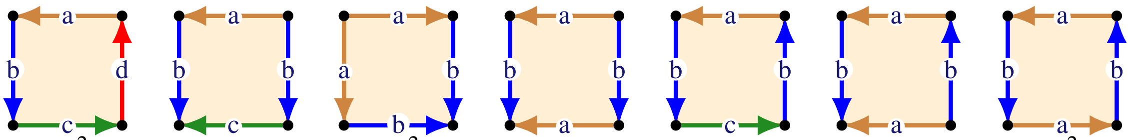
Tutorial 9

Weekly summary and definitions and results for this tutorial

a) Every connected surface has a polygonal decomposition with one polygon (and with some edges identified in pairs)

b) Polygonal decompositions, or identification spaces, for some important surfaces:



$\mathbb{D}^2$	$\mathbb{A}$	S^2	$\mathbb{T}$	$\mathbb{M}$	$\mathbb{K}$	$\mathbb{P}^2$
Disc	Annulus	Sphere	Torus	Möbius strip	Klein bottle	Projective plane

c) The **Euler characteristic** of a surface S with a polygonal decomposition (V, E, F) is:

$$\chi(S) = |V| - |E| + |F|.$$

d) The **boundary** of a surface S with a polygonal decomposition is the union of the unpaired edges. The boundary ∂S is a disjoint union of circles (or cyclic graphs), which are called the **boundary circles** of S.

e) **Surgery** Given any polygonal decomposition for a surface we can *cut* through the interior of a polygonal and then glue together along another paired edges to obtain an equivalent polygonal decomposition of the surface.

f) The **connected sum** of two surfaces S and T is the surface $S \# T$ obtained by cutting holes in the *interior* of S and T and then identifying the boundaries of these circles.

g) Given polygonal decompositions of S and T a polygonal decomposition of $S \# T$ can be obtained by breaking each of S and T apart at an *interior* vertex and then connecting the two ends of S and T to obtain a new polygon.

h) If S and T then $\chi(S \# T) = \chi(S) + \chi(T) - 2$.

i) If S, T and U are surfaces then $(S \# T) \# U \cong S \#(\mathbb{T} \# U)$. Write $\#^n S = \underbrace{S \# S \# \ldots \# S}_{n \text{ times}}$.

j) If S is a surface then $S \# \mathbb{D}^2$ is S with a puncture, or hole, and $S \# S^2 \cong S$.

Questions to complete *before* the tutorial

1. a) Find the Euler characteristic χ of the surfaces S^2, $\mathbb{D}^2$,$\mathbb{A}$, $\mathbb{M}$, $\mathbb{T}$, $\mathbb{K}$ and $\mathbb{P}^2$.
b) How many boundary circles do each of these surfaces have?

c) Are the Euler characteristic and the number of boundary circles enough to distinguish between the above surfaces? That is, can you conclude that no two of these surfaces are homeomorphic to each other?

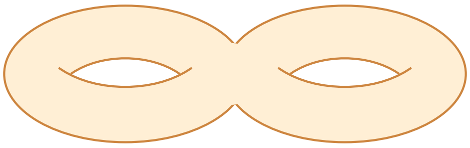
2. Solid models of the twenty-six letters of the alphabet

A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z.

are made from clay, shaped from hollow cylindrical pieces. Classify the surfaces of the resulting solids.

Questions to complete *during* the tutorial

3. Find a polygonal decomposition of the double torus:



4. Show that the Klein bottle $\mathbb{K}$ is the union of two copies of the Möbius band $\mathbb{M}$, joined along their boundary circles. (Start with the usual representation of $\mathbb{K}$ in terms of a rectangle with opposite sides identified, and divide the rectangle into three strips parallel to the side that is *not* reversed.)

5. a) Find a formula for the Euler characteristic of the surface $S_1 \# S_2 \# \dots \# S_n$, which is the connected sum of the surfaces $S_1, S_2, \dots, S_n$, in terms of the Euler characteristics $\chi(S_1), \chi(S_2), \dots, \chi(S_n)$ of the surfaces $S_1, \dots, S_n$.

b) Suppose that A and B are surfaces in $\mathbb{R}^n$, for some $n \geqslant 2$. Show that

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

6. **(The Euler characteristic of a connected surface.)** Find the Euler characteristic of the surface

$$S^2 \# \#^d \mathbb{D}^2 \# \#^t \mathbb{T} \# \#^p \mathbb{P}^2$$

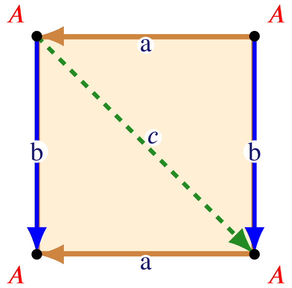
where $d, t, p \geq 0$ are integers.

7. How many *boundary circles* do the following surfaces have? Determine if the surfaces are orientable or non-orientable, carefully explaining your reasoning.

a) $S^2 \# \mathbb{D}^2 \# \mathbb{K} \# \mathbb{P}^2$,
b) $S^2 \# \mathbb{D}^2 \# \mathbb{A} \# \mathbb{K}$,
c) $S^2 \# \mathbb{D}^2 \# \mathbb{T} \# \mathbb{M}$.

8. A triangulation of a surface is polyhedral decomposition into triangles so that edges and triangles are uniquely determined by their vertex labels.

a) Verify that the tetrahedral decomposition of the sphere is a triangulation
b) Start with the standard representation of the torus by a rectangle with opposite sides identified and draw a diagonal that cuts the torus into two triangular pieces:



Explain why this is *not* a triangulation of the torus.

c) Find a triangulation of the torus using exactly 7 vertices. (It is known that a triangulation of the torus requires at least 7 vertices.) To do this start with the division of the identification space of the torus into triangles from part (b) and experiment to add new points and edges.