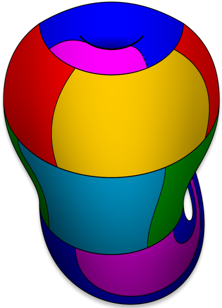
In fact, Franklin (1930) proved that $C(\mathbb{K}) = 6$





Using these maps you can show that $C(\mathbb{K}) \geq 6$

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By stereographic projection, it is enough to show that $C(S^2) \leq 5$

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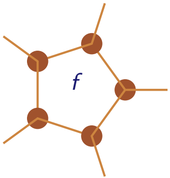

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As in the proof of Heawood's theorem, the idea is now to modify the 5-coloring on N to give a 5-coloring on M. This time the proof is more complicated and there are several cases to consider

Case 1: $\deg(f) < 5$

If $\deg(f) < 5$ then the 5-coloring of N has at most 4 colors for the faces in N around x

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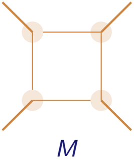

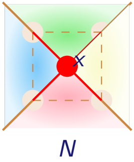

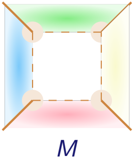

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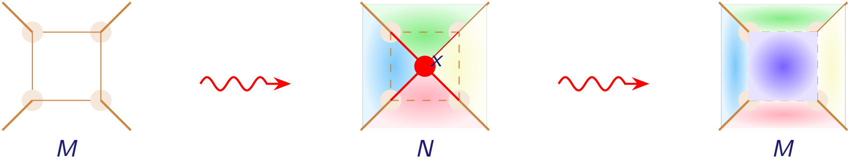

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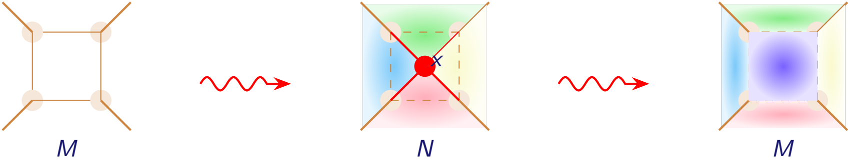

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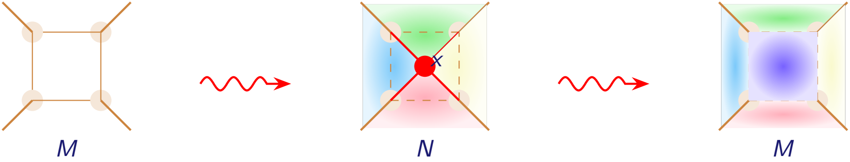

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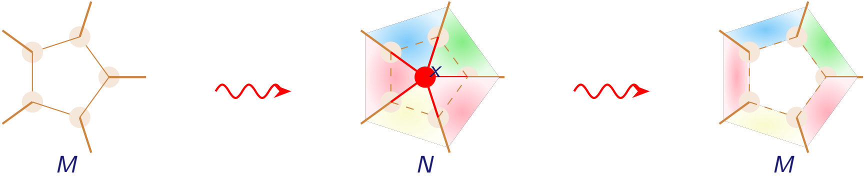

Proof of the Five color Theorem 2

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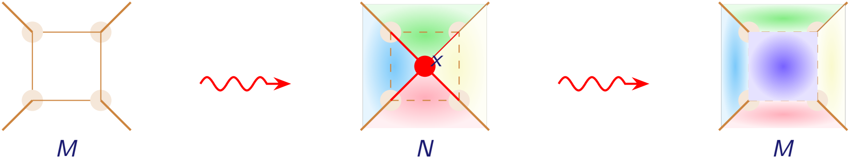

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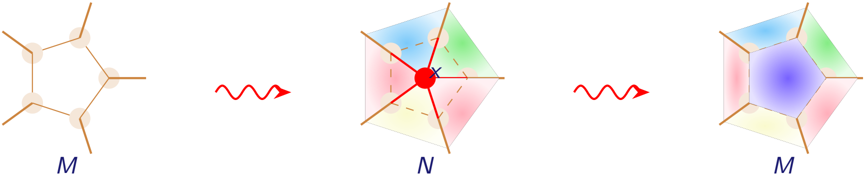
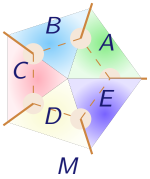


As we have used at most 4 colors in N around x, it follows that M is 5-colorable

Case 3: $\deg(f) = 5$ and all of the colors in N around x are different



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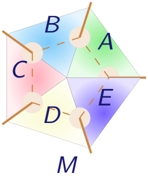

Label the regions A–E as shown.

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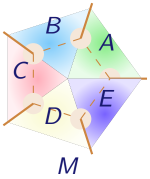

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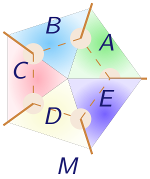

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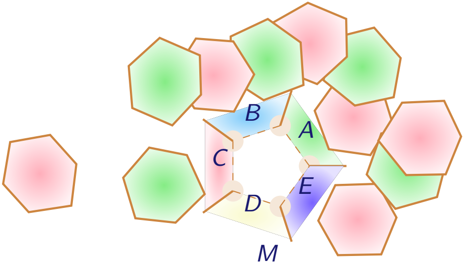

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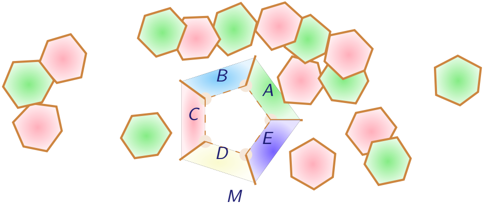

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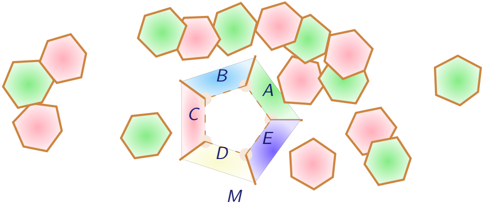
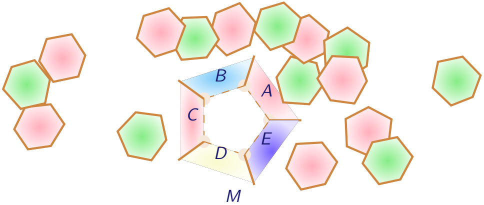


$\Longrightarrow$ Swapping the colors A and C in the connected component of P that contains A gives a new map N' with a valid coloring



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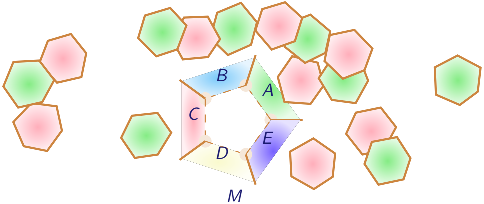

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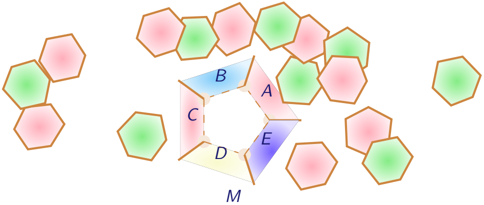

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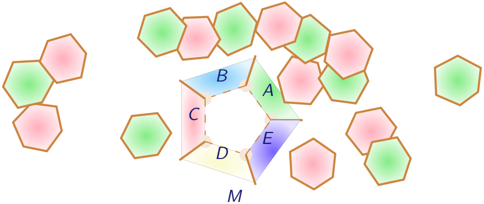

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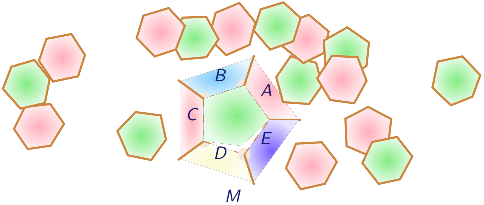

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Proof of the Five color Theorem 5

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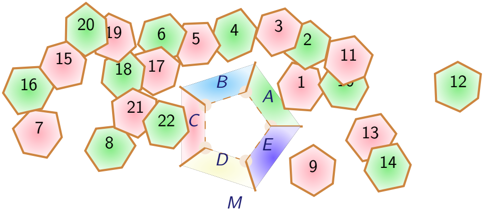

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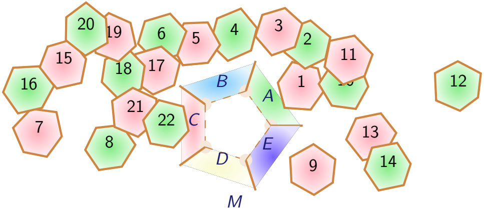

$\Longrightarrow$ As A and C are connected, B and E cannot be connected!

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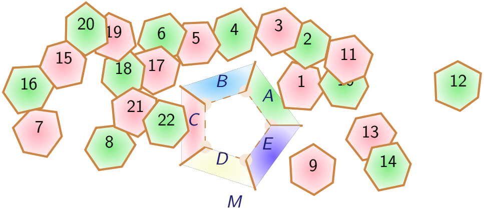

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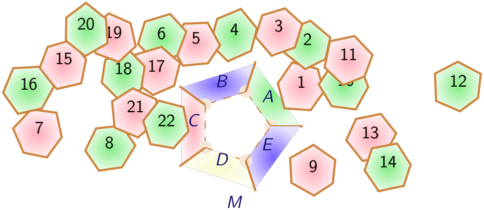

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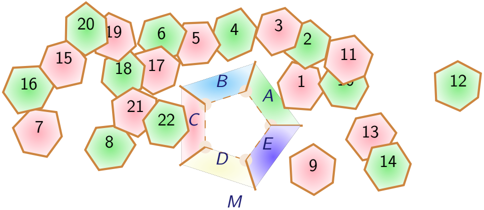

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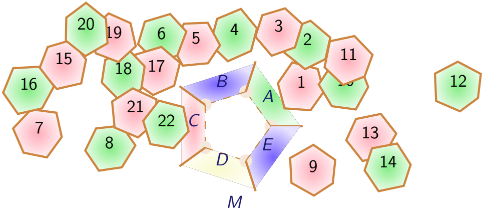

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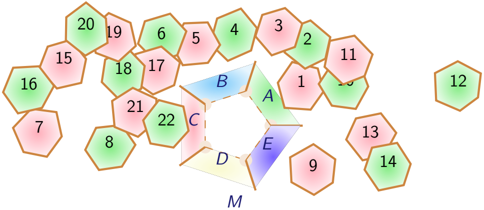

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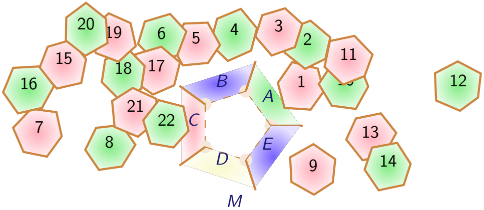

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This completes the proof of the Five color Theorem

Knots

Intuitive definition A knot is a piece of string with the ends tied together

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Definition

A knot is the image of an injective continuous map from S^1 into $\mathbb{R}^3$, where $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is the unit circle in $\mathbb{R}^2$

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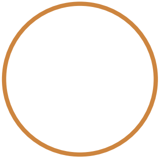

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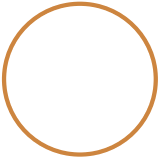

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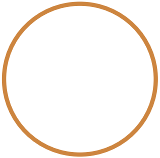

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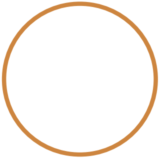

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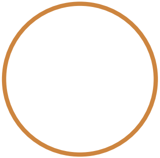

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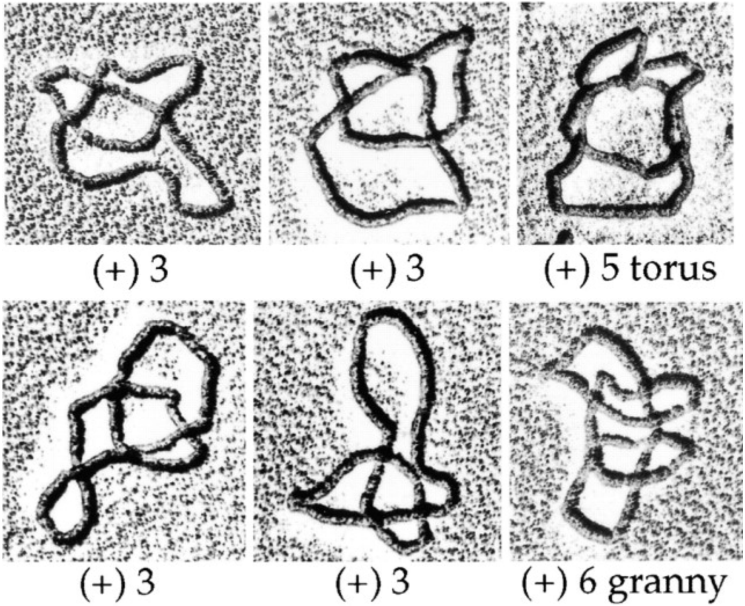
Reverse trefoil

Heart knot

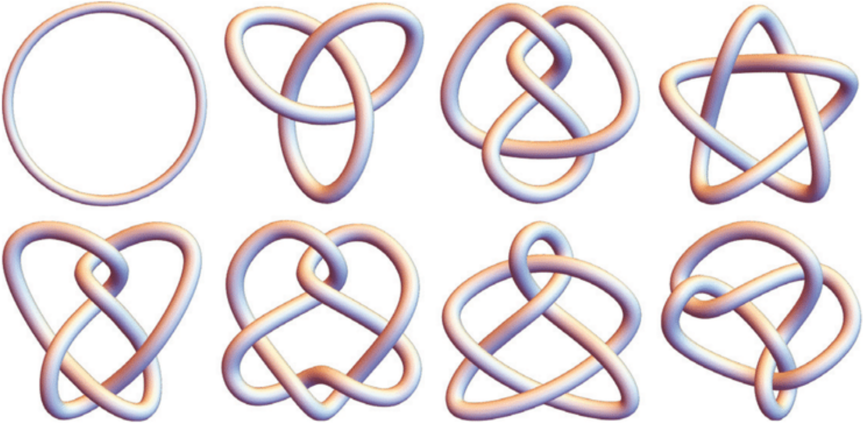
Knot theory is a beautiful mathematical subject with applications in mathematics, computer science, computer chip design, biology, ...

A picture of life



Another picture of life

More knots



Basic question in knot theory

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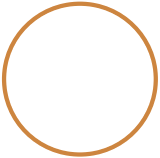

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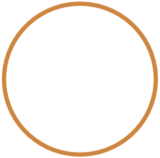

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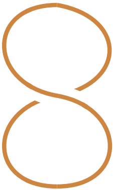

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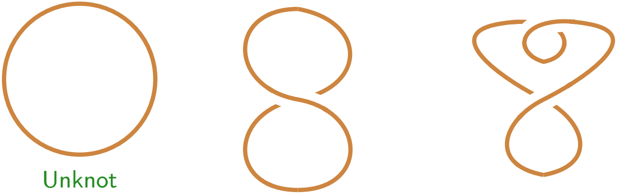

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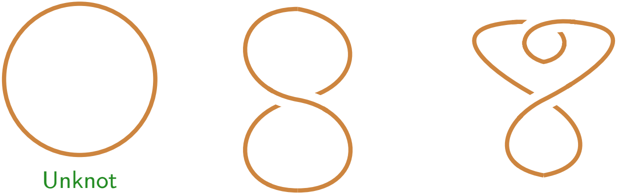

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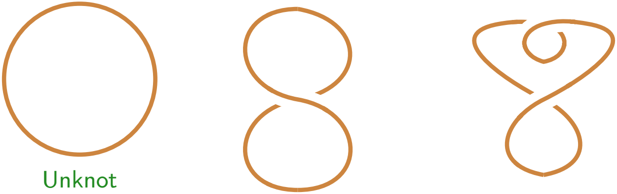

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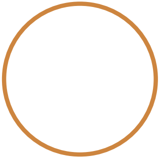

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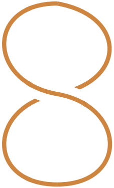
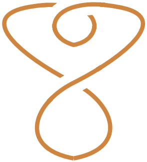
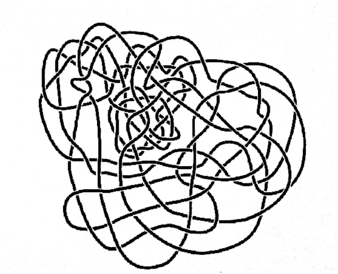




Another unknot

It is difficult to tell if a knot is the unknot



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Two knots K and L are equivalent, and we write $K \cong L$, if there exists a continuous map, or ambient isotopy, $f : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$ such that

1. for each $t \in [0,1]$ the map $\mathbb{R}^3 \to \mathbb{R}^3; x \mapsto f(x,t)$ is a homeomorphism
2. if $x \in K$ then $f(x,0) = x$, and
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Intuitively, f continuously deforms $K = f(K,0)$ into the knot $L = f(K,1)$

When are two knots the same?

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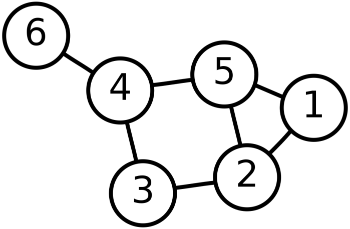
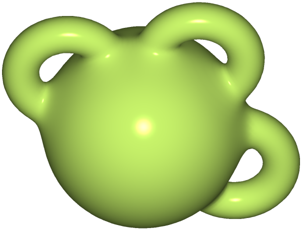
A knot K is trivial if it is equivalent to the unknot otherwise it is non-trivial

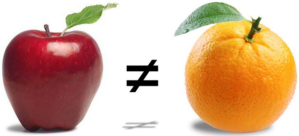
Different notions of “equal”

Objects	Graphs	Surfaces	Knots
Equivalence	Isomorphism of graphs	Homeomorphism	Equivalence of knots

In other words, graphs, surfaces and knots should never be directly compared – they are different beasts







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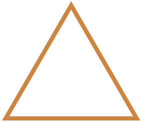

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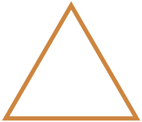

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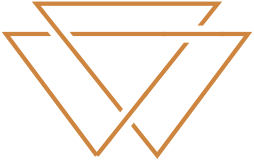

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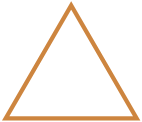

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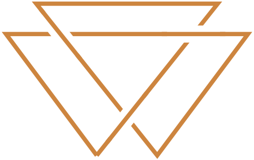

Trefoil

Figure eight

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Unknot

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Figure eight

Remark Two polygonal knots K and L are equivalent if they have a common subdivision

Only polygonal knots

From now on all knots are polygonal knots and we drop the adjective polygonal

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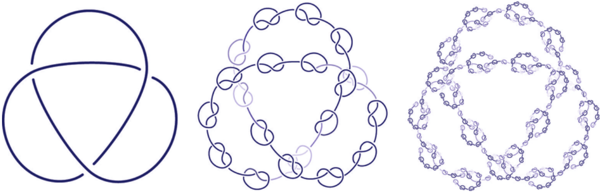
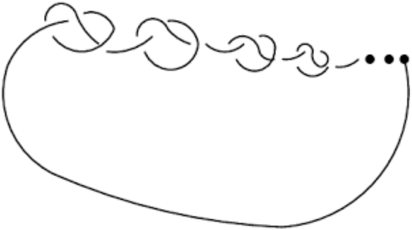
Good (but the limit is not):

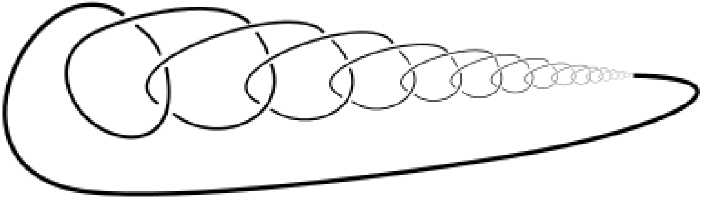


Not good:



Polygonal knots avoid pathologies

These are not polygonal knots:



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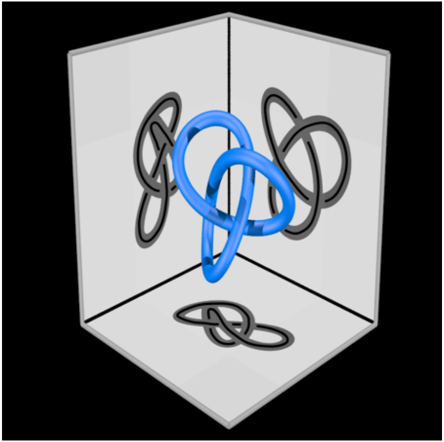
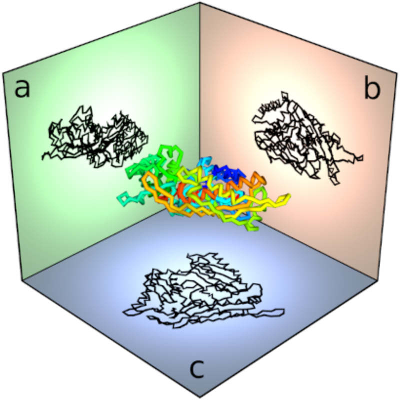
Knot projections are a convenient way of drawing knots but they involve a choice of projection

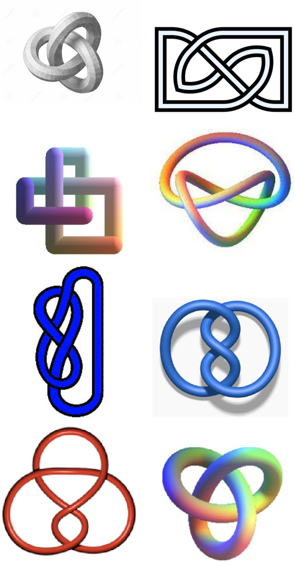
$\Longrightarrow$ Knot projections can be misleading so we have to check that our constructions are independent of the choice of knot projection

Projections = shadows





The trefoil knot times nine



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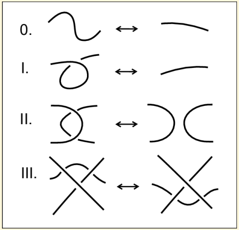
Two knot diagrams represent the same knot if and only if they are related by a (finite) sequence of moves of the following three types



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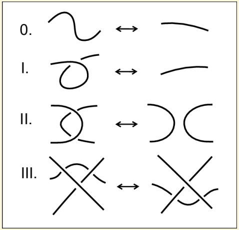

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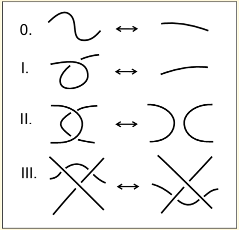

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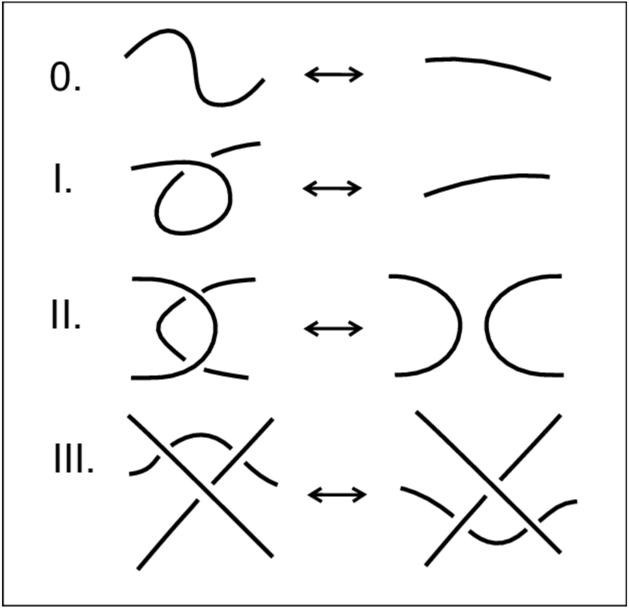
The point: Reidemeister's theorem reduces topology to combinatorics of diagrams

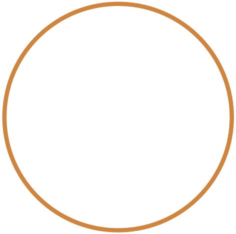
The Reidemeister moves on one slide

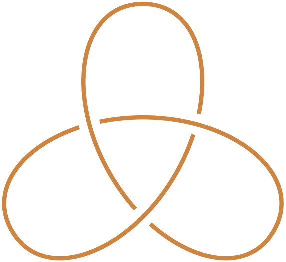


The knotty trefoil

Question

Is the trefoil knot equivalent to the unknot?





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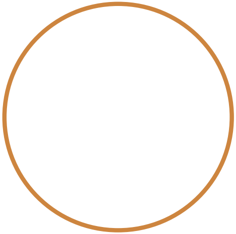

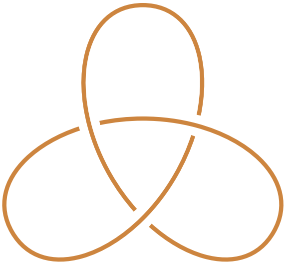

It seems clear that these two knots are different but, so far, we have not seen an easy way to distinguish between them

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Definition

A coloring of a knot (projection) is the assignment of colors to the different segments,or connected components, so that at each crossing all segments have either the same color or they all have different colors and at least two colors are used

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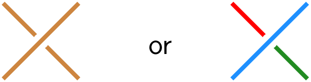

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- As soon as more than one color is used we must use all three colors, so K is 3 colorable if and only if $C_3(K) > 3$

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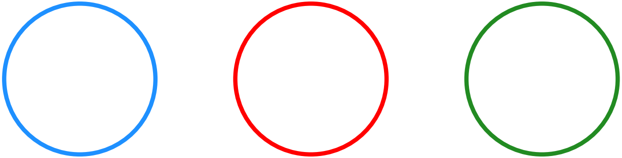
$\Longrightarrow$ $C_3(\text{Unknot}) = 3$ and the Unknot is not 3-colorable



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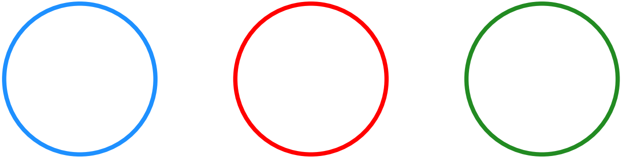
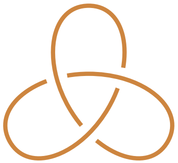


Which of the following are knots are 3-colorable?

coloring the trefoil knot

Question What is $C_3(T)$ if T is the trefoil knot?



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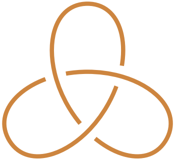

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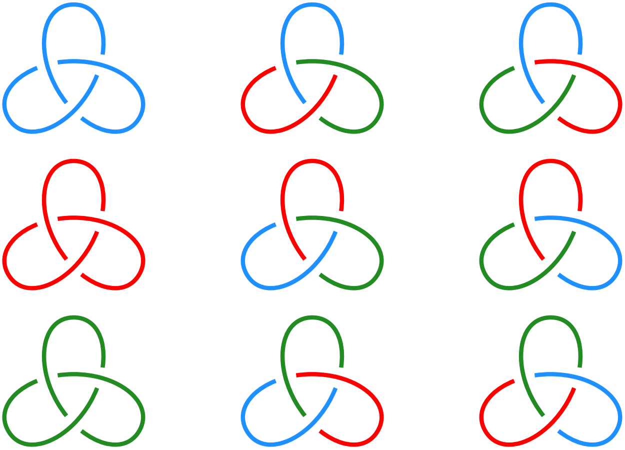

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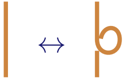
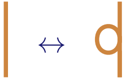
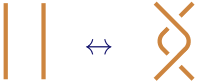
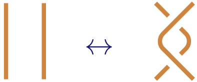
- Twisting  and 
- Looping  and 

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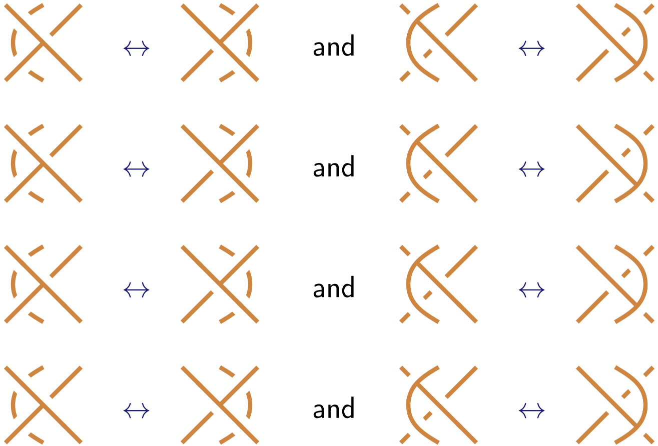


Key point For each Reidemeister move there is a unique way to complete any coloring given the existing colors of the segments going in and out

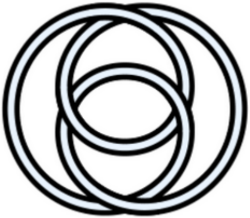
Topology – week 12

Math3061

Daniel Tubbenhauer, University of Sydney

Reidemeister moves are powerful but might be tricky

This is the unknot: $K =$ 

These two knots are equivalent: $K =$  , $K' =$ 

How to show that? Use Reidemeister moves (this is a strongly recommended exercise). But that might be tricky in general, so invariants is what we want.

Connected sums of knots

We used connected sums to construct and classify surfaces

We want an analog of connected sums for knots

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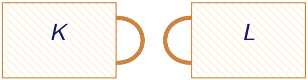

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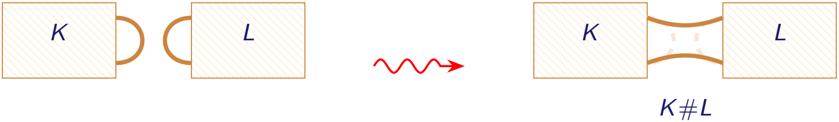

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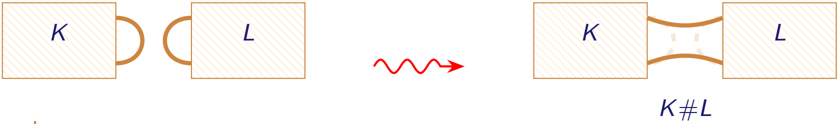

Remarks

- $\#$ does not depend on the choice of knot projections or where you cut either knot, and it is an "addition" or "multiplication":

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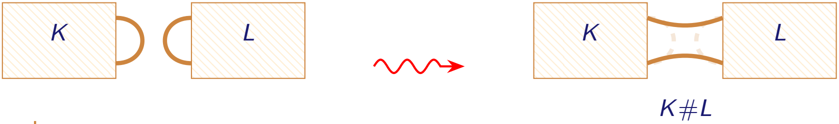

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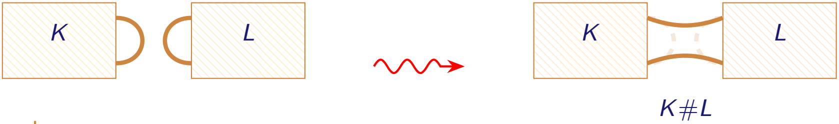

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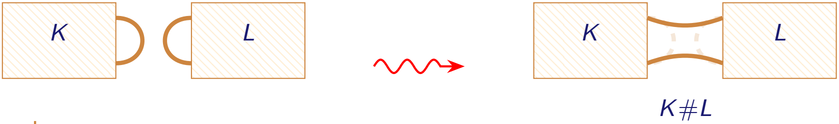

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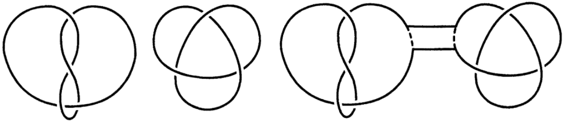

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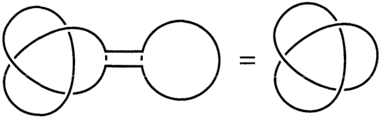
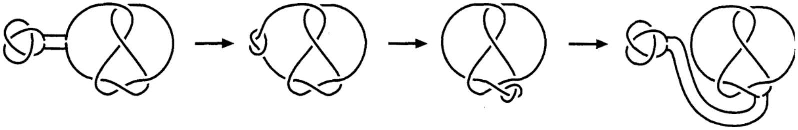


K unknot K



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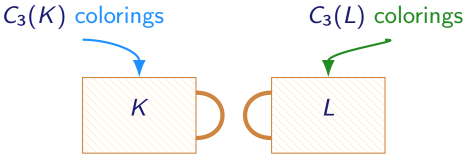

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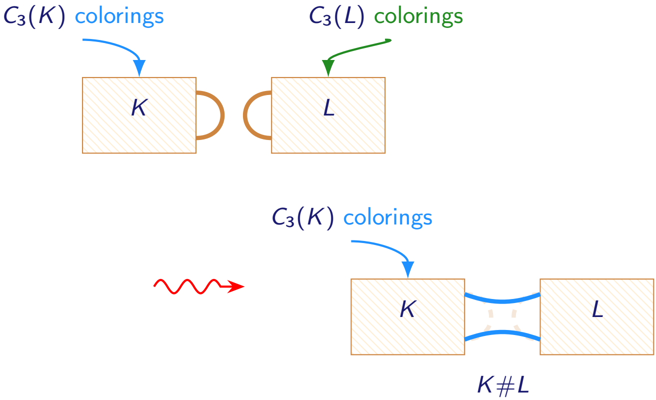

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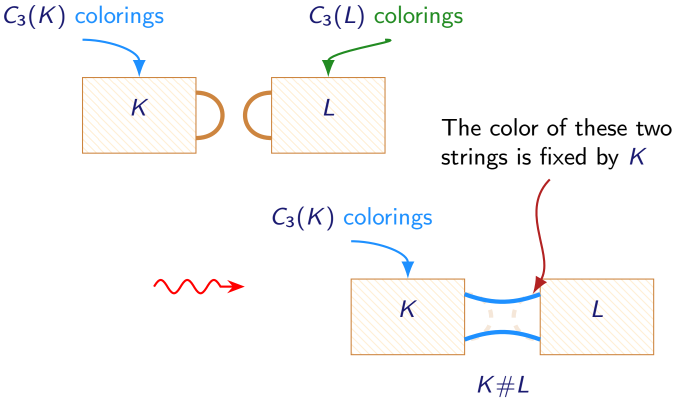

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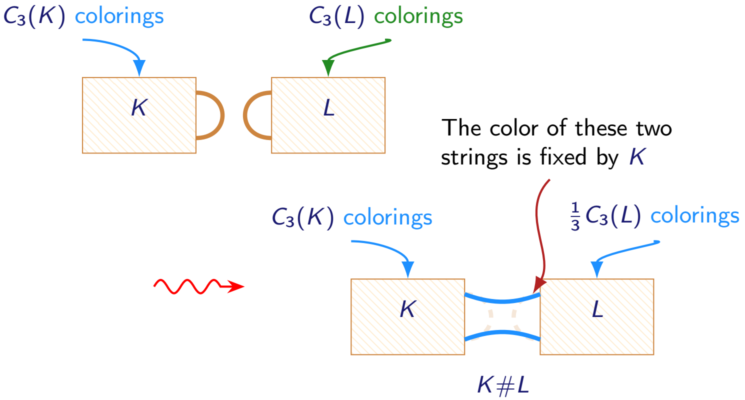

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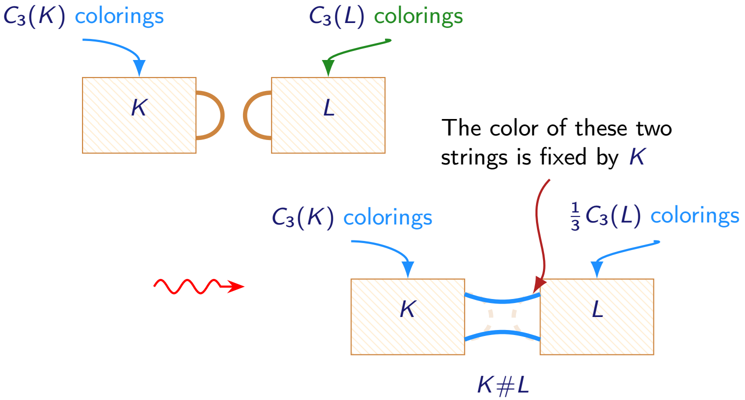

Since the colors of the connecting strands are fixed, there are only $\frac{1}{3}C_3(L)$ ways to 3-color the strands of L inside $K\#L$

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Corollary

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More generally, the same argument shows that if K is 3-colorable then the knots K, $\#^2 K$, $\#^3 K$,... are all inequivalent

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Definition

The knot K is a composite knot if it has a factorisation $K = L \# M$, where L and M are not the unknot
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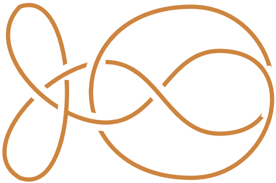

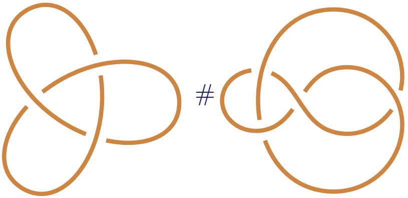

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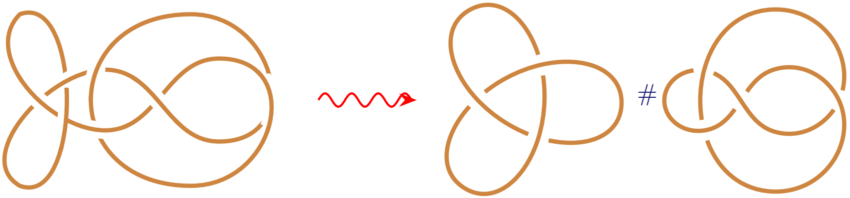

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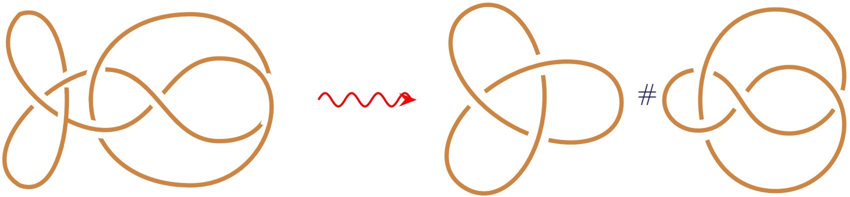

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In fact, we don't yet know that the figure eight knot is not the unknot!!

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Remark It is a big open question if $\mathrm{cross}(K\#L) = \mathrm{cross}(K) + \mathrm{cross}(L)$

This is only known to be true for certain types of knots such as alternating knots, which we will meet soon

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Conversely, we can ask how many prime knots there are

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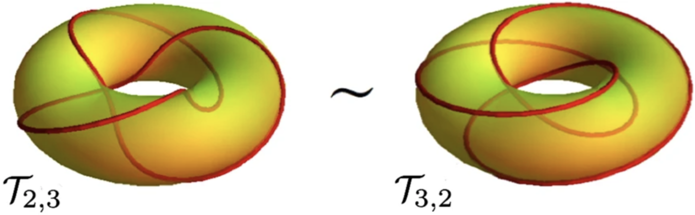

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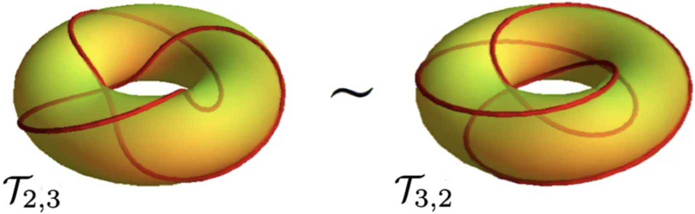

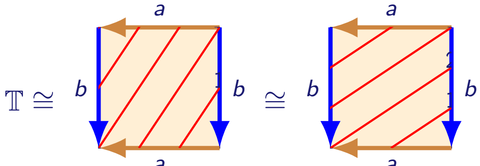

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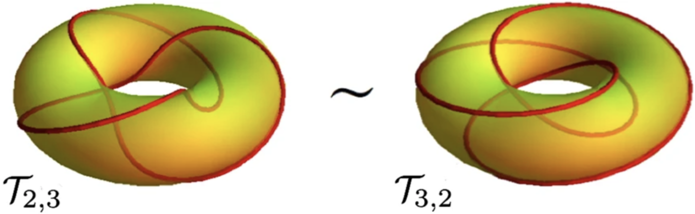

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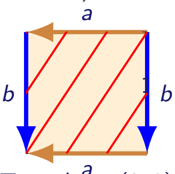

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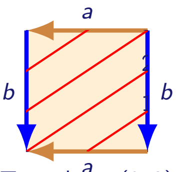

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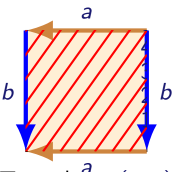

Torus knot $(5, 7)$

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The number of prime knots with n-crossings

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0	1	1	2	3	7	21	49	165	552	2176	9988	46972

As is common, knots and their mirror images are only counted once

Proof

For $p, g \geq 2$ let the (p, q)-torus knot K lie on an unknotted torus $T \subset S^3$ and let the 2-sphere S define a decomposition of K. We assume that S and T are in general position, that is, $S \cap T$ consists of finitely many disjoint simple closed curves.

Such a curve either meets K, is parallel to it or it bounds a disk D on T with $D \cap K = \emptyset$. Choose γ with $D \cap S = \partial D = \gamma$. Then γ divides S into two disks D', D'' such that $D \cup D'$ and $D \cup D''$ are spheres, $(\cup D') \cap (\cup D'') = D$; hence, D' or D'' can be deformed into D by an isotopy of S^3 which leaves K fixed. By a further small deformation we get rid of one intersection of S with T.

Torus knots are prime - proof sketch

Proof Continued

Consider the curves of $S \cap T$ which intersect K. There are one or two curves of this kind since K intersects S in two points only. If there is one curve it has intersection numbers $+1$ and -1 with K and this implies that it is either isotopic to K or nullhomotopic on T. In the first case K would be the trivial knot. In the second case it bounds a disk D_0 on T and $D_0 \cap T$, plus an arc on S, represents one of the factor knots of K; this factor would be trivial, contradicting the hypothesis.

Torus knots are prime - proof sketch

Proof Continued

The case remains where $S \cap T$ consists of two simple closed curves intersecting K exactly once. These curves are parallel and bound disks in one of the solid tori bounded by T. But this contradicts $p, q \geq 2$

Prime factorisation of knots

Theorem

Suppose that K is not the unknot. Then $K = P_1 \# P_2 \# \ldots \# P_n$, for prime knots $P_1, \ldots, P_n$. Moreover, the multiset of prime knots is a knot invariant

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This can be proved using Seifert surfaces (that we meet later)

Prime factorisation of knots

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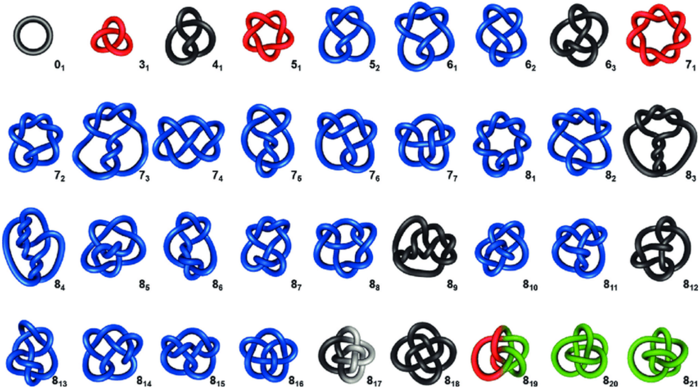
Here is a table of the unknot and the first 36 prime knots:



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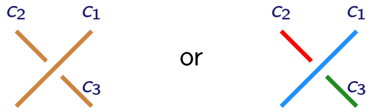
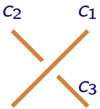
Question

What can we say about $c_1 + c_2 + c_3$ for a 3-coloring?



Possible colorings and the values of $c_1 + c_2 + c_3$

Allowed colorings



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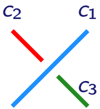
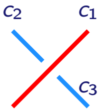


Disallowed colorings



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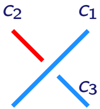

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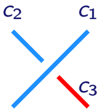
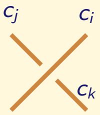


Knot colorings with p-colors

Definition

Let $p \in \mathbb{N}$. A p-coloring of a knot K is a coloring of the segments of K that using colors from $\{0, 1, \dots, p-1\}$ such that



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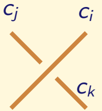

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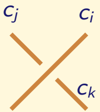

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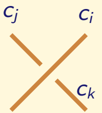

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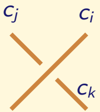

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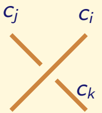

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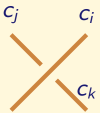

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$\Longrightarrow \quad C_p(K) \geq p$
$\Longrightarrow \quad K$ is p-colorable if and only if $C_p(K) > p$

Colorability is a knot invariant

Theorem

Suppose that $p \geq 3$. Then $C_p(K)$ and p-colorability are both knot invariants

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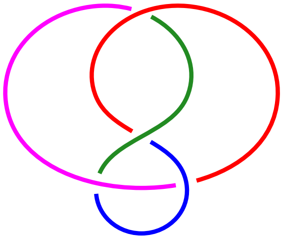
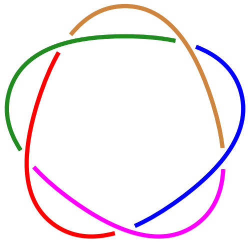
Question

Is there an easy way to tell if a knot is p-colorable?

Are the following knots 4-colorable, 5-colorable, ... ?





Examples of p-colorings

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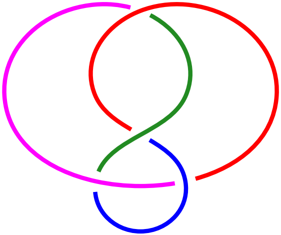

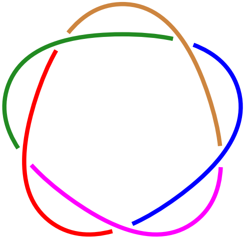

We need a better way to determine if a knot is p-colorable!

$\rightsquigarrow$ Use linear algebra!

The trefoil knot is knotted!

Corollary

The trefoil knot is not the unknot

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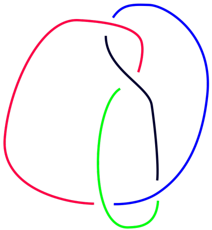
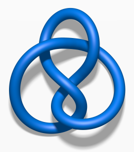


The trefoil knot in comparison

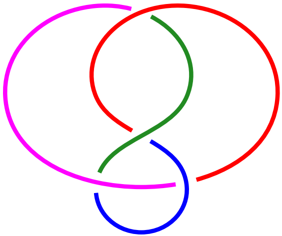
$\neq$

or



Colorful linear algebra

Consider the figure eight knot.



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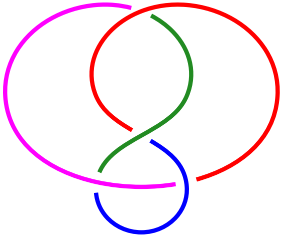
Label the segments c_1, c_2, c_3, c_4 in traveling order around the knot



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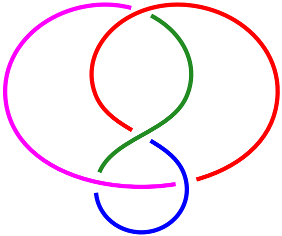

In matrix form this becomes $M_K \underline{C} \equiv \underline{0} \pmod{p}$, where

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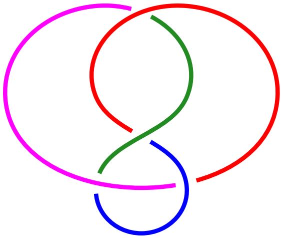

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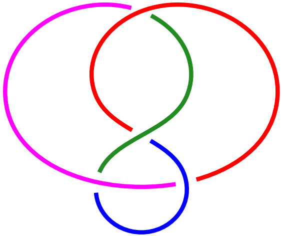

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We have reduced finding $c_1, \ldots, c_4$ to linear algebra!

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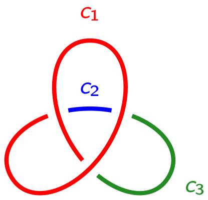
An atypical example



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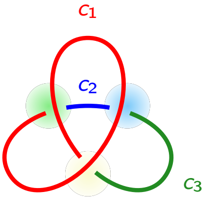

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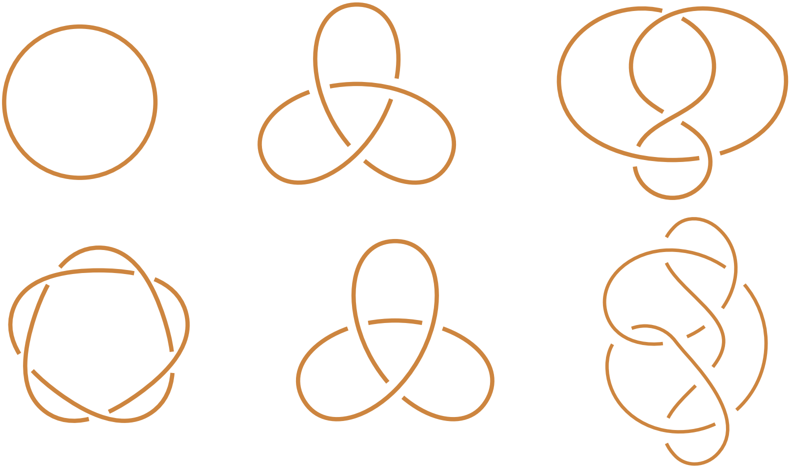

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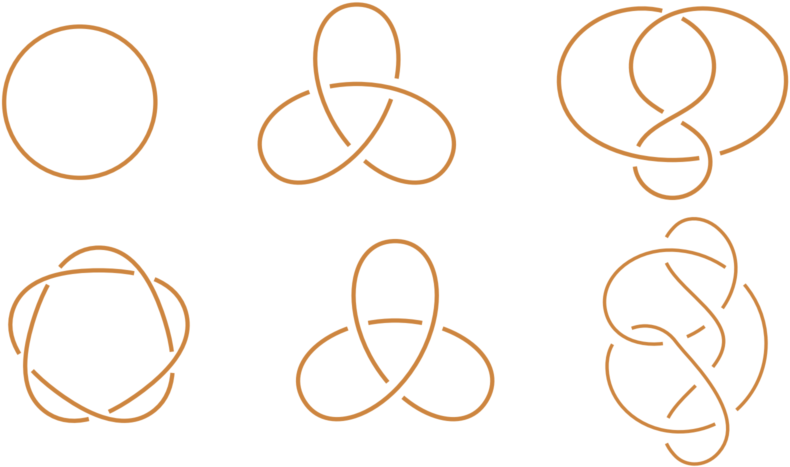

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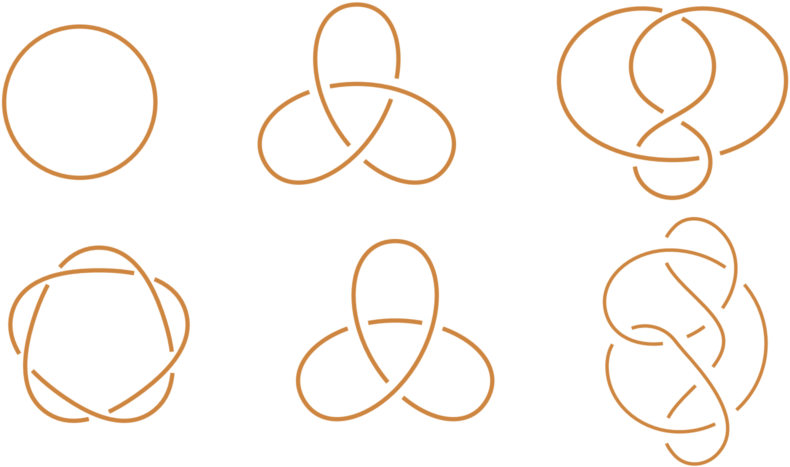

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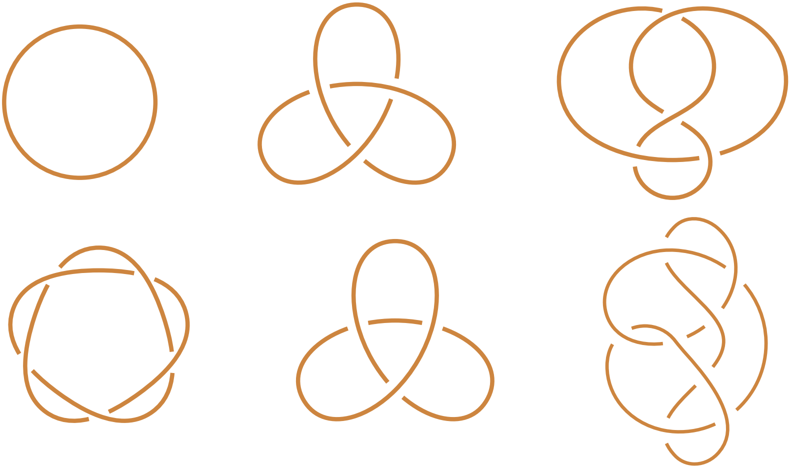
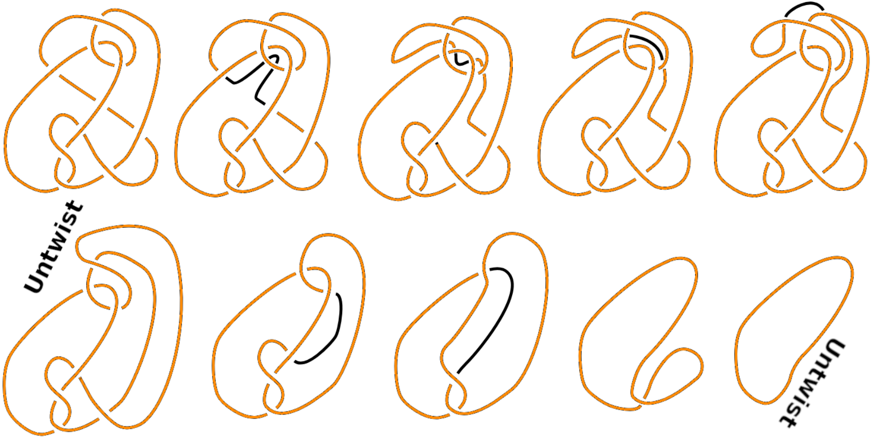


$\Longrightarrow$ Being alternating is not a knot invariant

Alternating knots – careful with projections

The unknot is alternating, but it can have non-alternating projections:



Similarly, for other knots

Knot matrices for alternating knots

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Knot matrices for alternating knots

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$\Longrightarrow$ every segment starts as an under-string, becomes an over-string and finishes as an under-string

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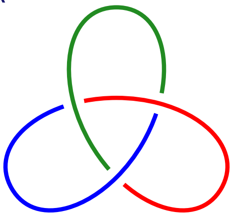
We will mainly consider colorings of alternating knots

Knot matrix examples

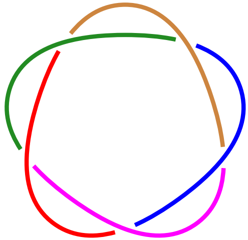
$$M_K = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$K =$



$$M_L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L =$



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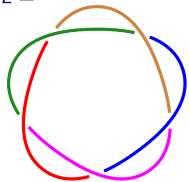
Proof

(1) Since the knot is alternating every colored strand contributes 2 once and -1 twice (see below) and dually from crossings

$$M_L = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$L =$ 

Properties of the knot matrix

Proof Continued

(2) By (1), the respective vector is an eigenvector with eigenvalue zero

(3) By (2) there is an zero eigenvector, so the kernel is nontrivial

Minors of a matrix

The (r, c)-minor of an $n \times n$ matrix M is the $(n-1) \times (n-1)$-matrix M_{rc} obtained by deleting row r and column c from M)

$$M = \begin{bmatrix} a_{11} & \cdots\cdots & a_{1c} & \cdots\cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots\cdots & a_{rc} & \cdots\cdots & a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots\cdots & a_{nc} & \cdots\cdots & a_{nn} \end{bmatrix}$$

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Then $$\det(M + \mathbb{I}) = \det \begin{bmatrix} m_{11}+1 & m_{12}+1 & \cdots & m_{1n}+1 \\ m_{21}+1 & m_{22}+1 & \cdots & m_{1n}+1 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1}+1 & m_{n2}+1 & \cdots & m_{nn}+1 \end{bmatrix}$$

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By the same argument, if $1 \le r, c \le n$ then

$$\det(M + \mathbb{I}) = (-1)^{r+c} n^2 \det M_{rc}$$

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Hence, K is p-colorable if and only if and only if there exist $c_2, \dots, c_n$ such that

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$$M_K \begin{bmatrix} 0 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \equiv 0 \pmod p \quad \Longleftrightarrow \quad (M_K)_{11} \begin{bmatrix} c_2 \\ \vdots \\ c_n \end{bmatrix} \equiv 0 \pmod p$$

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Hence, K is p-colorable if and only if and only if there exist $c_2, \ldots, c_n$ such that

$$M_K \begin{bmatrix} 0 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \equiv 0 \pmod{p} \iff (M_K)_{11} \begin{bmatrix} c_2 \\ \vdots \\ c_n \end{bmatrix} \equiv 0 \pmod{p}$$

$$\iff \det(K) \neq 0 \pmod{p}$$

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3. If K is not alternating then the row sums of M_K are still 0. Therefore, the argument used to prove the theorem shows that K is p-colorable if and only if p divides $(M_K)_{rc}$, for some r, c.

Colorability of the figure eight knot

Summary of how to determine p-colorability

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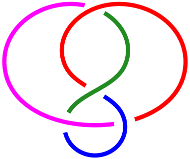
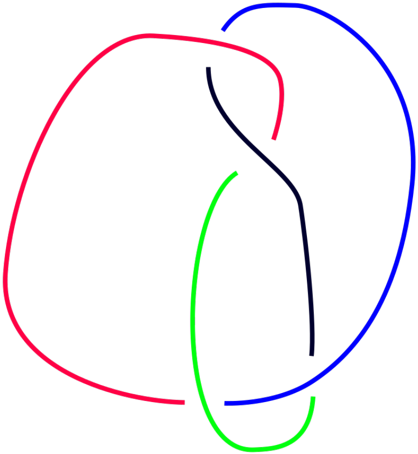
3. Compute the knot determinant $\det(K) = |\det(M_K)_{11}|$
4. Check if p divides $\det(K)$

$$M_K = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \end{pmatrix}$$

$K =$ 

The determinant is five, so the figure eight knot is five-colorable (and only five colorable)



Thus, the figure eight knot is not trivial (it has strictly more than five 5-colorings) and also not the trefoil knot

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We will prove this result by giving an algorithm for constructing a Seifert surface for any knot

Proof Real world version

Take a knot, build out of wire, and put it into soap

The minimal surface you get is a Seifert surface

Constructing Seifert surfaces

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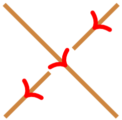
Step 2 At each crossing cut the over-string and join the incoming and outgoing strings; the knot is then a disjoint union of Seifert circles



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Constructing Seifert surfaces

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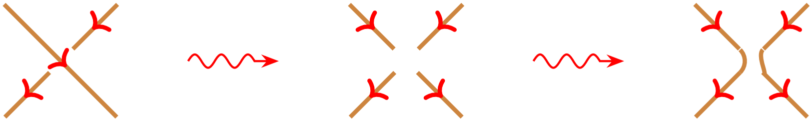

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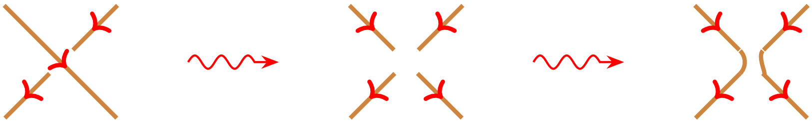

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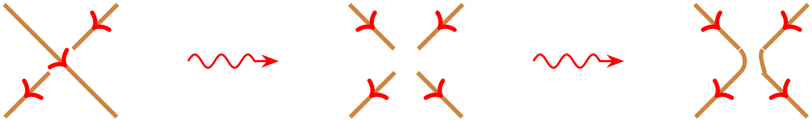

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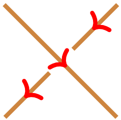

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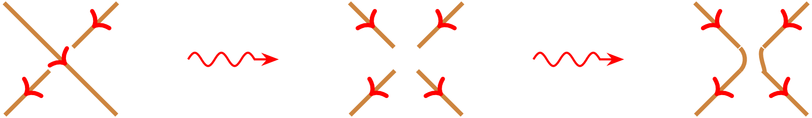

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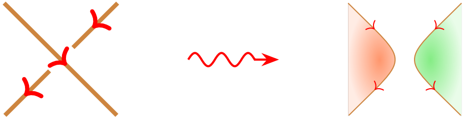

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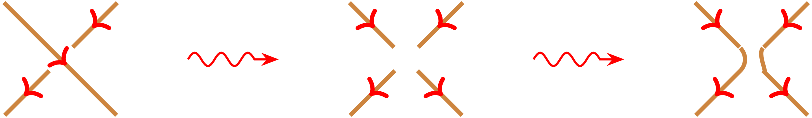

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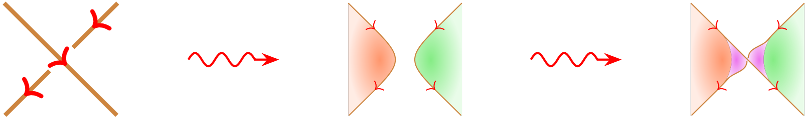
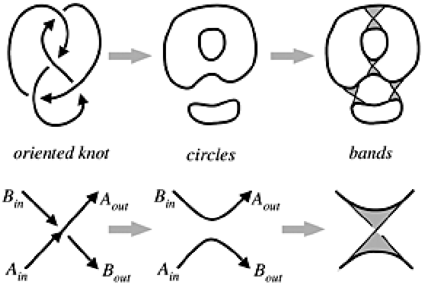
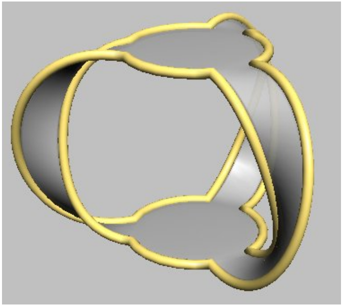
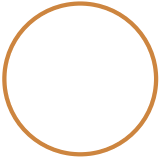


The platform construction





Examples of Seifert surfaces

- Unknot: 

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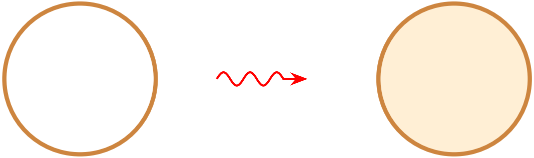
- Unknot: 

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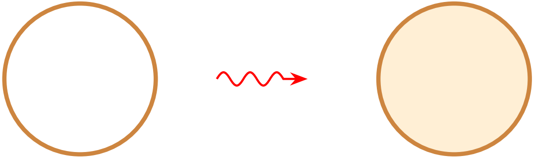
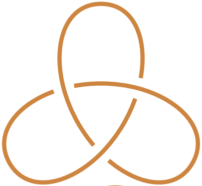
- Unknot: 
- Trefoil 
- Figure eight 

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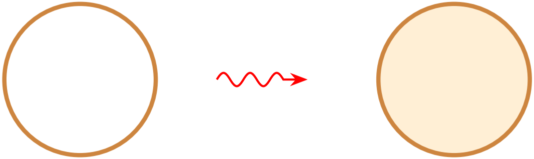

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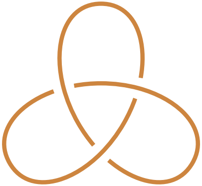

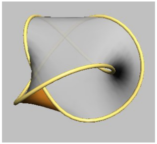

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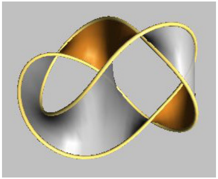

More examples of Seifert surfaces

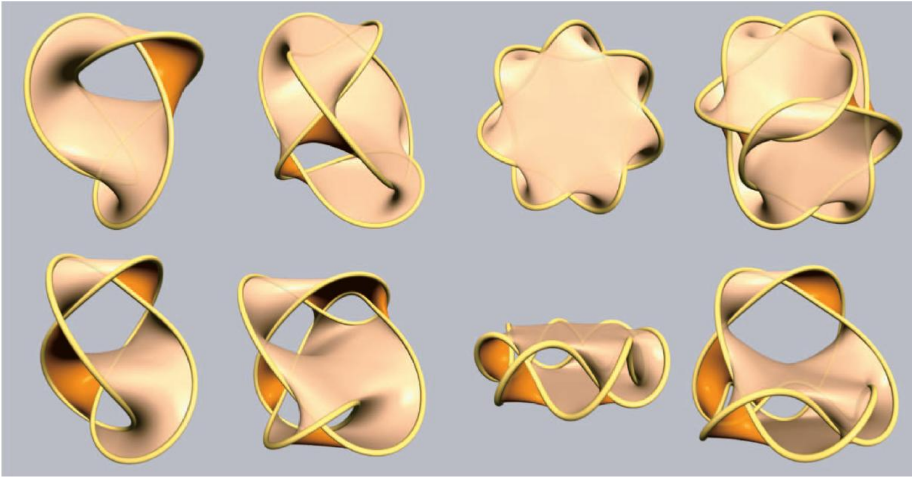

Figure 8$= 4_1$ 6_1 7_1 8_5

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The genus of K is $g(K) = \min\left\{ \frac{1-\chi(S)}{2} \,\middle|\, S \text{ a Seifert surface of } K \right\}$

Remark Used to prove uniqueness of factorization of prime knots

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- $K = \bigcirc$ $\implies$ $g(K) = 0$ as $S \cong \mathbb{D}^2$ and g cannot be smaller, so just checking this one diagram $\bigcirc$ is sufficient

Fact $g(K) = 0 \iff K = \bigcirc$

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Fact $g(K) = 0 \iff K =$ 

Problem K is the trefoil:  ...not very clear how to calculate $g(K)$!

Calculating the knot genus

Proposition

Let S be the Seifert surface with s Seifert circles that is constructed from a knot projection for a knot K with c crossings.
Then $\chi(S) = s - c$ and $g(K) \leq \frac{1+c-s}{2}$

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Proof Recall from tutorials that $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$

Write $S = A \cup B$, where A the union of the Seifert circles and B the union of the twists in S

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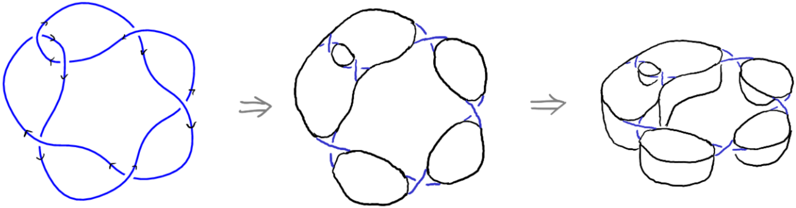
Hence, $g(K) \leq \frac{1-\chi(S)}{2} = \frac{1+c-s}{2}$

Genus of trefoil and figure eight knots

If K has c crossings and s Seifert circles then $g(K) \leq \frac{1+c-s}{2}$

So $g(K) \leq \frac{1+4-3}{2} = 1$



genus=1

Bad news: It can happen that $g(K) < \frac{1-\chi(S)}{2}$!!

Genus of alternating knots

Bad news: It can happen that $g(K) < \frac{1-\chi(S)}{2}$!!

The good news is that there is no bad news for alternating knots

Theorem

Let S be the Seifert surface constructed from an alternating knot projection of K. Then $g(K) = \frac{1-\chi(S)}{2}$

Proof Nontrivial and omitted!

Knot genus is additive

Theorem

Let K and L be knots. Then $g(K\#L) = g(K) + g(L)$

Start of proof It is not hard to see that $S_{K\#L} \cong S_K \#_{\text{strip}} S_L$ (connected sum along a strip connecting the surfaces and boundary cycles). This implies that $g(K\#L) \leq g(K) + g(L)$. The reverse implication is much harder!

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Corollary

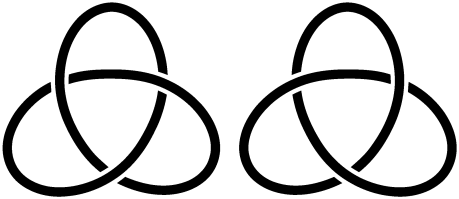
Let K and L be knots, which are not the unknot. Then $K \ncong (K\#L)\#M$ *for any knot* M

Proof If such a knot M existed then

$$g(K) = g\big((K\#L)\#M\big) = g(K) + g(L) + g(M)$$
$$\implies \quad g(M) = -g(L) < 0 \qquad ⚡⚡⚡$$

Left = right-handed trefoil? No idea...

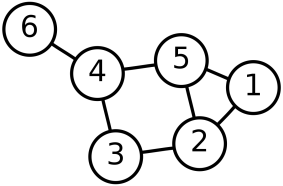
No method we have seen distinguishes these two fellows:

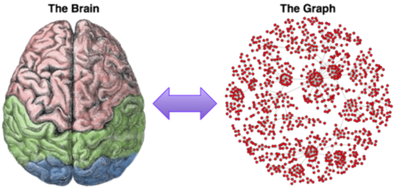


But that has to wait for another time...

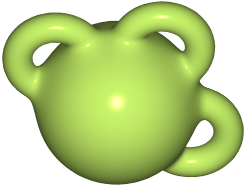
A few take away pictures

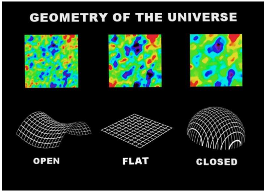
Topic 1: graphs!





Topic 2: surfaces!





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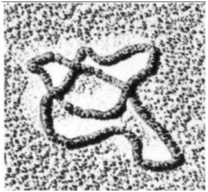


This was my last slide!

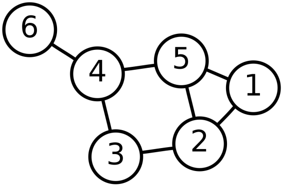
Topology – recollection

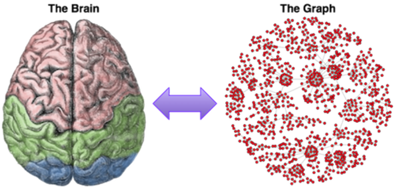
Math3061

Daniel Tubbenhauer, University of Sydney

The three main topics

Topic 1: graphs!





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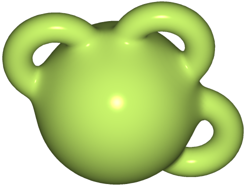

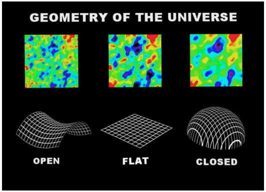

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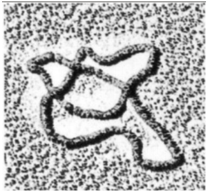

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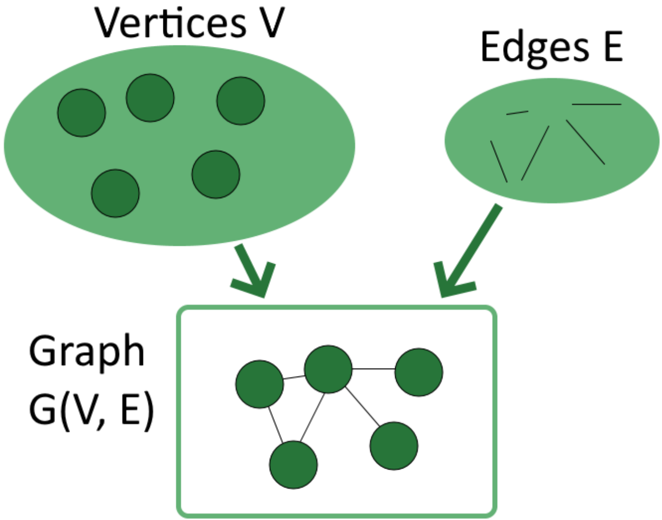


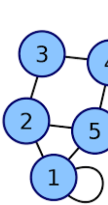
Questions we ask about a graph G

1. Have we seen G before? Is it one of the standard ones (lines, cycles, complete graphs, complete bipartite graphs)?
2. How many vertices and edges does G have?
3. What is its Euler characteristic?
4. Is G connected? How many connected components does G have?
5. Is G a tree? If not, then can we find a spanning tree?
6. What are its paths (start and endpoint might be different)? What are its circuits?
7. Does G have an Eulerian circuit? Does G have an Eulerian path?
8. Is G planar, i.e. does it embed into the plane = the disc = S^2?
9. Does G embed into other surfaces?
10. How many colors do we need to color maps defined by G?

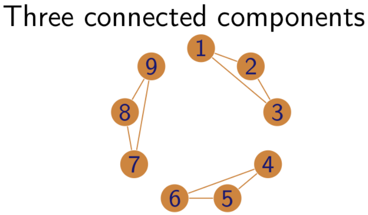
Let us answer 1-10 for the Pappus graph

But before, let us recall what the above are!

A connected graph with $|V| = 6$, $|E| = 8$, $\chi = -2$ and one loop:



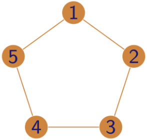
A non-connected graph with $|V| = 9$, $|E| = 9$, $\chi = 0$:



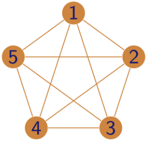
Standard graphs

Line:

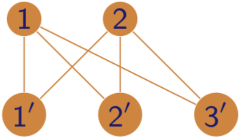
Cycle:



Complete:

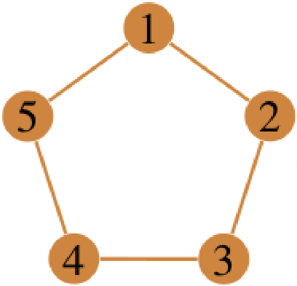


Complete bipartite:

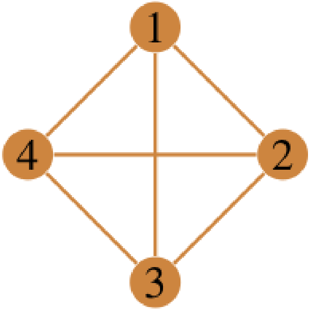


Standard graphs – part 2

Exercise Check whether you understand how the various standard graphs are related and what properties they have. For example, which ones are subgraphs, which ones are planar etc.

$G =$ 

,

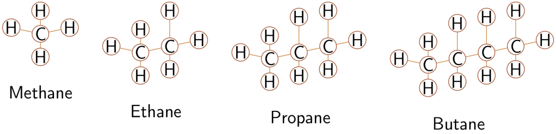
$H =$ 

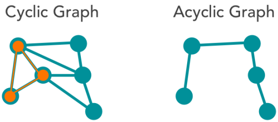
Trees are acyclic, so only the right graph below is a tree:

A tree is a connected graph that has no non-trivial circuits

Examples

- Saturated hydrocarbons





Trees satisfy many properties and are always amenable for induction, e.g. prove the following as an exercise:

Corollary

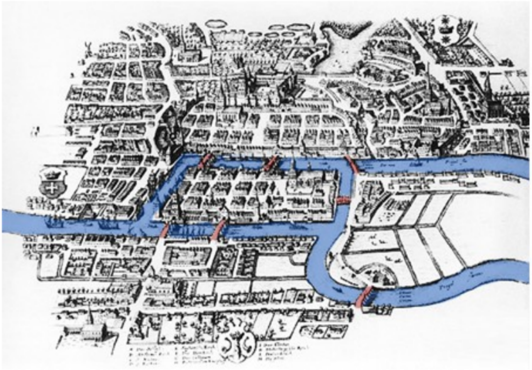
Suppose that $T = (V, E)$ is a tree. Then $|V| = |E| + 1$.

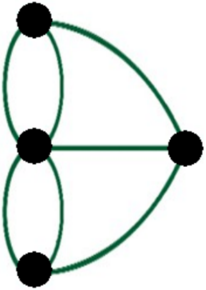
Euler and cycles

Euler's famous criterion:

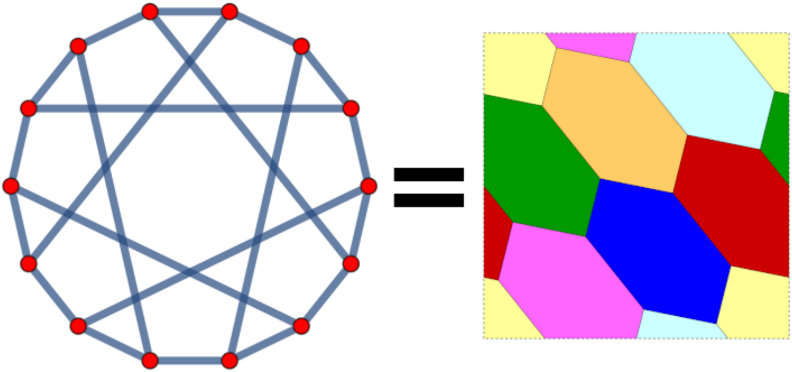
Theorem

Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex has even degree





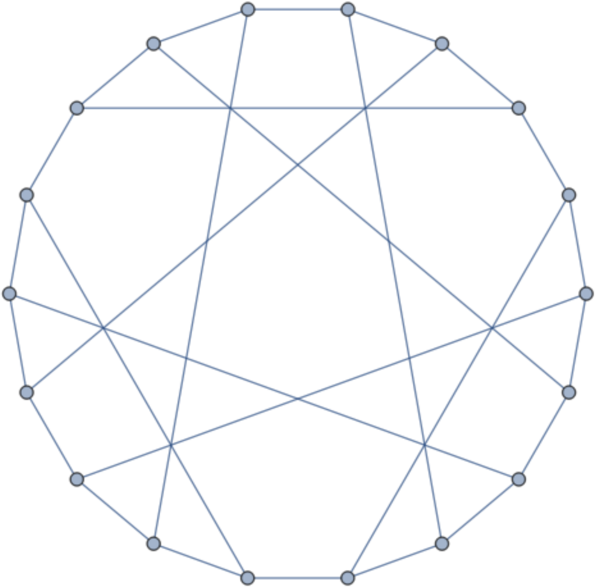
Embeddings on surfaces



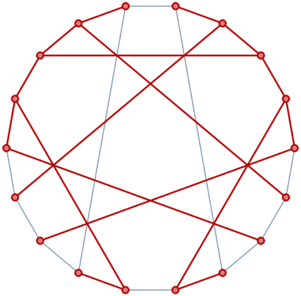
Heawood's coloring formula:

$$c = \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor$$

The Pappus graph G



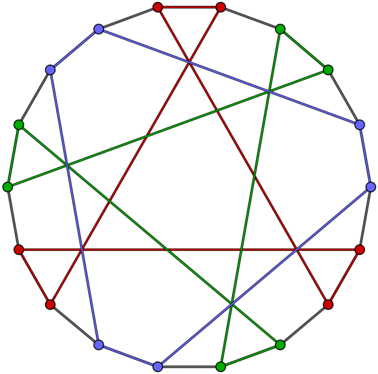
The Pappus graph G – answering 1–10, part 1



The Pappus graph is a not a standard graph – it is neither a line nor a cycle nor complete nor complete bipartite

We clearly have $|V| = 18$ and $|E| = 27$, so that $\chi(G) = |V| - |E| = -9$, and G is connected

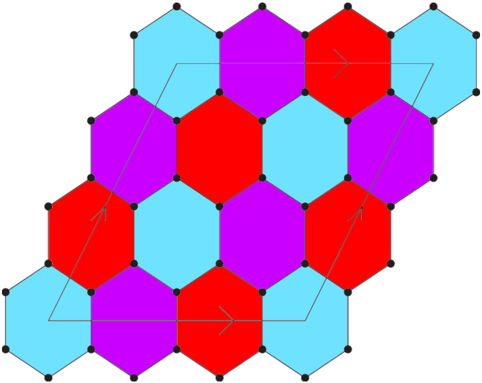
The Pappus graph is not a tree and a spanning tree is illustrated above (there are many more spanning trees)



The Pappus graph has many cycles that are hexagons, as illustrated above. In fact, one checks that the length of the smallest cycle is 6

Every vertex in the Pappus graph is of degree 3, so there are neither Eulerian circuits nor paths

The Pappus graph G – answering 1–10, part 3

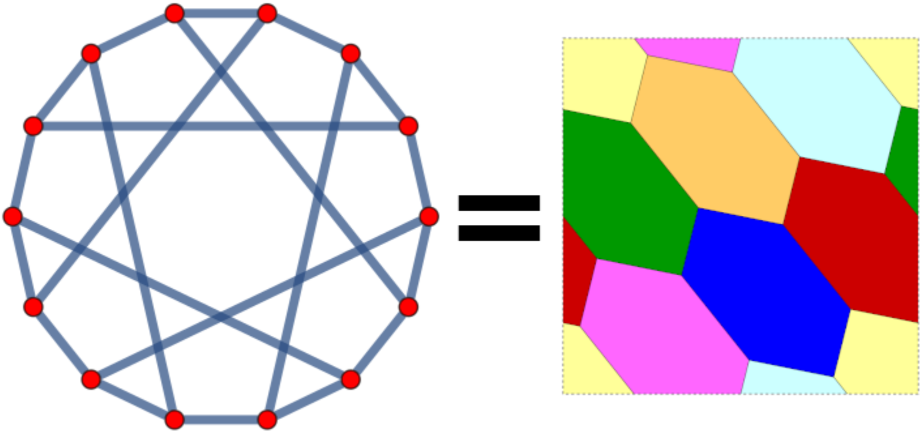


The Pappus graph does not embed into S^2

G embeds onto the torus and then needs 3 colors to color it, see above

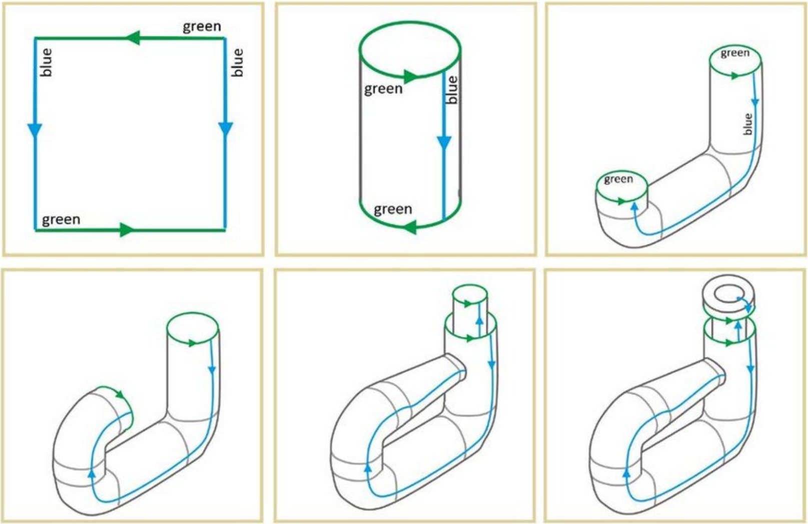
Heawood's theorem for the torus would give $\lfloor\frac{7+\sqrt{49-24\cdot 0}}{2}\rfloor = 7$ as the number of colorings needed in the worst case, so G does better

The Heawood graph – answer 1–10 as an exercise



The above graph is called the Heawood graph – **try yourself!**

Topic 2: surfaces!



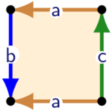
Questions we ask about a surface S

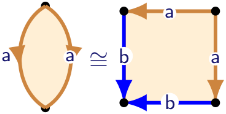
1. Have we seen S before? Is it one of the standard ones (sphere, torus, Klein bottle, projective plane etc.)?
2. How many boundary cycles = punctures does S have?
3. What is its Euler characteristic?
4. Is S connected? How many connected components does S have?
5. Is S orientable?
6. Can we find a polygonal form of S?
7. What is its standard form?
8. How many cross-caps are there in standard form?
9. How many handles are there in standard form?
10. If $d = 0$, then what is the chromatic number of S?

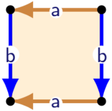
Let us answer 1-10 for a randomly generated polygonal form

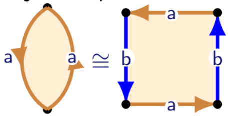
But before, let us recall what the above are!

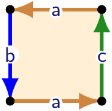
The standard surfaces in polygonal form

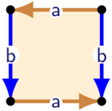
- Annulus $\mathbb{A} \cong$ 

- Sphere $S^2 \cong$ 

- Torus $\mathbb{T} \cong$ 

- Projective plane $\mathbb{P}^2 \cong$ 

- Möbius strip $\mathbb{M} \cong$ 

- Klein bottle $\mathbb{K} \cong$ 

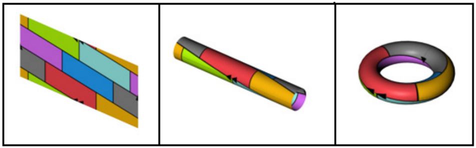
These are 2 dimensional objects, e.g. the torus is hollow:

$\mathbb{T} =$ 

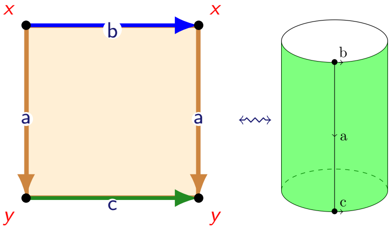
From polygons to surfaces

Recall that one goes from a polygon to a surface by identifying paired edges

For the torus that means e.g.



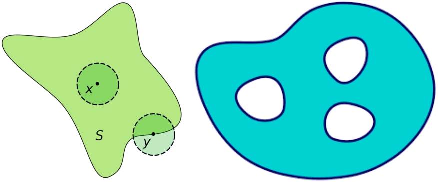
For an annulus one gets



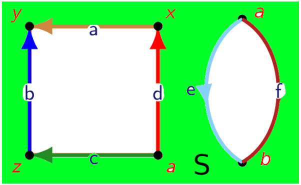
One can build of most these, e.g. a Möbius, strip out of paper

The boundary

Boundary points have neighborhoods that are half-discs; all other point have disc neighborhoods

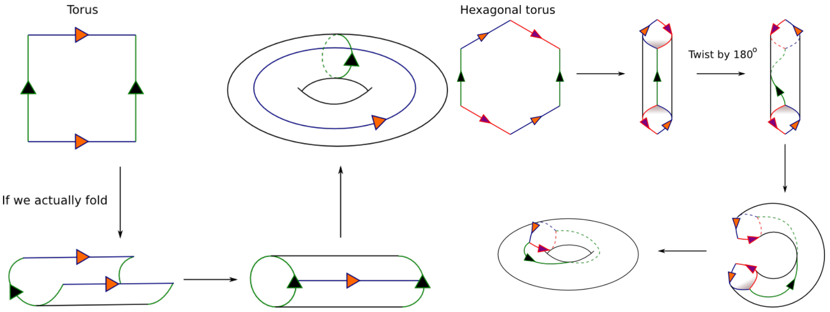


In a polygonal form, the free edges wrap around boundary components:



Note that the surface S is on the outside in these pictures

Every surface S has infinitely many polygonal forms and they might look wildly different, e.g.:

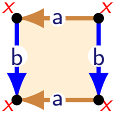


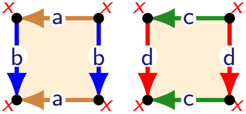
The Euler characteristic $\chi(S) = |V| - |E| + |F|$ is the same for any polygonal form

$$\text{left: } \chi(\mathbb{T}) = 1 - 2 + 1 = 0, \quad \text{right: } \chi(\mathbb{T}) = 2 - 3 + 1 = 0$$

Euler characteristic – only almost perfect

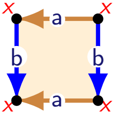
We have $\chi(S) \neq \chi(T) \Rightarrow S \ncong T$ but the converse is not true:

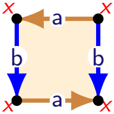
$\mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T}) = 0,\ \mathbb{T} \cup \mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T} \cup \mathbb{T}) = 0$

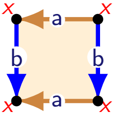
Fix: check connectivity

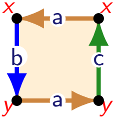
$\mathbb{T} \leftrightsquigarrow$ 

$, \chi(\mathbb{T}) = 0, \quad \mathbb{K} \leftrightsquigarrow$ 

$, \chi(\mathbb{K}) = 0$

Fix: check orientability

$\mathbb{T} \leftrightsquigarrow$ 

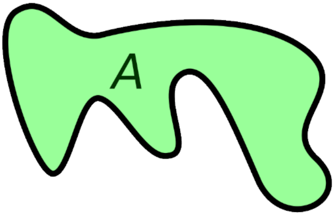
$, \chi(\mathbb{T}) = 0, \quad \mathbb{A} \leftrightsquigarrow$ 

$, \chi(\mathbb{A}) = 0$

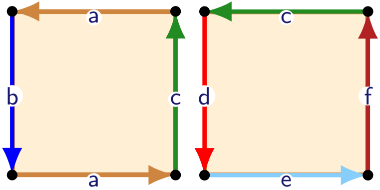
Fix: check boundary

Connected = we can go from every point of S to any other point of S

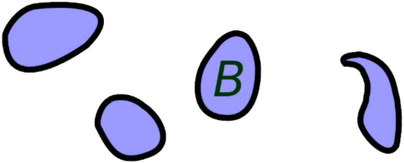
Connected:



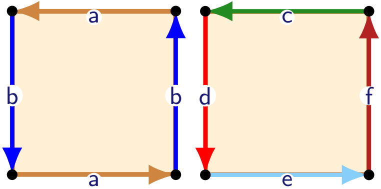
or



Not connected:

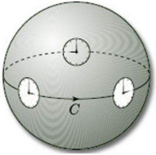


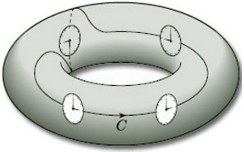
or

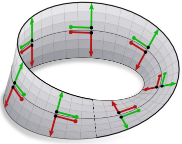


Orientable = consistent choice of a coordinate system

Top: orientable, bottom: not orientable





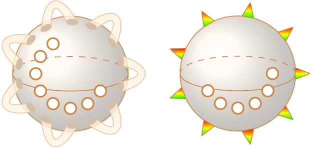


This is hard to check on the surface itself but:

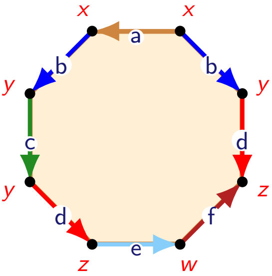
- Words encode orientability
  - Orientable: $\dots a \dots \bar{a} \dots$ or $\dots \bar{a} \dots a \dots$
  - Non-orientable: $\dots a \dots a \dots$ or $\dots \bar{a} \dots \bar{a} \dots$

Boundary = punctures = holes

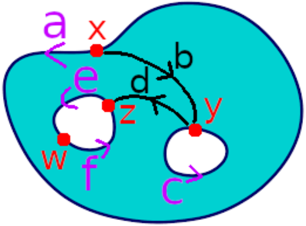
Eight and six boundary components, respectively:



On the polygon this is the free-edge game: identify free edges, and check what cycles they form, e.g.:



$\leftrightsquigarrow$



The classification theorem

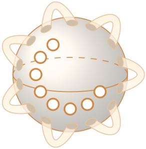
Theorem

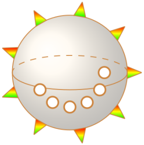
Let S be a connected surface. Then there exist non-negative integers d, p and t such that

1. $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$
2. *the boundary of S is the disjoint union of d circles*
3. *S is orientable if and only if $p = 0$*

Moreover, we can assume that $pt = 0$, in which case S is uniquely determined up to homeomorphism by (d, p, t)

Thus, every surfaces is of either of the following two forms, called standard:

$\#^8 \mathbb{D}^2 \# \#^7 \mathbb{T} \quad \cong$ 

$\#^6 \mathbb{D}^2 \# \#^9 \mathbb{P}^2 \quad \cong$ 

$d \leftrightsquigarrow$ punctures=boundary=holes

$p \leftrightsquigarrow$ projective planes=cross caps

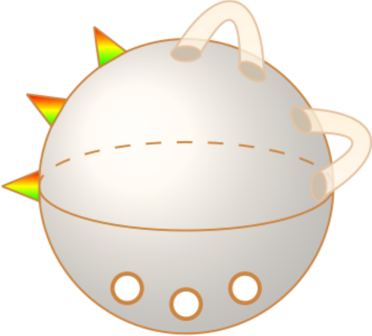
$t \leftrightsquigarrow$ handles=tori

Handles and cross-caps do not want to go along

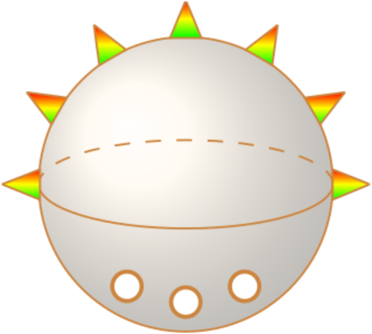
$$\mathbb{T} \# \mathbb{P}^2 \cong \mathbb{K} \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \leftrightsquigarrow \text{“}t = 2p\text{”}$$

Not true: $\mathbb{T} \cong \mathbb{K}$

We can use this to always get rid of all tori in the presence of $\mathbb{P}^2$, e.g.:



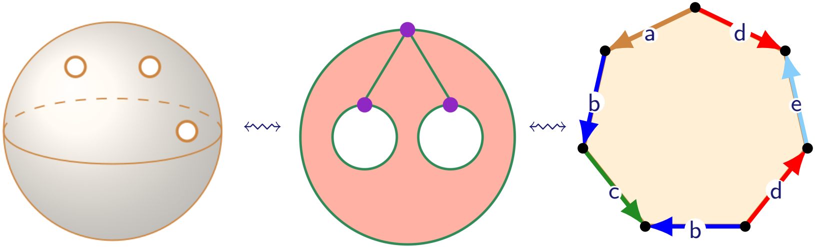
$\cong$



The left-hand surface is not in standard form

From a surface to a polygon

Here is an example how to find a word for the 3-times punctured sphere:



In general, using the classification theorem, we had standard words that we can paste together:

$$\#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \dots\ a_t\ b_t\overline{a_t}\ \overline{b_t}$$

$$\#^p\mathbb{P}^2 = a_1\ a_1\ a_2\ a_2\ \dots\ a_p\ a_p$$

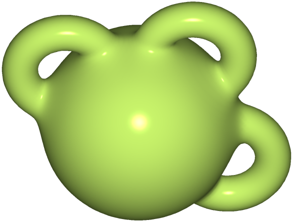
$$\#^d\mathbb{D}^2 = a_1\ b_1\ a_2\ b_2\ \dots\ b_{d-1}a_d\ \overline{b}_{d-1}\ \dots\overline{b}_2\ \overline{b}_1$$

For a connected closed surface $S \ncong \mathbb{K}$ we have that the chromatic number $C(S)$ is

$$C(S) = \left\lfloor \tfrac{1}{2}(7 + \sqrt{49 - 24\chi(S)}) \right\rfloor$$

Additionally $C(\mathbb{K}) = 6$

Example

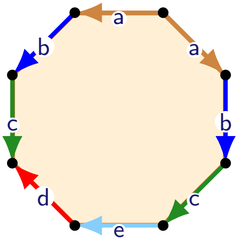
$S =$  $, \chi(S) = -4, C(S) = \lfloor 9.5208 \rfloor = 9$

recall the formula:

$$\chi(S) = 2 - d - p - 2t$$

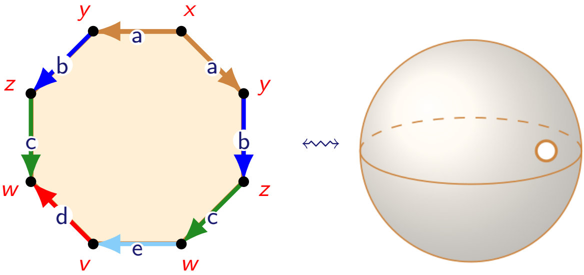
for $S \cong S^2 \# \#^d \mathbb{D}^2 \# \#^p \mathbb{P}^2 \# \#^t \mathbb{T}$

A random example



To find (d, p, t) for S we go through a list of steps:

1. Identify vertices and count them $\Rightarrow |V|$
2. Count edges and faces $\Rightarrow |E|$ and $|F|$
3. Compute $\chi(S) = |V| - |E| + |F|$
4. Check how free edges arrange themselves in cycles $\Rightarrow d$
5. Check for $a...a$ and $\overline{a}...\overline{a}$; if we find them, then $t = 0$ otherwise $p = 0$ $\Rightarrow$ we get either p or t
6. Use $\chi(S) = 2 - d - p - 2t$ to determine the remaining entry t or p

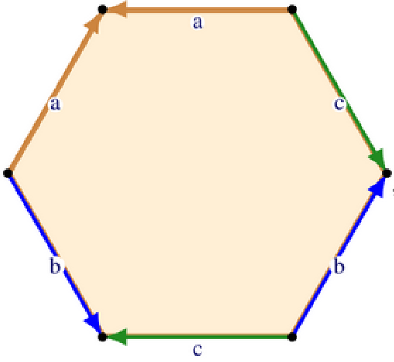


Lets do it!

1. From the above we get $|V| = 5$
2. Counting edges and faces gives $|E| = 5$ and $|F| = 1$
3. We get $\chi(S) = |V| - |E| + |F| = 1$
4. The only free edges $d\colon v \to w$ and $e\colon w \to v$ form one cycle, so $d = 1$
5. No pairs $a...a$ or $\bar{a}...\bar{a}$, so $p = 0$
6. $1 = \chi(S) = 2 - 1 - 0 - 2t$ gives $t = 0$

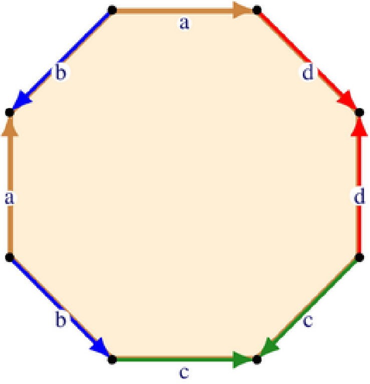
More examples – answer 1–10 as an exercise

$S =$



,

$S' =$



These two surfaces are well-known and want to be identified – **try yourself!**

Exercise Write down some word representing a polygonal form and identify its corresponding standard form, meaning (d, p, t)

Topic 3: knots!

Questions we ask about a knot K

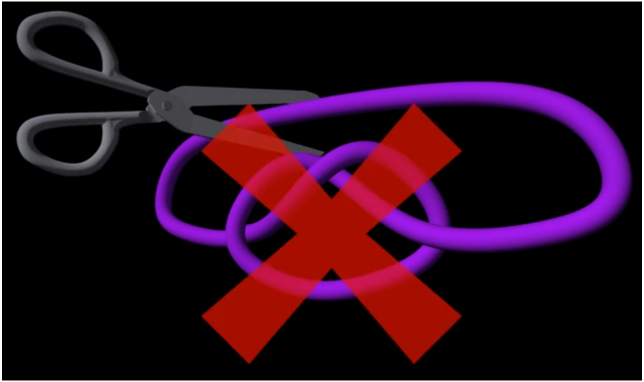
1. Have we seen K before? Is it one of the standard ones, i.e. for low crossing number?
2. Can the diagram(=projection) of K that we see be simplified?
3. Is K the unknot a.k.a. trivial?
4. What is the crossing number of K?
5. Is K alternating?
6. Is K three colorable?
7. Is K p-colorable for $p > 3$?
8. What is the knot determinant of K?
9. Can we explicitly compute a Seifert surface for a diagram of K?
10. What is the genus of K?

Let us answer 1-10 for the knot 5_1

But before, let us recall what the above are!

Knots

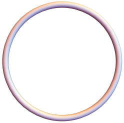
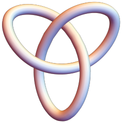
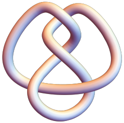
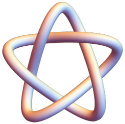
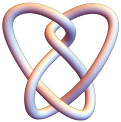
A knot is an embedding of S^1 into $\mathbb{R}^3$ and we study these up to equivalence, i.e. continuous deformation without cutting



Note that all knots are homeomorphic, so this is the wrong notion of equivalence for knots

The periodic table of knots

A main point of knot theory is to have a table of knots up to mirror images:

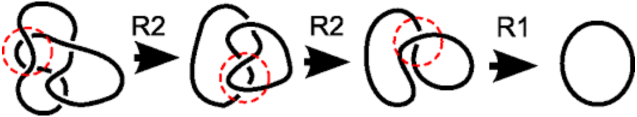
knot					
name	unknot	trefoil	figure 8	cinquefoil	three-twist
notation	0_1	3_1	4_1	5_1	5_2
$cross(K)$	0	3	4	5	5
$\det(K)$	1	3	5	5	7
$g(K)$	0	1	1	2	1
prime?	yes	yes	yes	yes	yes
alternating?	yes	yes	yes	yes	yes

Google The Rolfsen Knot Table or use e.g. KnotData of Mathematica

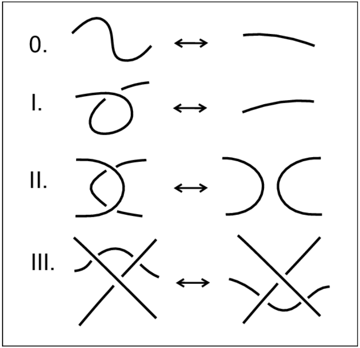
Mirror images (=flipped crossings) cannot be detected by our invariants

Simplify diagrams using Reidemeister moves

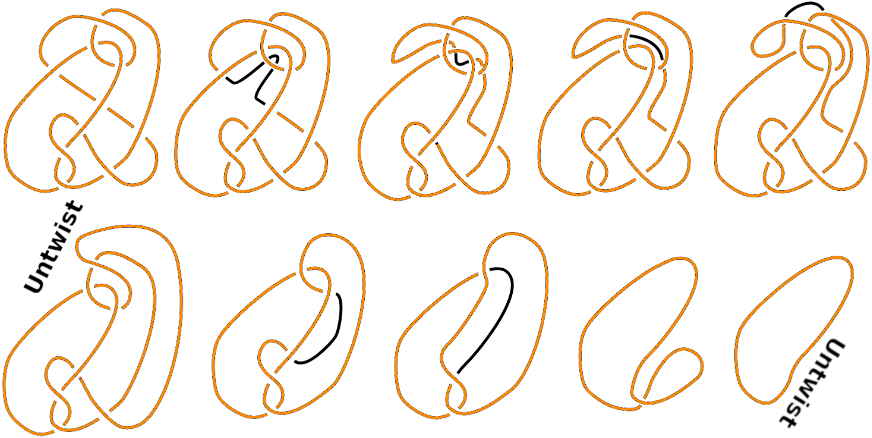
A first step is to check whether there are any "obvious" simplifications:



Recall that two knot diagrams represent the same knot if and only if we can relate them by the Reidemeister moves:

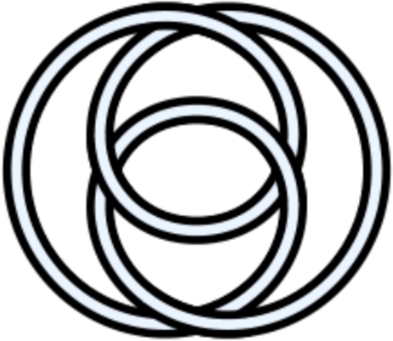


Sometimes diagrams drastically simplify:



Exercise Check whether you understand the Reidemeister moves used for the culprit on the previous slide

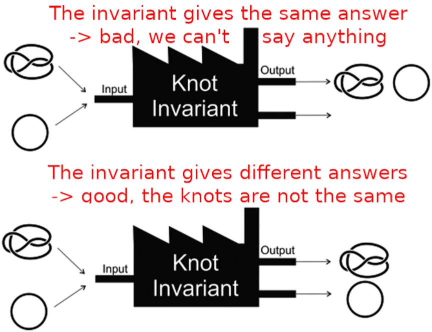
Exercise Check using isotopies and Reidemeister moves whether these two beasts are the same knot:

$K =$  , $K' =$ 

The main question...

...is always: are two knot diagrams representing the same knot?

We want knot invariants to do this!

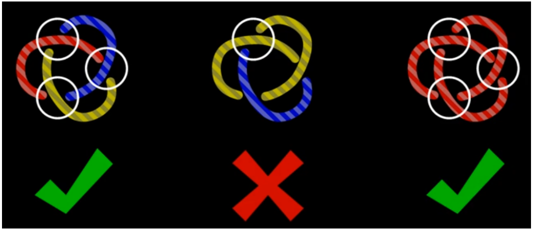


We had essentially two ways to decide that

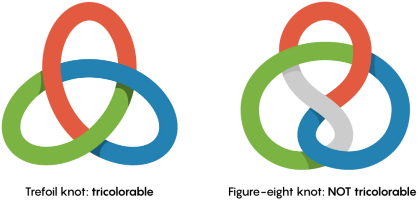
- Knot invariant 1: colorability
- Knot invariant 2: genus

p-colorable; here only $p = 3$

Coloring = each segment gets a color such that we have 3-colored crossings or monochromatic crossings

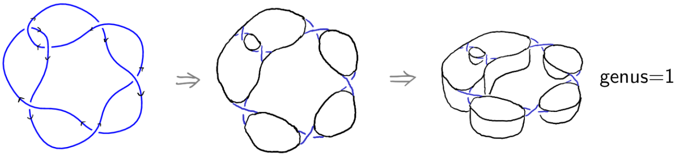


A knot is 3-colorable if it admits a non-monochromatic coloring



The genus: great to check whether a knot is trivial

Genus = the minimal t of all Seifert surfaces; to compute it for an alternating knot run Seifert's algorithm:

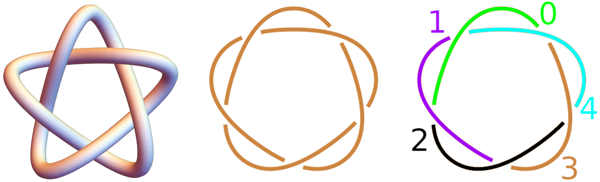


Then $t = \frac{1}{2}(1 + c - s)$ where c is the number of crossings and s the number of Seifert circles

Cool fact (verify "$\Leftarrow$" as an exercise):

$$g(K) = 0 \Leftrightarrow K \cong \text{unknot}$$

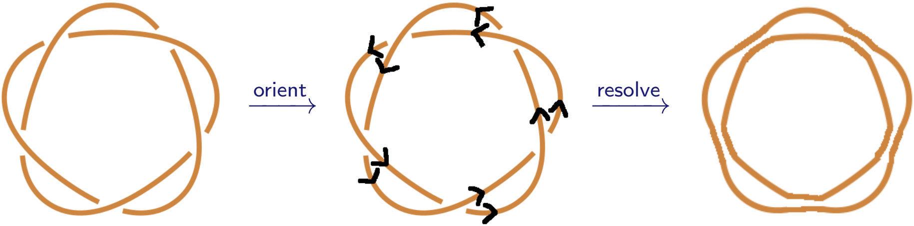
The knot $K = 5_1$



Let us go through the list of steps:

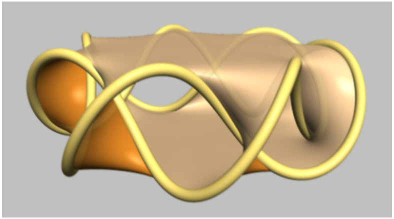
1. We have seen it before, it is 5_1
2. The diagram cannot be made simpler in any obvious way
3. The knot is not trivial, see next slide or coloring above
4. Since the diagram is alternating $cross(K) = 5$
5. The diagram is clearly alternating
6. No, K is not 3-colorable see above
7. Yes, K is 5-colorable, see above
8. We have $\det(K) = 5$ by computation
9. Yes, Seifert surfaces are easy to get, see next slide
10. $g(K) = 1$, see next slide

The knot $K = 5_1$ – Seifert surfaces and genus

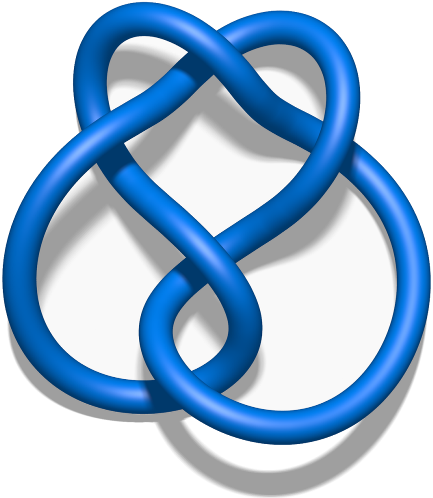


Thus, $c = 5$ and $s = 2$ gives $\chi(S) = s - c = -3$ and
$g = \frac{1}{2}(1 - \chi(S)) = \frac{1}{2}(1 + c - s) = 2$

Putting in the twists gives:



Another knot – answer 1–10 as an exercise



This is knot 5_2 – **try yourself!**

I hope you enjoyed topology!