Tutorial 11 — Solutions

Weekly summary and definitions and results for this tutorial

a) If $G = (V, E)$ is a connected graph embedded in $\mathbb{D}^2$ then $|V| - |E| + |F| = 2$, where F is the set of disconnected regions, or faces, in $\mathbb{D}^2 \setminus G$.
b) The complete graphs K_n, for $n \geqslant 5$, are not planar.
c) **Face-degree equation**: Let S be a polygonal surface without boundary, with e edges, vertex set V and F the set of faces. Then $\sum_{x\in V} \deg x = 2e = \sum_{y\in F} \deg y$.
d) A **platonic solid** is a solid made by gluing together regular n-gons of the same size with p edges meeting at every vertex. If the regular solid has vertex set V, edge set E and face set F then

$$\frac{1}{p} + \frac{1}{n} = \frac{1}{2} + \frac{1}{|E|} > \frac{1}{2}.$$

As a consequence, we saw that there are exactly five platonic solids:

Solid	n	p	$v = \frac{2e}{p}$	e	$f = \frac{2e}{p}$
Tetrahedron	3	3	4	6	4
Octahedron	3	4	6	12	8
Icosahedron	3	5	12	30	20
Cube	4	3	8	12	6
Dodecahedron	5	3	20	30	12

e) A **map** on a closed polygonal surface S is polygonal decomposition such that all vertices have degree at least 3, no region (or face), borders itself, no region contains a hole or another region and no internal region has only two borders.
f) A *colouring* of a map on a surface S is a colouring of the faces of the map so that polygons sharing a common edge (a.k.a countries that share a border) have different colours.
g) The **chromatic number** $C_M(S)$ of the *map* M is the minimum number of colours needed to colour M. The **chromatic number** of the *surface* S is

$$C(S) = \max\{C_M(S) \mid M \text{ a map on } S\}.$$

h) Heawood's estimate says that

$$C(S) \leqslant \begin{cases} 6, & \text{if } S = S^2 \text{ or } S = \mathbb{P}^2, \\ \frac{7+\sqrt{49-24\chi(S)}}{2}, & \text{otherwise} \end{cases}$$

The key to proving this when $\chi(S) \leqslant 0$ is that $\partial_F \leqslant 5$, where $\partial_F = \frac{2|E|}{|F|}$ is the average degree of a face.
Heawood's estimate is *sharp* (i.e. exactly right), except when $S = S^2$ or $S = \mathbb{K}$. We proved that every map on S^2 or, equivalently (by stereographic projection), a map on $\mathbb{D}^2$, requires at most 5 colours. In fact, every map on S^2 is 4-colourable.
i) A **knot** is a closed path in $\mathbb{R}^3$ with no self-intersections.
j) A **knot projection** is a drawing of a knot in $\mathbb{R}^2$ with over and under crossings being used to indicate the relative positions of the strings and with no more than two strands meeting at any crossing.
k) A **polygonal decomposition** of a knot is a sequence of line segments with consecutive endpoints identified. Any knot is equivalent to a polygonal knot. Two polygonal knots are **equivalent** if there exists a polygonal knot that is a subdivision of both knots.
l) Two knot projections correspond to equivalent knots if and only if one can be transformed into the other using the three Reidemeister moves: twisting, looping and sliding.
m) The **segments** of a knot projection are the connected components of the knot projection.

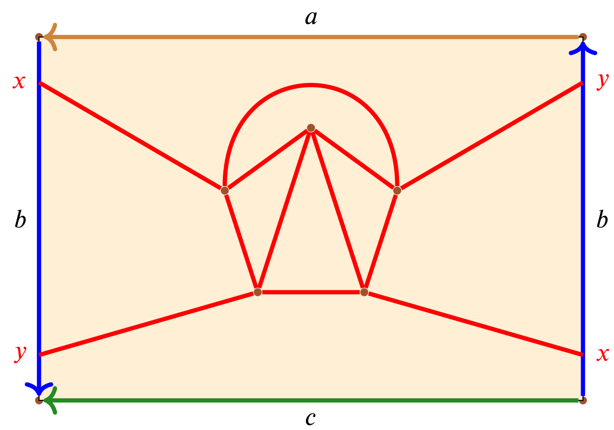
Questions to complete *during* the tutorial

1. Recall that the complete graph K_5 on 5 vertices is not planar. That is, K_5 cannot be drawn on the plane or on the sphere without edge crossings.

a) Is it possible to draw K_5 without edge crossings on the Möbius band $\mathbb{M}$?
b) Is it possible to draw K_5 without edge crossings on the annulus $\mathbb{A}$?
[*Hint:* Argue by contradiction thinking about the relationship between $\mathbb{A}$ and S^2.]

Solution

a) Using the standard polygonal decomposition of $\mathbb{M}$ it is not hard to draw K_5 on $\mathbb{M}$:

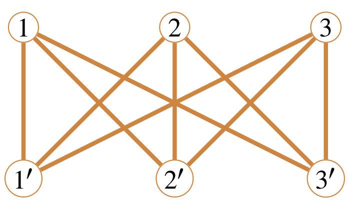


where the red edges are all distinct, The key point is that the left and right-hand edges of the Möbius band have opposite orientations. Therefore, the top edge leaving the Möbius band on the left-hand side at the vertex x is connected to the bottom edge on the right-hand side. Similarly, the bottom left-hand edge that leaves $\mathbb{M}$ at y is connects to top right-hand edge.

b) It is not possible to draw K_5 on the annulus without edge crossings. As $\chi(\mathbb{A}) = 0 = \chi(\mathbb{M})$ we cannot argue as we did in lectures to show that this is impossible. Instead, as suggested by the hint, we argue by contradiction and assume that we can draw K_5 on the annulus without edge crossings. Now, the annuls $\mathbb{A}$, or cylinder, is a twice punctured sphere. Therefore, as we are assuming that we can draw K_5 on $\mathbb{A}$ without edge crossings, by filling in the two punctures in $\mathbb{A}$ we can draw K_5 on the sphere without edge crossings. We saw in lectures, however, that this is impossible, so this is a contradiction. Hence, we cannot draw K_5 on $\mathbb{A}$ without edge crossings either.

More generally, by the same argument, we cannot draw K_5 on $S^2 \# \#^d \mathbb{D}^2$, the sphere with d-punctures where $d \geqslant 0$, without edge crossings. Indeed, it is not hard to show that a graph is planar if and only if it can be drawn on $S^2 \# \#^d \mathbb{D}^2$ without edge crossings.

2. Show that the complete bipartite graph $K_{3,3}$



is not planar.
[*Hint:* Argue by contradiction and first show that each face has four or six edges.]

Solution Suppose by way of contradiction that $K_{3,3}$ is planar. Then there is a polygonal decomposition of the sphere with 6 vertices and 9 edges. Let the number of faces in this decomposition be F. Then $2 = \chi(S^2) = 6 - 9 + F$, so that $F = 5$. Now, observe that because $K_{3,3}$ is bipartite it can only have cycles of even length. Therefore, as it has no cycles of length 2, all of the cycles in $K_{3,3}$ have length 4 or 6 since there can be no cycles of length greater than 6 since $K_{3,3}$ has only 6 vertices. Hence, in the polygonal decomposition of S^2 coming from $K_{3,3}$, each face of has either 4 or 6 edges. Let F_4 be the number of faces with 4 edges and F_6 be the number of faces with 6 edges. Then $F_4 + F_6 = 5$ and, since each edge must meet two faces, $4F_4 + 6F_6 = 2\#\text{edges} = 18$, so that $2F_4 + 3F_6 = 9$. Hence,

$$9 = 2F_4 + 3F_6 \geqslant 2(F_4 + F_6) = 2 \times 5 = 10. \quad !!$$

This is a contradiction, so we conclude that $K_{3,3}$ is not planar.

3. A ball is constructed from squares and regular hexagons sewn along edges such that at each vertex 3 edges meet. Each square is surrounded by hexagons, and each hexagon by 3 squares and 3 hexagons. How many squares and hexagons are used in the construction?

Solution Let the number of squares be F_4 and the number of hexagons be F_6 and let V, E and F be the number of vertices, edges and faces of the decomposition, respectively. Then $F = F_4 + F_6$ and $3V = 2E$. Counting the edges of the faces counts each edge twice so

$$4F_4 + 6F_6 = 2E.$$

Each square meets 4 hexagons and each hexagon meets 3 squares, so $4F_4 = 3F_6$. Hence $9F_6 = 2E = 3V$. Therefore,

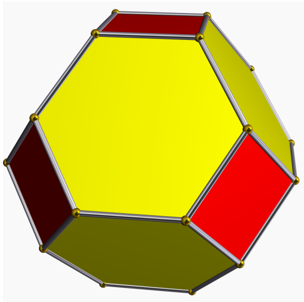
$$E = \frac{9}{2}F_6, \quad V = 3F_6, \quad F = F_6 + \frac{3}{4}F_6 = \frac{7}{4}F_6.$$

Using the Euler characteristic equation $V - E + F = \chi(S^2) = 2$ shows that

$$12F_6 - 18F_6 + 7F_6 = 8$$

Hence $F_4 = 6$ and $F_6 = 8$. Such a surface can realised as a regular truncated octahedron.

See http://en.wikipedia.org/wiki/File:Truncatedoctahedron.gif for an animated rotating image.



4. a) Show that there is no regular polygonal decomposition of the torus by pentagons.
b) For which n is there a regular polygonal decomposition of the torus into n-gons?

Solution

a) Suppose we have a decomposition of the torus with pentagonal faces with p pentagons meeting at each vertex. Let v, e and f be the number of vertices, edges and faces in this decomposition. The graph of vertices and edges of the decomposition has all vertices of degree p. Hence,

$$pv = \sum_{\text{vertices } x} \deg x = 2e.$$

Each pentagon has 5 edges so counting over all faces $5f = 2e$, since each edge is counted twice. Therefore, since the torus has Euler characteristic zero,

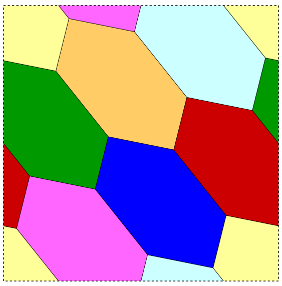
$$0 = \chi(\mathbb{T}) = v - e + f = \frac{2e}{p} - e + \frac{2e}{5} = e\left(\frac{2}{p} - 1 + \frac{2}{5}\right) = e\left(\frac{2}{p} - \frac{3}{5}\right).$$

Hence, $\frac{2}{p} - \frac{3}{5} = 0$ after dividing by e (which is necessarily non-zero), so that $p = \frac{10}{3}$. This is nonsense, however, because p is an integer. Therefore, it is not possible to find a decomposition of the torus using pentagons.

b) Arguing as in part (a), but now assuming that p n-gons meet in each vertex, we obtain

$$pv = 2e = nf.$$

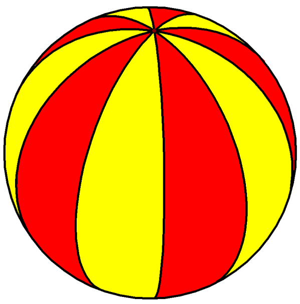
Therefore, $v = \frac{2e}{p}$ and $f = \frac{2e}{n}$ so that $0 = \chi(\mathbb{T}) = \frac{2e}{p} - e + \frac{2e}{n} = 2e(\frac{1}{p} + \frac{1}{n} - \frac{1}{2})$. We can divide by e since it is non-zero so, clearing denominators, we require that $np - 2n - 2p = 0$ or, equivalently, that $(n-2)(p-2) = 4$. As the only positive integer factorisations of 4 are $4 = 4 \times 1 = 2 \times 2 = 1 \times 4$ this means that the only possible solutions are $(n, p) = (6, 3)$, $(n, p) = (4, 4)$ or $(n, p) = (3, 6)$, respectively. All of these decompositions can be realised.
The last paragraph gives *necessary* conditions on p and n for the existence of a regular decomposition of the torus, but we have not yet shown that these decompositions exist in any of the three cases $(n, p) = (6, 3)$, $(n, p) = (4, 4)$ or $(n, p) = (3, 6)$. The only way to prove that such a decomposition exists is to produce one. In fact, it turns out that in each case there are *infinitely* many different regular polygonal decompositions of the torus by triangles, squares and hexagons. The regular decompositions of the torus can be found at www.weddslist.com/groups/genus/1/
Recall that we have seen one of these, namely the one with seven touching hexagons:



5. The Degenerate Regular Decompositions of the Sphere

a) Show that for each integer $p \geq 2$ there is regular decomposition of the sphere into p two sided polygons.
b) Dually, show that for each integer $n \geq 2$ there is a regular decomposition of the sphere into 2 polygons with n sides.

Solution To get regular decomposition of the standard sphere into p two sided polygons, mark a north pole N and south pole S on the sphere and then draw p longitudinal semi-great circles with angle between lines of longitude each $2\pi/p$. Then this is a polygonal decomposition of sphere into 2-gons with vertices N and S. This decomposition into 2-gons is regular the same number p, of the 2-gons, meets at each vertex. You have all seen this before:



To get a regular decomposition of the sphere into two polygons each with n mark n equally spaced points on the great circle. Then gives a polygonal decomposition of the sphere into two n-gons whose vertices are the marked points, edges the great circle arcs joining them and faces the northern and southern hemispheres. This is regular decomposition into n-gons because the same number two, of n-gons, meets at each vertex.