Potential master or Ph.D. project of I WANT YOU in 2023

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Key information

Candidate. I WANT YOU.

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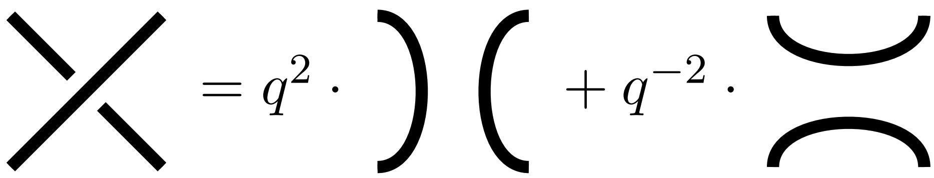
Research areas. Low dimensional topology, representation theory, and graph theory, more specifically diagrammatic categories and their relation to graph coloring problems.

Title. *"Representation theory of $SO(3)$ and the four color theorem"*.

First read. [Abr08] for $SL(2)$ and [MPS17] for $SO(3)$.

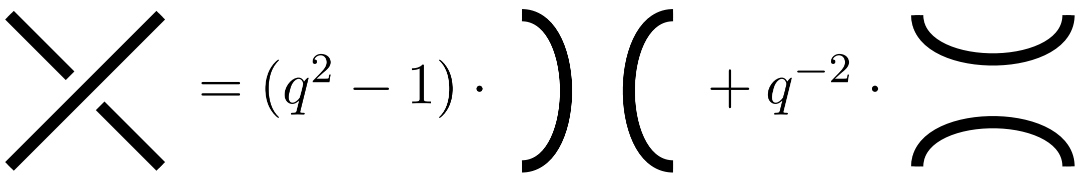
One of the most famous skein relation is the one coming from the Jones polynomial:

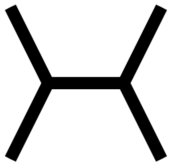


This relation is the key to define the Jones polynomial and it holds in the representation category associated to $SL(2)$. Indeed, the pictures above describe $SL(2)$ which is modeled on crossingless matchings.

The importance of skein relations is that it allows you to make teleological connections between knot polynomials and other, seemingly distant areas of mathematics like representation theory, quantization, algebraic geometry, gauge theory, and low dimensional geometry and topology.

This is the main point of the current project: we use the representation category associated to $SO(3)$ to study the four color theorem. The relevant skein relation is now:



$$-(q^2 + q^{-2}) \cdot$$ 

In this relation a new type of object appears: a web = a trivalent graph (the right-hand picture). In fact, the representation category associated to $SO(3)$ is modeled by trivalent graphs called webs. These graphs combinatorially describe $SO(3)$ and four colorings of planar graphs at the same time.

Minimal goal. Write a concise and self-contained summary of the representation category of $SO(3)$.
Average goal. Make the connection to the four color theorem by explaining Tait colorings and their relation to $SO(3)$ webs.
Optimal goal. To to the categorified picture or address some of the open questions mentioned below.
Key. Use diagrammatic methods and combinatorial ways to give algorithm to calculate the number of four colorings of planar graphs.

The thesis in details – minimal goal

The minimal master project should be structured as follows.

- Write an introduction and explain the main ideas of monoidal categories, see *e.g.* [EGNO15] or [TV17]. Explain how your master thesis fits into this framework, *i.e.* in what sense diagrammatic monoids
- Summarize basics about diagrammatic algebra, see *e.g.* [TV17, Chapter I].
- Explain the diagrammatic representation category of $SO(3)$ in [MPS17, Section 4].

The thesis in details – average goal

As above, but add:

- Define the evaluation algorithm (by removing of faces) and explain why every web can be evaluated using face removals up to a pentagon only. The argument, in some form, appeared in [Kup94, Lemma 2.2], but can be cleaned a bit.
- Explain the relation ot Tait colorings, and how Tait colorings fit to the four color theorem, see for example [FF98, Section 4.8].

The thesis in details – optimal goal

Here are some open question which (if time suffices) deserve further study.

- Can one prove the four color theorem for a subclass of graphs using webs?
- What about the categorification as in [KM19].

References

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