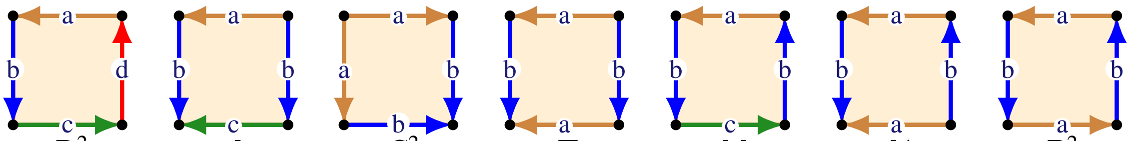
Tutorial 9 — Solutions

Weekly summary and definitions and results for this tutorial

a) Every connected surface has a polygonal decomposition with one polygon (and with some edges identified in pairs)

b) Polygonal decompositions, or identification spaces, for some important surfaces:



$\mathbb{D}^2$	$\mathbb{A}$	S^2	$\mathbb{T}$	$\mathbb{M}$	$\mathbb{K}$	$\mathbb{P}^2$
Disc	Annulus	Sphere	Torus	Möbius strip	Klein bottle	Projective plane

c) The **Euler characteristic** of a surface S with a polygonal decomposition (V, E, F) is:

$$\chi(S) = |V| - |E| + |F|.$$

d) The **boundary** of a surface S with a polygonal decomposition is the union of the unpaired edges. The boundary ∂S is a disjoint union of circles (or cyclic graphs), which are called the **boundary circles** of S.

e) **Surgery** Given any polygonal decomposition for a surface we can *cut* through the interior of a polygonal and then glue together along another paired edges to obtain an equivalent polygonal decomposition of the surface.

f) The **connected sum** of two surfaces S and T is the surface $S\#T$ obtained by cutting holes in the *interior* of S and T and then identifying the boundaries of these circles.

g) Given polygonal decompositions of S and T a polygonal decomposition of $S\#T$ can be obtained by breaking each of S and T apart at an *interior* vertex and then connecting the two ends of S and T to obtain a new polygon.

h) If S and T then $\chi(S\#T) = \chi(S) + \chi(T) - 2$.

i) If S, T and U are surfaces then $(S\#T)\#U \cong S\#(\mathbb{T}\#U)$. Write $\#^n S = \underbrace{S\#S\#\dots\#S}_{n \text{ times}}$.

j) If S is a surface then $S\#\mathbb{D}^2$ is S with a puncture, or hole, and $S\#S^2 \cong S$.

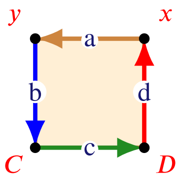
Questions to complete *before* the tutorial

1. a) Find the Euler characteristic χ of the surfaces S^2, $\mathbb{D}^2$,$\mathbb{A}$, $\mathbb{M}$, $\mathbb{T}$, $\mathbb{K}$ and $\mathbb{P}^2$.
b) How many boundary circles do each of these surfaces have?

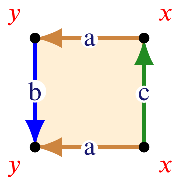
c) Are the Euler characteristic and the number of boundary circles enough to distinguish between the above surfaces? That is, can you conclude that no two of these surfaces are homeomorphic to each other?

Solution

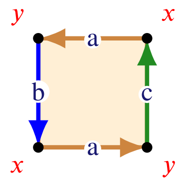
a) We found in lecture that $\chi(S^2) = 2$. Drawing identification diagrams for the remaining six surfaces makes it easy to compute the Euler characteristic:



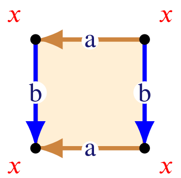
$\chi(\mathbb{D}^2) = 4 - 4 + 1 = 1$



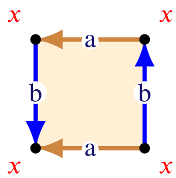
$\chi(\mathbb{A}) = 2 - 3 + 1 = 0$



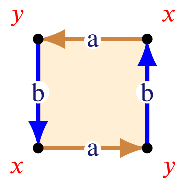
$\chi(\mathbb{M}) = 2 - 3 + 1 = 0$



$\chi(\mathbb{T}) = 1 - 2 + 1 = 0$



$\chi(\mathbb{K}) = 1 - 2 + 1 = 0$



$\chi(\mathbb{P}^2) = 2 - 2 + 1 = 1.$

Note that each of these diagrams is made up of the *vertices* and *edges*, with the specified identifications, together with the enclosed *faces* of the diagrams.

b) We saw in lecture that S^2 has no boundary circles. The number of boundary circles is the number of different circles in the boundary of the surfaces. Hence, the number of boundary circle in the remaining six surfaces are 1,2,1,0,0 and 0, respectively.

c) We have computed the following values:

Surface	Euler characteristic	Boundary circles
S^2	2	0
$\mathbb{D}^2$	1	1
$\mathbb{A}$	0	2
$\mathbb{M}$	0	1
$\mathbb{T}$	0	0
$\mathbb{K}$	0	0
$\mathbb{P}^2$	1	0

Therefore, these numbers are sufficient to distinguish between these surfaces except for the torus and Klein bottle. Hence, we have shown that the listed surfaces are pairwise non-homeomorphic *except for*, possibly, the torus and Klein bottle. Note that the Klein bottle is not orientable because it contains a Möbius band by Question 3. We still do not know whether or not the torus is orientable, so until we prove this we will not know whether or not the torus and Klein bottle are homeomorphic.

2. Solid models of the twenty-six letters of the alphabet

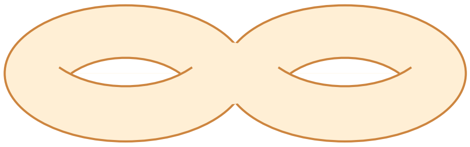
A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z.

are made from clay, shaped from hollow cylindrical pieces. Classify the surfaces of the resulting solids.

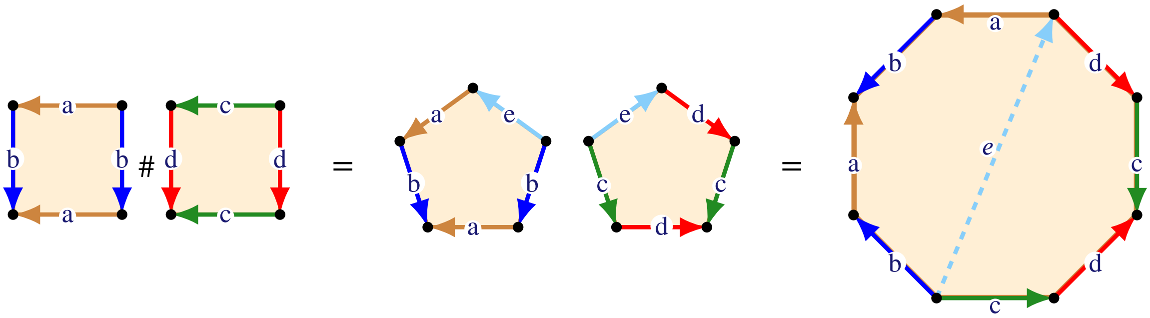
Solution Each of the letters C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z is homeomorphic to a sphere. The letters A,D,O,P,Q R are homeomorphic to a torus. The remaining letter B is homeomorphic to the double torus $\mathbb{T}\#\mathbb{T}$.

Questions to complete *during* the tutorial

3. Find a polygonal decomposition of the double torus:

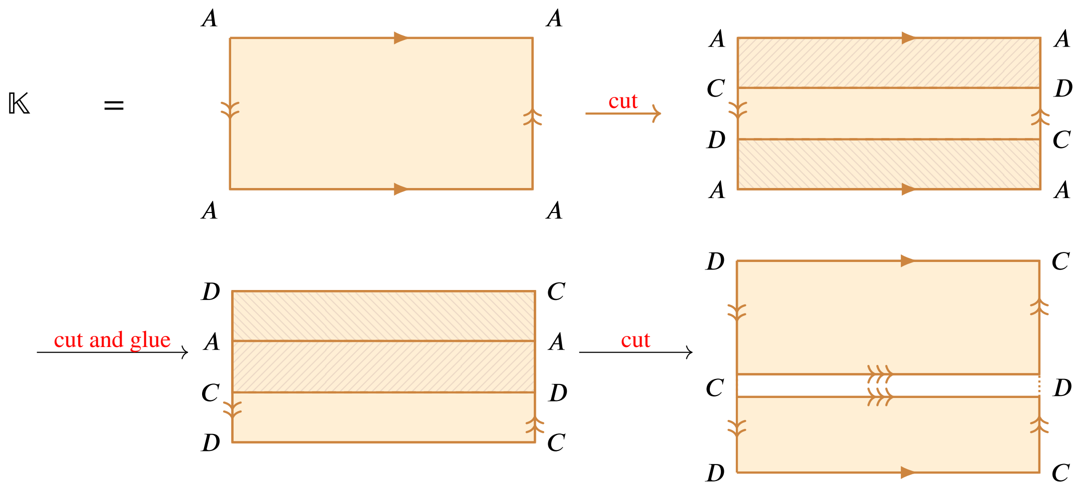


Solution Think about stretching the torus so that it looks like two tori joined by a very thin cylinder in the middle. Then it becomes clear that we can just put the polygonal decompositions of two tori together to give one for the double torus. Hence, a polygonal decomposition of the double torus is:



4. Show that the Klein bottle $\mathbb{K}$ is the union of two copies of the Möbius band $\mathbb{M}$, joined along their boundary circles. (Start with the usual representation of $\mathbb{K}$ in terms of a rectangle with opposite sides identified, and divide the rectangle into three strips parallel to the side that is *not* reversed.)

Solution Starting with the standard drawing of the Klein bottle and then cutting and re-gluing gives (the shading is for illustrative purposes only):



Both of the diagrams on the right-hand side are Möbius strips. Therefore, we have shown that the Klein bottle is the union of two Möbius strips, as required.

5. a) Find a formula for the Euler characteristic of the surface $S_1 \# S_2 \# \dots \# S_n$, which is the connected sum of the surfaces $S_1, S_2, \dots, S_n$, in terms of the Euler characteristics $\chi(S_1), \chi(S_2), \dots, \chi(S_n)$ of the surfaces $S_1, \dots, S_n$.

b) Suppose that A and B are surfaces in $\mathbb{R}^n$, for some $n \geqslant 2$. Show that

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Solution

a) In lectures we saw that $\chi(S \# T) = \chi(S) + \chi(T) - 2$. Therefore,

$$S_1 \# S_2 \# \cdots \# S_n = \chi(S_1) + \cdots + \chi(S_n) - 2(n-1).$$

b) Choose polygonal subdivisions for A and B such that $A \cap B$ is also a union of polygons in this decomposition. For a surface X let v_X, e_X and f_x be the number of vertices, edges and faces in a polygonal decomposition of X. Then $v_{A\cup B} = v_A + v_B - v_{A\cap B}$, $e_{A\cup B} = e_A + e_B - e_{A\cap B}$ and $f_{A\cup B} = f_A + f_B - f_{A\cup B}$. Therefore,

$$\begin{aligned}\chi(A \cup B) &= v_{A\cup B} - e_{A\cup B} + f_{A\cup B}\\ &= (v_A + v_B - v_{A\cap B}) - (e_A + e_B - e_{A\cap B}) + (f_A + f_B - f_{A\cap B})\\ &= (v_A - e_A + f_A) + (v_B - e_B + f_B) - (v_{A\cap B} - e_{A\cap B} + f_{A\cap B})\\ &= \chi(A) + \chi(B) - \chi(A \cap B),\end{aligned}$$

as required.

6. (The Euler characteristic of a connected surface.) Find the Euler characteristic of the surface

$$S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T} \# \#^p \mathbb{P}^2$$

where $d, t, p \geq 0$ are integers.

Solution

a) In lectures we saw that if S and T are surfaces then $\chi(S \# T) = \chi(S) + \chi(T) - 2$. More generally, it follows easily by induction that if $S_1, \ldots, S_r$ are surfaces then

$$\chi(S_1 \# \ldots \# S_r) = \chi(S_1) + \cdots + \chi(S_r) - 2(r-1).$$

Therefore, if $r \geqslant 1$ then

$$\chi(S \# \#^d \mathbb{D}^2) = \chi(S) + d\,\chi(\mathbb{D}^2) - 2d = \chi(S) - d$$
$$\chi(S \# \#^t \mathbb{T}) = \chi(S) + t\,\chi(\mathbb{T}) - 2t = \chi(S) - 2t,$$

since $\chi(\mathbb{D}^2) = 1$ and $\chi(\mathbb{T}) = 0$. Therefore, the Euler characteristic of the d-times punctured sphere with t handles is

$$\chi(S^2 \# \#^t \mathbb{T} \# \#^d \mathbb{D}^2) = 2 - d - 2t.$$

Similarly, $\chi(S \# \#^p \mathbb{P}^2) = 2 - p$, so adding a crosscap reduces Euler characteristic by 1. Therefore,

$$S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T} \# \#^p \mathbb{P}^2 = 2 - d - p - 2t.$$

7. How many *boundary circles* do the following surfaces have? Determine if the surfaces are orientable or non-orientable, carefully explaining your reasoning.

a) $S^2 \# \mathbb{D}^2 \# \mathbb{K} \# \mathbb{P}^2$,
b) $S^2 \# \mathbb{D}^2 \# \mathbb{A} \# \mathbb{K}$,
c) $S^2 \# \mathbb{D}^2 \# \mathbb{T} \# \mathbb{M}$.

Solution

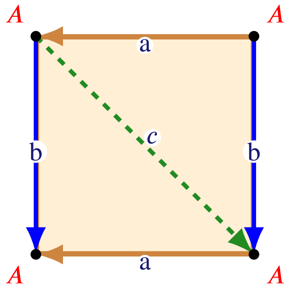
a) The surface $S^2 \# \mathbb{D}^2 \# \mathbb{K} \# \mathbb{P}^2 = \mathbb{D}^2 \# \mathbb{K} \# \mathbb{P}^2$ has 1 boundary circle because $\mathbb{D}^2$ has one boundary circle, and the number of boundary circles in a connected sum is the sum of the number of boundary circles in the pieces. This surface is *non-orientable* because the projective plane and Klein bottle are both non-orientable.

b) The surface $S^2 \# \mathbb{D}^2 \# \mathbb{A} \# \mathbb{K} = \mathbb{D}^2 \# \mathbb{A} \# \mathbb{K}$, has 3 boundary circles because $\mathbb{D}^2$ has one and $\mathbb{A}$ has two boundary circles, and the number of boundary circles in a connected sum is the sum of the number of boundary circles in the pieces. This surface is *non-orientable* because the Klein bottle is non-orientable.

c) The surface $S^2 \# \mathbb{D}^2 \# \mathbb{T} \# \mathbb{M} = \mathbb{D}^2 \# \mathbb{T} \# \mathbb{M}$ has 2 boundary circles $\mathbb{D}^2$ has one boundary circle because $\mathbb{D}^2$ and $\mathbb{M}$ both have one boundary circle. This surface is *non-orientable* because it contains a Möbius band.

8. A triangulation of a surface is polyhedral decomposition into triangles so that edges and triangles are uniquely determined by their vertex labels.

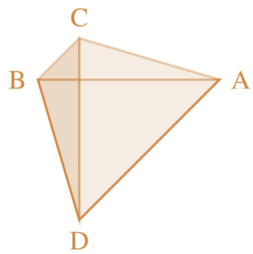
a) Verify that the tetrahedral decomposition of the sphere is a triangulation
b) Start with the standard representation of the torus by a rectangle with opposite sides identified and draw a diagonal that cuts the torus into two triangular pieces:



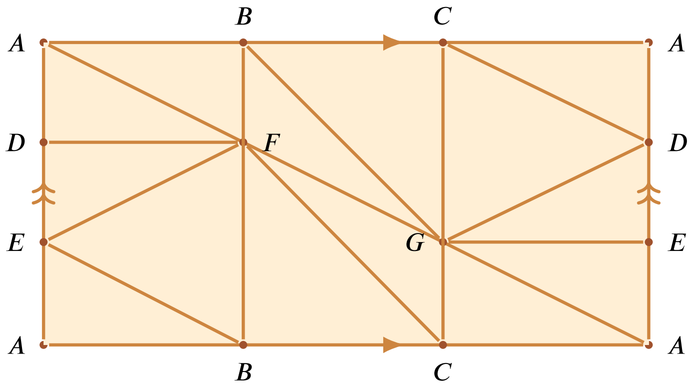
Explain why this is *not* a triangulation of the torus.
c) Find a triangulation of the torus using exactly 7 vertices. (It is known that a triangulation of the torus requires at least 7 vertices.) To do this start with the division of the identification space of the torus into triangles from part (b) and experiment to add new points and edges.

Solution

a) Labelling the vertices in the tetrahedral decomposition as A, B, C, D gives four triangles:



Namely, the triangles BCD, ACD, ABD and ABC. Hence, all of the triangles are uniquely determined by their vertex set so they give a triangulation of the sphere.
b) All of the corners in this polygonal decomposition of the torus of the tetrahedron represent the same vertex. In addition, the two triangles have the same edges. As a triangulation decomposes a surface into a disjoint union of *distinct* triangles, this is not a triangulation.
c) In order for each edge of the polygonal decomposition be uniquely determined by its end points, we need to divide each edge into at least 3 segments. Therefore, we should add new vertices B, C to the top (or, equivalently, bottom edge), C, D to the sides and vertices G, H on the diagonal. Together these points give seven distinct vertices on the torus. A little experiment shows that you can add enough edges so that all the faces are triangles and that no two triangles have the same set of three vertices. For example, here is one way to do this: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