# Tutorial 8

## *Weekly summary and definitions and results for this tutorial*

a) A **Eulerian circuit**, or **Eulerian cycle**, in a connected graph $\mathcal{G}$ is a circuit in $\mathcal{G}$ that goes through every edge exactly once.

b) Theorem A graph has a Eulerian circuit if and only if it is connected and every vertex has even degree.

c) Suppose that $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. Then $X$ and $Y$ are **homeomorphic** if there exist continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, where $\mathrm{id}_X$ and $\mathrm{id}_Y$ are the identity maps on $X$ and $Y$. We write $X \cong Y$.

d) If $a < b$ and $c < d$ are real numbers then $(a, b) \cong (c, d) \cong \mathbb{R}$, $[a, b) \cong [c, d) \cong (c, d]$ and $[a, b] \cong [c, d]$. Any two (filled in) polygons are homeomorphic: a (filled in) triangle is homeomorphic to a (filled in) square, pentagon, hexagon, … and all of these are homeomorphic to the disc $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$.

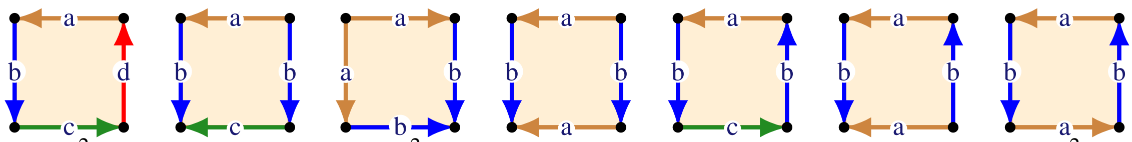
e) *Informal definition of a surface*: Let $n \geqslant 0$. A **surface** is a subset of $\mathbb{R}^n$ such that, *locally*, $X$ looks like the graph of a function $z = f(x, y)$.

f) Examples of surfaces: planes, spheres, cubes, the tori (or donuts), coffee cups, double torus, Möbius strips, Klein bottles, projective planes, cylinders, annuli, …

g) An **identification space** for a surface $T$ is a collection of surfaces $X_1, \dots, X_r$ together with a continuous map $f : \bigcup_r X_r \longrightarrow T$. So $f$ *identifies* $x \in X_i$ and $y \in X_j$ whenever $f(x) = f(y)$.

h) *Formal definition of the surfaces considered in this course:* A surface with a **polygonal decomposition** is the identification space given by a (finite) collection of polygons with at most two polygons being identified along any edge.

i) Polygonal decompositions, or identification spaces, for some important surfaces:




$\mathbb{D}^2$ Disc | $\mathbb{A}$ Annulus | $S^2$ Sphere | $\mathbb{T}$ Torus | $\mathbb{M}$ Möbius strip | $\mathbb{K}$ Klein bottle | $\mathbb{P}^2$ Projective plane

j) The **Euler characteristic** of a surface $S$ with polygonal decomposition $(V, E, F)$ is:

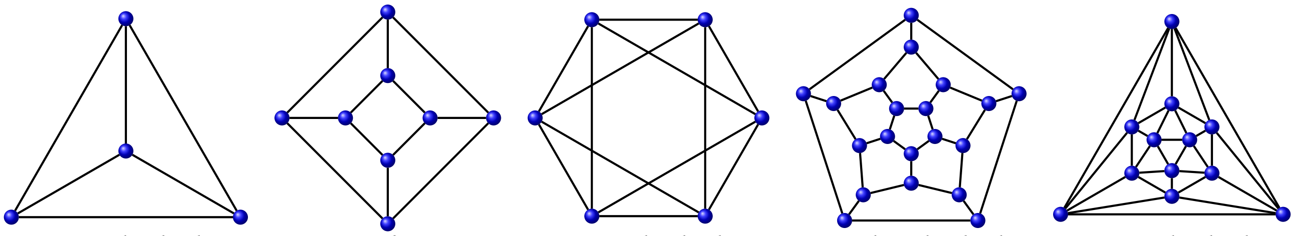
$$\chi(S) = |V| - |E| + |F|.$$

k) The **boundary** of a surface $S$ with a polygonal decomposition is the union of the free, or unpaired, edges. The boundary $\partial S$ is a disjoint union circuits, which are the **boundary circles** of $S$.

## Questions to complete *before* the tutorial

**1.** In the tutorial of week 7, we talked about the projections of the five Platonic solids onto the plane:




We found a Eulerian cycle for the octahedral graph last week. Determine, for each of the graphs of the platonic solids, whether they have a Eulerian cycle.

**2.** Recall that $K_n$ is the complete graph on $n$ vertices, for $n \geqslant 1$.

a) Determine the Euler characteristic of $K_n$.
b) For which values of $n$ is $K_n$ a tree?
c) For which values of $n$ does $K_n$ have a Eulerian circuit?

**3.** Recall that the complete bipartite graph $K_{m,n}$, for $m, n \geq 1$, has vertex set $V = M \sqcup N$ (disjoint union), where $m = |M|$, $n = |N|$, and with edge set $\{(x, y) \mid x \in M \text{ and } y \in N\}$. That is, $K_{m,n}$ has $mn$ edges that connect every element of $M$ with every element of $N$.

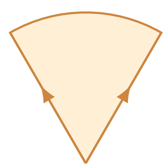
a) Find a formula for $\chi\left(K_{m,n}\right)$.
b) For which values of $m$ and $n$ does $K_{m,n}$ have a Eulerian circuit?

## Questions to complete *during* the tutorial

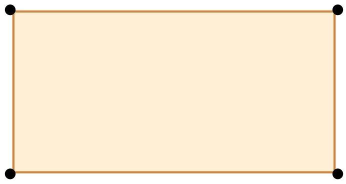
**4.** a) Show that a connected graph with $n \geqslant 3$ and every vertex of degree 2 is isomorphic to the cyclic graph $C_n$.
b) Deduce a graph has every vertex of degree two if and only if it is a disjoint union of cycle graphs.
c) Show that any connected graph in which the vertex degrees are 1 or 2 is a path graph or a cycle graph.
d) Deduce that a graph has every vertex of degree one or two if and only if it is a disjoint union of cycles graphs and path graphs. In this case show the Euler characteristic of the graph counts the number of path graph components.

**5.** Check that you understand the polygonal decompositions of the surfaces $\mathbb{D}^2$, $\mathbb{A}$, $S^2$, $\mathbb{T}$, $\mathbb{M}$, $\mathbb{K}$ and $\mathbb{P}^2$ given in the lecture summary at the start of the tutorial.

**6.** Let $W$ be a sector of the disc between two distinct radii $OP$ and $OQ$. What surface do we get when we identify $OP$ with $OQ$?

$W =$ 


**7.** Label the vertices of a rectangle $A, B, C, D$ as we move anticlockwise around the sides.



What surfaces do we get when we identify:

a) $AB$ with $AD$?
b) $AB$ with $AD$ and $CB$ with $CD$?

## Questions to complete *after* the tutorial

**8.** In lectures it was explained in an intuitive way why the annulus $\mathbb{A}$ and the cylinder are homeomorphic. Up to homeomorphism, the annulus is the set

$$A = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \tfrac{1}{2} \leqslant x^2 + y^2 \leqslant 1 \right\} \subseteq \mathbb{R}^2$$

and up to homeomorphism the cylinder is the set

$$C = \left\{ (x, y, z) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1 \text{ and } \tfrac{1}{2} \leqslant z \leqslant 1 \right\} \subseteq \mathbb{R}^3.$$

In particular, the annulus $A$ embeds in $\mathbb{R}^2$ whereas the cylinder $C$ embeds in $\mathbb{R}^3$.

a) Draw the sets $A$ and $C$ and verify that they are the annulus and the cylinder, respectively.
b) Show that $A \cong C$ by constructing explicit continuous maps $f : A \longrightarrow C$ and $g : C \longrightarrow A$ such that $f \circ g = \mathrm{id}_C$ and $g \circ f = \mathrm{id}_A$.