EQUIVARIANT NEURAL NETWORKS AND PIECEWISE LINEAR REPRESENTATION THEORY – DIHEDRAL GROUPS

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Contents

We assume familiarity with the main paper's notation [GTW23].

1. Example: dihedral groups

Let $D_{2\cdot n}=\mathbb{Z}/n\mathbb{Z}=\langle a,b|a^n=b^2=(ab)^2=1\rangle=\langle s,t|s^2=t^2=(st)^n=1\rangle$ be the ***dihedral group*** of order $2n\in\mathbb{Z}_{\geq 1}$. (The second presentation is the ***Coxeter presentation*** of $D_{2\cdot n}$ where ab is replaced by s.) The ***regular*** $D_{2\cdot n}$***-representation*** is the $2n$ dimensional $\mathbb{R}$-vector space $R=\mathbb{R}\{x_0,...,x_{n-1},y_0,...,y_{n-1}\}$ with (left) action given by

$$a\centerdot x_i=x_{i+1}, a\centerdot y_i=y_{i-1}, b\centerdot x_i=y_i, b\centerdot y_i=x_i, \tag{1.1}$$

where we again read indexes modulo n. We denote by $\mathtt{M}_a$ and $\mathtt{M}_b$ the associated action matrices.

We also need the ***right action*** of $D_{2\cdot n}$ on its regular representation. We denote this action in matrices by $\mathtt{M}_a^r$ and $\mathtt{M}_b^r$. When comparing to Example 1.3, the difference is that the south-east block matrix of $\mathtt{M}_a$ becomes the reverse cyclic matrix in $\mathtt{M}_a^r$, and the two antidiagonal block matrices of $\mathtt{M}_b$ are identity matrices in $\mathtt{M}_b^r$. Reading indices modulo n as before, this action is given by

$$x_i\centerdot a=x_{i+1}, y_i\centerdot a=y_{i+1}, x_i\centerdot b=y_{n-i}, y_i\centerdot b=x_{n-i}. \tag{1.2}$$

Example 1.3. For $n=3$ the action matrices one gets from (1.1) and (1.2) are:

$$\mathtt{M}_a=\left(\begin{array}{c|c}\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}&0\\\hline 0&\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\end{array}\right),\quad \mathtt{M}_b=\left(\begin{array}{c|c}0&\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\\\hline \begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}&0\end{array}\right),\quad \mathtt{M}_a^r=\left(\begin{array}{c|c}\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}&0\\\hline 0&\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix}\end{array}\right),\quad \mathtt{M}_b^r=\left(\begin{array}{c|c}0&\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\\\hline \begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}&0\end{array}\right).$$

In general, $\mathtt{M}_a$ and $\mathtt{M}_a^r$ have cyclic matrices along the diagonal and $\mathtt{M}_b$ is the long permutation while $\mathtt{M}_b^r$ has identity block matrices on the antidiagonal. $\diamond$

1A. Simple representations, projections and inclusions.

If n is even, then the dihedral group $D_{2\cdot n}$ has four nonequivalent one dimensional representations on $\mathbb{R}\{v\}$ given by

$$L_0\colon a\centerdot v=1, b\centerdot v=1,\quad L_m\colon a\centerdot v=-1, b\centerdot v=1,$$
$$L_0^\star\colon a\centerdot v=1, b\centerdot v=-1,\quad L_m^\star\colon a\centerdot v=-1, b\centerdot v=-1.$$

If n is odd, then only L_0, the ***trivial*** $D_{2\cdot n}$***-representation***, and $L_0^\star$ exist. Different from the cyclic group case, the $D_{2\cdot n}$-representation $L_0^\star$ is the ***sign representation*** (of the Coxeter presentation).

There are also two dimensional $D_{2\cdot n}$-representations that we, similarly as in [GTW23, ???], denote by $L_k=\mathbb{R}\{v_k,v_{-k}\}$. The action of a on L_k is as before and we additionally have

$$b\centerdot v_{\pm k}=v_{\mp k}\rightsquigarrow b\mapsto \mathtt{N}_k^b=\begin{pmatrix}0&1\\1&0\end{pmatrix}\in\mathrm{End}_{\mathbb{R}}(L_k).$$

Lemma 1A.1. *As real $D_{2\cdot n}$-representations we have*

$$n\ \textit{even}\colon R\cong L_0\oplus L_0^\star\oplus L_1^{\oplus 2}\oplus\ldots\oplus L_{m-1}^{\oplus 2}\oplus L_m\oplus L_m^\star,\quad n\ \textit{odd}\colon R\cong L_0\oplus L_0^\star\oplus L_1^{\oplus 2}\oplus\ldots\oplus L_m^{\oplus 2}.$$

All of the appearing real $D_{2\cdot n}$-representations are simple and pairwise nonisomorphic. All simple real $D_{2\cdot n}$-representations appear in this way.

Proof. Since there is no difference between simple real and complex $D_{2\cdot n}$-representations, this can be found in many textbooks, e.g. [Ser77, Section 5.3]. $\square$

Notation 1A.2. All two dimensional simple real $D_{2\cdot n}$-representations in Lemma 1A.1 appear twice, and we use $+1$ and -1 as a subscript to distinguish them. We use the same notation for the corresponding projectors in Lemma 1A.6 below.

Lemma 1A.3. *We have the following character tables for the dihedral group: Firstly, if n is even then:*

	1	a	a^2	a^3	$\dots$	a^{m-1}	a^m	b	ab
	1	2	2	2	$\dots$	2	1	m	m
χ_0	1	1	1	1	$\dots$	1	1	1	1
$\chi_0^\star$	1	1	1	1	$\dots$	1	1	-1	-1
χ_1	2	$2\cos(\theta)$	$2\cos(2\theta)$	$2\cos(3\theta)$	$\dots$	$2\cos\big((m-1)\theta\big)$	-2	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_{m-1}	2	$2\cos\big((m-1)\theta\big)$	$2\cos\big(2(m-1)\theta\big)$	$2\cos\big(3(m-1)\theta\big)$	$\dots$	$2\cos\big((m-1)^2\theta\big)$	$(-1)^{m-1}2$	0	0
χ_m	1	-1	1	-1	$\dots$	$(-1)^{m-1}$	$(-1)^m$	-1	1
$\chi_m^\star$	1	-1	1	-1	$\dots$	$(-1)^{m-1}$	$(-1)^m$	1	-1

Second, if n is odd, then:

	1	a	a^2	a^3	$\dots$	a^{m-1}	a^m	b
	1	2	2	2	$\dots$	2	2	n
χ_0	1	1	1	1	$\dots$	1	1	1
$\chi_0^\star$	1	1	1	1	$\dots$	1	1	-1
χ_1	2	$2\cos(\theta)$	$2\cos(2\theta)$	$2\cos(3\theta)$	$\dots$	$2\cos\big((m-1)\theta\big)$	$2\cos(m\theta)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$
χ_m	2	$2\cos(m\theta)$	$2\cos(2m\theta)$	$2\cos(3m\theta)$	$\dots$	$2\cos\big((m-1)m\theta\big)$	$2\cos(m^2\theta)$	0

Here χ_i denotes the simple character associated to L_i, and similarly for the star versions. The first row is a representative of the associated conjugacy class, and the second row is the number of elements in the class.

Proof. Well-known, e.g. this follows from [Ser77, Section 5.3], and omitted. □

Notation 1A.4. In this and the next section we will abuse notation and use the same symbol for the idempotents in the group ring and the corresponding projectors from R to the simple summands.

As we have seen above, the real and complex $D_{2\cdot n}$-representations are essentially the same, so we can use the classical formulas for the central idempotents in $\mathbb{R}[D_{2\cdot n}]$ which give us the projectors $p_L^c\colon R \twoheadrightarrow L^{\dim_{\mathbb{R}}(L)}$:

$$p_L^c = \tfrac{\dim_{\mathbb{R}}(L)}{|D_{2\cdot n}|}\cdot \sum_{g\in D_{2\cdot n}} \chi(g^{-1})g. \tag{1A.5}$$

These project to the isotypic components of R, in particular, we need to make them finer for the two dimensional simple real $D_{2\cdot n}$-representations to have projectors $p_L\colon R \twoheadrightarrow L$ for every appearing summand separately.

Solving this for the one dimensional simple real $D_{2\cdot n}$-representations gives:

Lemma 1A.6. *We have the following formulas for the primitive idempotents in $\mathbb{R}[D_{2\cdot n}]$.*

(a) *For the one dimensional summands:*

$$n \text{ even or odd}\colon p_{L_0^{(\star)}} = \frac{1}{2n}\sum_{i=1}^{n}(a^i + \epsilon a^i b),$$

$$n \text{ even only}\colon p_{L_m^{(\star)}} = \frac{1}{2n}\big(1 + \sum_{i=1}^{n}(-1)^i\epsilon a^i b + \sum_{i=1}^{m-1}(-1)^i(a^i + a^{-i}) + (-1)^m a^m\big).$$

Here $(\star)$ means either star or not, with $\epsilon = 1$ for the non-star and $\epsilon = -1$ for the star case.

(b) *Let $\epsilon \in \{+1,-1\}$. We have the following formulas for the primitive idempotents in $\mathbb{R}[D_{2\cdot n}]$:*

$$p_{L_k}^{\epsilon} = \frac{1}{n}\big(id_{2\cdot n} + \sum_{i=1}^{n}\cos(ik\theta)a^i + \epsilon\sin(ik\theta)a^i b\big).$$

The sum $p_{L_k}^{+1} + p_{L_k}^{-1}$ is the central projector obtained from (1A.5).

Proof. (a). Directly from (1A.5) and the character table in Lemma 1A.3.

(b). The primitive central idempotents have their formulas given by (1A.5) and Lemma 1A.3 implies that, for n even, we get

$$p_{L_k}^{central} = \frac{1}{m}\sum_{i=1}^{n}\cos(ik\theta)a^i.$$

Thus, we get $p_{L_k}^{central} = p_{L_k}^{+1} + p_{L_k}^{-1}$. Moreover, a direct computation verifies $p_{L_k}^{x}p_{L_k}^{y} = \delta_{x,y}p_{L_k}^{x}$ where $x,y \in \{+1,-1\}$, and we are done. The n odd case can be shown verbatim. □

Lemma 1A.7. *Substituting $a \mapsto \mathtt{M}_a^r$, $b \mapsto \mathtt{M}_b^r$ in Lemma 1A.6 gives projectors $p_L\colon R \twoheadrightarrow L$.*

Proof. Immediate. □

Example 1A.8. For $n=3$ we have

$$p_{L_1^{+1}}=\begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad p_{L_1^{-1}}=\begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix},$$

as the projectors to the two dimensional summand. Moreover, the matrix for p_{L_0} has the entry $1/6$ everywhere, while $p_{L_0^{(\star)}}$ has the entries $\pm 1/6$ everywhere with the positive entry in the block diagonal, and with the negative entry in the block antidiagonal. ◇

The change-of-basis matrix $\mathtt{Q}\in\mathrm{End}_{\mathbb{R}}(R)$ between the matrices obtained from (1.1) and the decomposition in Lemma 1A.1, denote the matrices $\mathtt{N}_a$ and $\mathtt{N}_b$, is now given as follows.

Consider the order of decomposition as in Lemma 1A.1, where we order L_i^{+1} before L_i^{-1}. For every projector to a one dimensional simple real $D_{2\cdot n}$-representation choose a row, and for every projector to a two dimensional simple real $D_{2\cdot n}$-representation choose two linearly independent rows. We have thus chosen $2n$ rows that we put, in the above order, into a matrix $\mathtt{Q}^{-1}$.

Lemma 1A.9. *Let n be even.*

- **(a)** *Let $\mathtt{D}=\mathrm{diag}(1/2n,1/2n,1/n,\dots,1/n,1/2n,1/2n)$ be a diagonal matrix with the indicated diagonal entries. The matrix $\mathtt{Q}^{-1}$ is invertible, with inverse denoted by $\mathtt{Q}$, and we have $\mathtt{Q}^{-1}=\mathtt{D}\mathtt{Q}^T$.*
- **(b)** *The matrix $\mathtt{Q}$ is $D_{2\cdot n}$-equivariant and satisfies $\mathtt{N}_a=\mathtt{Q}^{-1}\mathtt{M}_a\mathtt{Q}$ and $\mathtt{N}_b=\mathtt{Q}^{-1}\mathtt{M}_b\mathtt{Q}$.*

Similarly for n is odd.

Proof. Note that the construction of $\mathtt{Q}^{-1}$ involves choosing $\dim_{\mathbb{R}} L$ linearly independent vectors for each L. Lemma 1A.6 then ensures that $\mathtt{Q}^{-1}$ realizes the base change for the Artin–Wedderburn decomposition, cf. [Ser77, Section 6.2]. In particular, $\mathtt{Q}^{-1}$ is invertible and (b) follows. The equation $\mathtt{Q}^{-1}=\mathtt{D}\mathtt{Q}^T$ is then easy to verify by hand.

The case n odd works verbatim. □

Notation 1A.10. To be completely explicit below, in the definition of $\mathtt{Q}$ we now fix the first row for all projectors and additionally the last row for the projectors to the two dimensional summands.

Example 1A.11. For $n=3$ we get

$$\mathtt{Q}=\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & -1 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad \mathtt{Q}^{-1}=\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 \\ 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\ 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix},$$

as one easily checks. Note that $\mathtt{Q}\mathtt{Q}^T=\mathrm{diag}(1/6,1/6,1/3,1/3,1/3,1/3)$. ◇

Let us describe $\mathtt{Q}$ more explicitly, and this works mutatis mutandis as for the cyclic group. Precisely, assume that n is even. For $L_0^{(\star)}$ and $L_m^{(\star)}$ we use the length $2n$ vectors $w_0=(1,\dots,1)$, $w_0^{\star}=(1,\dots,1,-1,\dots,-1)$ and $w_m=(1,-1\dots,1,-1,-1,1,\dots,-1,1)$ with the change of pattern in the middle, and $w_m^{\star}=(1,-1\dots,1,-1)$, respectively. For $L_k^{\pm 1}$ we take

$$+1\colon \begin{cases} w_k=\big(\cos((-j+1)k\theta)\big)_{j\in\{1,\dots,n\}}\cup\big(\sin(jk\theta)\big)_{j\in\{1,\dots,n\}}, \\ w_{-k}=\big(\sin((j-1)k\theta)\big)_{j\in\{1,\dots,n\}}\cup\big(\cos(-jk\theta)\big)_{j\in\{1,\dots,n\}}, \end{cases}$$
$$-1\colon \begin{cases} w_k=\big(\cos((-j+1)k\theta)\big)_{j\in\{1,\dots,n\}}\cup\big(-\sin(jk\theta)\big)_{j\in\{1,\dots,n\}}, \\ w_{-k}=\big(-\sin((j-1)k\theta)\big)_{j\in\{1,\dots,n\}}\cup\big(\cos(-jk\theta)\big)_{j\in\{1,\dots,n\}}, \end{cases}$$

where $\cup$ means concatenation. Similarly for n odd.

Lemma 1A.12. *For n even we have $\mathtt{Q}=(w_0,w_0^{\star},w_1^{+1},w_{-1}^{+1},\dots,w_{m-1}^{-1},w_{-(m-1)}^{-1},w_m,w_m^{\star})$. For n odd we have $\mathtt{Q}=(w_0,w_0^{\star},w_1^{+1},w_{-1}^{+1},\dots,w_m^{-1},w_{-m}^{-1})$. Here we read $\mathtt{Q}$ as a sequence of column vectors.*

Proof. By construction. □

1B. **ReLU and dihedral groups.** We redefine the order. Some special cases for $n \equiv 0 \bmod 4$ appear:

Definition 1B.1. The ***order*** $\mathrm{ord}(L)$ of L is defined as the pair of the orders of the action matrix for a and b. We then define $\mathrm{ord}'(L)_a$ as in [GTW23, ???] and

$$\mathrm{ord}'(L)_b = \begin{cases} \mathrm{ord}(L)_b & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even and } (L = L_m \text{ or } \mathrm{ord}'(L)_a \equiv 0 \bmod 4), \\ 5 & \text{if } n \text{ is even and } (L = L_m^* \text{ or } \mathrm{ord}'(L)_a \equiv 2 \bmod 4), \end{cases}$$

where we use subscripts to indicate the a and b component of $\mathrm{ord}(L)$.

With the adjusted definition of the order we get the analog of [GTW23, ???] which reads almost exactly the same.

Theorem 1B.2. *Consider the interaction graph Γ_{ReLU}. Every vertex has a loop. Moreover, there is a non-loop edge from L to K if and only if $\mathrm{ord}(K)$ divides $\mathrm{ord}'(L)$ componentwise.*

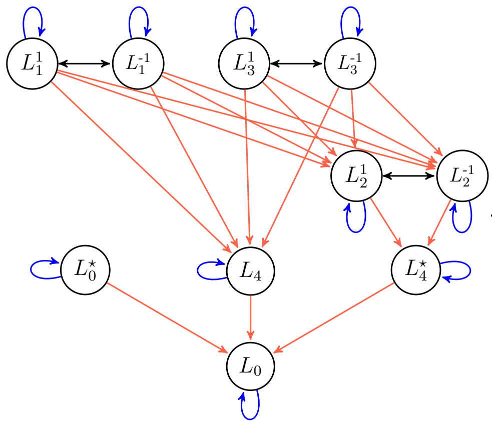
The isotypic interaction graph $\mathrm{i}\Gamma_{\mathsf{ReLU}}$ is obtained from Γ_{ReLU} by identifying the vertices for L_i^{+1} and L_i^{-1}.

Note that this implies that most example essentially stay the same as for the cyclic group, but some of the modules are doubled. The main difference comes from the one dimensional real $D_{2\cdot n}$-representations, which is what the next example focuses on.

Example 1B.3. Take $n = 8$. Then we have

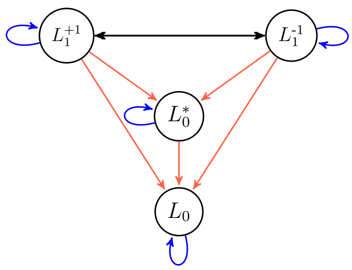
$$R \cong L_0 \oplus L_0^\star \oplus L_1^1 \oplus L_1^{-1} \oplus L_2^1 \oplus L_2^{-1} \oplus L_3^1 \oplus L_3^{-1} \oplus L_4 \oplus L_4^\star.$$

The orders and adjusted orders, from left to right, are $(1,1)$, $(1,2)$, $(8,2)$, $(8,2)$, $(4,2)$, $(4,2)$, $(8,2)$, $(8,2)$, $(2,1)$, $(2,2)$, and $(1,1)$, $(1,2)$, $(4,3)$, $(4,3)$, $(2,5)$, $(2,5)$, $(4,3)$, $(4,3)$, $(2,3)$, $(2,5)$. Then Theorem 1B.2 gives

$\Gamma_{\mathsf{ReLU}} =$  .

We omitted the arrows going from $L_k^{\pm 1}$ to L_0 to avoid clutter. The top part of this graph is a form of doubling of the cyclic case. ◇

Example 1B.4. For $n = 3$ we get

$\Gamma_{\mathsf{ReLU}} =$  ,

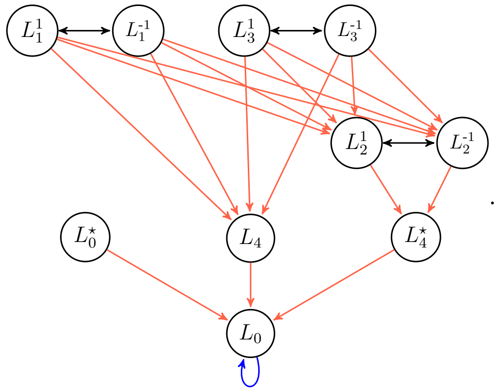
as a calculation verifies. ◇

1C. **Absolute value and dihedral groups.** For the absolute value we get almost the same theorem:

Theorem 1C.1. *There is an edge from L to K in Γ_{Abs} if and only if $\mathrm{ord}(K)$ divides $\mathrm{ord}'(L)$ componentwise.*

The isotypic interaction graph $\mathrm{i}\Gamma_{\mathsf{Abs}}$ is obtained from Γ_{Abs} by identifying the vertices for L_i^{+1} and L_i^{-1}.

Example 1C.2. Again omitting some arrows to L_0, for $n = 8$ one gets

$\Gamma_{\mathsf{Abs}} =$ 

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which is almost the same as in Example 1B.3. ◇

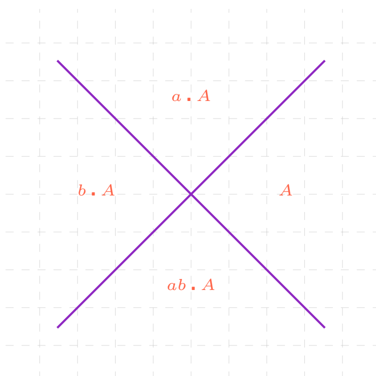
1D. **The piecewise linear maps for the dihedral group.** The analysis of the various piecewise linear maps is similar to [**GTW23**, ???] and we will be brief and focus on the differences.

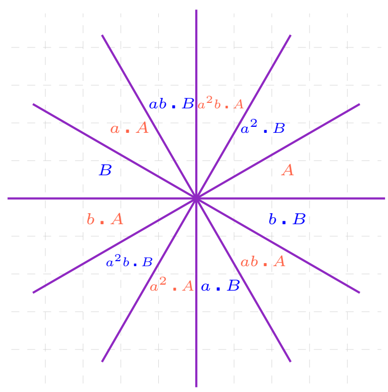
Definition 1D.1. The ***hyperplane arrangment*** associated to L_i is almost in [**GTW23**, ???] but with the following adjustments:

- **(a)** For all one dimensional simple real $D_{2\cdot n}$-representations we take the hyperplane arrangment of L_0 from [**GTW23**, ???].
- **(b)** Let $\dim_{\mathbb{R}} L_k = 2$. For $n \equiv 2 \bmod 4$ or n odd we take twice as many hyperplanes compared to [**GTW23**, ???], namely the ones for $\zeta_k^{1/2}$ in case $n \equiv 2 \bmod 4$ and the ones for $\zeta_k^{1/4}$ for n odd.

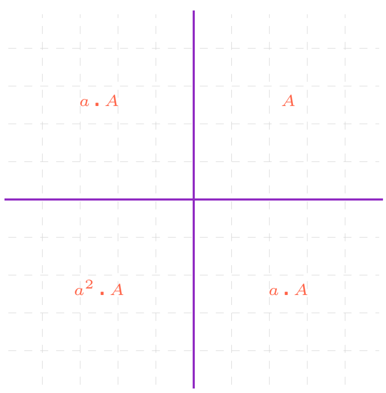
The case $n \equiv 0 \bmod 4$ remains the same.

Example 1D.2. For $n \in \{2, 3, 4\}$ the hyperplane arrangements for L_1 are as follows:

$n = 2\colon$ 

, $n = 3\colon$ 

,

$n = 4\colon$ 

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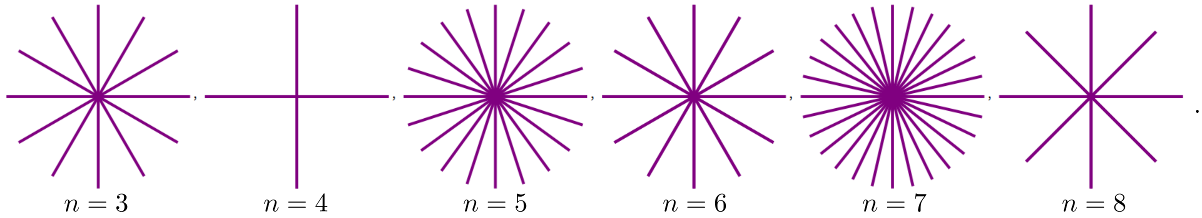
For later use, we have also indicated the $D_{2\cdot n}$-orbits. Note that we do not need the generator b for $n = 4$ to get one orbit. We also list all hyperplane arrangements for $n \in \{3, ..., 8\}$:



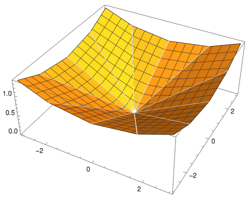
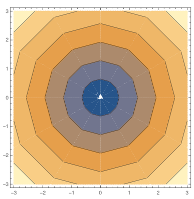
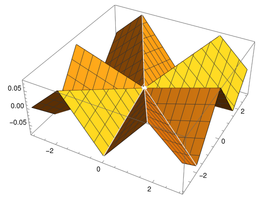
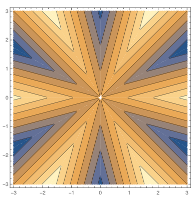
This should be compared to [**GTW23**, ???]. ◇

Proof of Theorem 1B.2 and Theorem 1C.1. As a first step, we get analogs of [**GTW23**, ???] and [**GTW23**, ???] by using the same arguments for the hyperplane arrangements in Definition 1D.1. The only new observation in this case is that for $n \equiv 2 \bmod 4$ or n odd the extra generator (when compared to C_n) b doubles the size of orbits, but we also double the number of hyperplanes, so the overall statement remains the same.

Using Lemma 1A.9 and Lemma 1A.12, we then continue along the same lines as in the proof in [**GTW23**, ???].

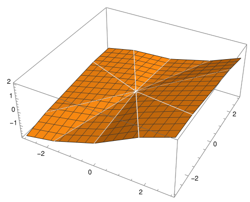
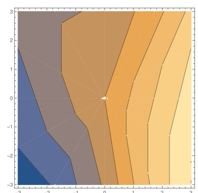
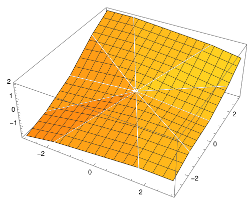
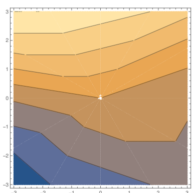
That $i\Gamma_{\mathsf{ReLU}}$ is as claimed can be proven mutatis mutandis. □

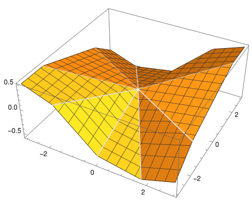
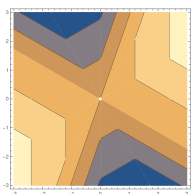
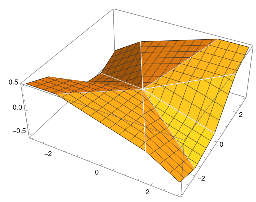
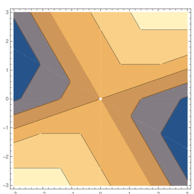
The explicit description of the various piecewise linear maps is similar to the cyclic group case discussed in [**GTW23**, ???]. Let us therefore just exemplify the result. For example, for $n = 3$ we have $R \cong L_0 \oplus L_0^* \oplus L_1^{+1} \oplus L_1^{-1}$ of dimension $1, 1, 2, 2$. For L_1^{+1} we therefore have two piecewise linear maps that we can illustrate using level sets, namely:

ReLU^0_{1+} :  ,  , ReLU^{0*}_{1+} :  ,  ,

where we use the evident adjustment of [**GTW23**, ???]. These piecewise linear maps are almost the same as the ones for C_n, but with doubled hyperplanes.

Finally, let us look at the two piecewise linear maps ReLU^{1+}_{1+} and ReLU^{1-}_{1+}. These are maps $\mathbb{R}^2\mathbb{C} \cong\to \mathbb{C} \cong \mathbb{R}^2$, and we illustrate the real part (left) and imaginary part (right) of them:

ReLU^{1+}_{1+} :  ,  ,  ,  ,

ReLU^{1-}_{1+} :  ,  ,  ,  .

Note that these are quite different maps.

References

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