Tutorial 12

Weekly summary and definitions and results for this tutorial

a) A **p-colouring** of a knot is a colouring of the segments of the knots by colours $0, 1, 2, \dots, p-1$ in such a way that for each crossing $2c_i \equiv c_j + c_k \pmod{p}$. Let $C_p(K)$ be the number of p-colourings of the knot K.

b) A knot K is ***p*-colourable** if it has a p-colouring that uses at least two colours. Equivalently, K is p-colourable if and only if $C_p(K) > 0$.

c) Given a knot projection, read the segments $c_1, \dots, c_n$ in a direction around the knot. The **knot matrix** $M_K = (m_{ij})$ is the $n \times n$ matrix where m_{ij} is the sum of contributions of the jth segment to the ith crossing given by

$$m_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } c_j \text{ goes under } c_i \text{ at the } i^{\text{th}} \text{ crossing}, \\ 0, & \text{otherwise.} \end{cases}$$

If K is alternating then the row and column sums in M_K are all zero.

d) If M is an $n \times n$ matrix and $1 \leqslant r, c \leqslant n$ then the (r, c)-minor of M is the $(n-1) \times (n-1)$ matrix obtained by removing all entries in row r and column c from M.

e) The **knot determinant** of an alternating knot is $\det(K) := |\det(M_K)_{rc}|$, where $(M_K$ is *any minor* of the knot matrix M_K. If p is a prime then the alternating knot K is p-colourable if and only if p divides $\det(K)$.

f) The **crossing number** $\mathrm{cross}(K)$ of a knot K is the minimum number of crossings in any knot projection of K. By definition, $\mathrm{cross}(K)$ is a knot invariant but it is difficult to calculate. We saw that $\mathrm{cross}(K) = 0$ if and only K is the unknot and $\mathrm{cross}(K \# L) \leqslant \mathrm{cross}(K) + \mathrm{cross}(L)$.

g) A knot K is a **composite knot** if $K = L \# M$ for non-trivial knots L and M. The knot K is **prime** if it is not composite.

h) Every knot can be written, uniquely, as a connected sum of prime knots.

i) A knot is **alternating** is the under and over crossings alternate as you you travel around the knot in a fixed direction.

j) Let K be a not. A **Seifert surface** for K is any surface that has K as its boundary. Seifert surfaces of K always exist but they are not unique. We gave an algorithm for constructing the Seifert surface of a knot given by putting an orientation on the knot, cutting the over-strings and then rejoining the using the orientation, gluing **Seifert circles** onto the result circles and then added **twists** with boundaries given by the previous crossings.

k) The **genus** of the knot S is the knot invariant

$$g(K) = \min\Big\{ \tfrac{1}{2}\big(1 - \chi(S)\big) \Bigm| S \text{ is a Seifert surface of } K \Big\}.$$

For knots K and L, $g(K \# L) = g(K) + g(L)$

l) If K has a knot projection with c crossings and the corresponding Seifert surface has s Seifert circles then $g(K) = \frac{1}{2}(1 + c - s)$.

Questions to complete *before* the tutorial

1. Let $K = 4_1$ be the figure of eight knot:

$4_1 \quad =$ 

Show that 4_1 is not 3-colourable.

2. Compute the knot determinants of the knots:



Questions to complete *during* the tutorial

3. Let K be a knot with n crossings, $p > 2$ and suppose that $c_1, \dots, c_n$ is a p-colouring of K.

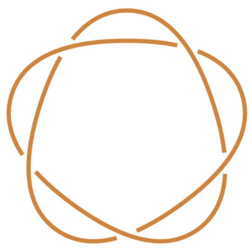
a) Let $c_k' = c_k + 1 \pmod{p}$, for $1 \leqslant k \leqslant n$. Show that $c_1', \dots, c_n$; is a p-colouring of K.

b) Using (a), or otherwise, give five different 5-colourings that use two or more colourings for the two knots:

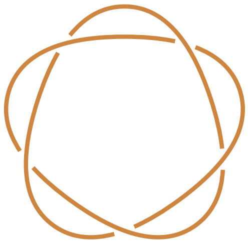
$4_1 =$  $5_2 \ =$ 

c) For knots K and L show that $C_p(K \# L) = \frac{1}{p} C_p(K) C_p(L)$.

4. Find the determinants of the knots 4_1, 5_1 and 5_2 and determine for which odd primes p they are p-colourable.

$4_1 \quad =$  $5_1 \quad =$  $5_2 \quad =$ 

5. a) Calculate the genus of the three knots:

$4_1 =$  $5_1 =$  $5_2 =$ 

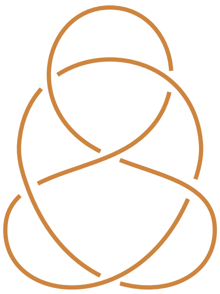
b) Using part (a), or otherwise, show that all of these knots are prime.

6. a) What is the Euler characteristic of the double torus?

b) What is the minimum number of colours needed to be able to colour any map on the double torus so that no two adjacent regions have the same colour?

Questions to complete *after* the tutorial

7. Find the determinants of the knots 6_1, 6_2 and 6_3 and determine for which odd primes p they are p-colourable.

$6_1 =$  $6_2 =$  $6_3 =$