MAT572: SEMINAR DIAGRAMMATIC ALGEBRA: A PROTOTYPICAL EXAMPLE – PLAN

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Disclaimer

Please do not hesitate to contact me in case of questions. (The contact data can be found below.) Never forget to give many examples during your talk!

What?

The theory of Soergel bimodules emerged in the work of Wolfgang Soergel in the 1990s and 2000s. Soergel bimodules are certain algebraic objects, and the collection of these forms a monoidal category under tensor product. Soergel bimodules are fairly elementary objects, yet they have deep links to representation theory, topology and geometry, and have a remarkably rich internal structure.

The purpose of this seminar is to provide a comprehensive introduction to the theory of Soergel bimodules, in particular, using diagrammatic methods. Indeed, it is not an exaggeration to say that the computational power afforded by diagrammatics was the key breakthrough that allowed many recent advances on the one hand, but is beautiful in its own right on the other hand.

The seminar follows the book [EMTW20].

Who?

BSC or MSC or PhD students in Mathematics interested in a mixture of linear algebra and combinatorics, but everyone is welcome.

Preliminaries?

Some linear algebra, algebra and category theory. More specifically: Algebra, means modules, bimodules, tensor products, linear algebra *etc.*. Category theory, means equivalence of categories, isomorphisms of functors, adjunctions *etc.*. But we can also learn these as we move along.

Where and when?

- ▶ Time and date.
  - Every Monday from 13:00–14:45.
  - All talks will be given using zoom https://zoom.us/download. Zoom links will be sent to participants shortly before the seminar starts.

- First meeting: Monday 21.Sep.2020. Last meeting: Monday 23.Nov.2020.

▶ Website http://www.dtubbenhauer.com/seminar-soergel-2020.html

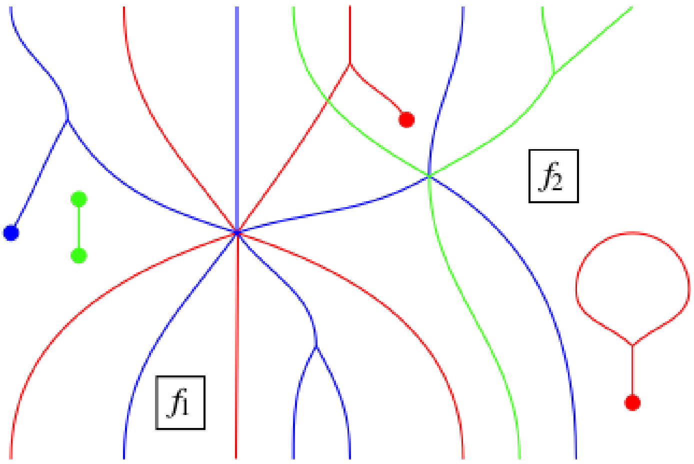

FIGURE 1. Soergel diagrams. (Picture from the course book.)

Schedule and some details.

General rule

Proofs are supposed to be done "behind closed doors", *i.e.* they can only be really understood by line-to-line reading and not during a 60 minute online talk. Instead your goal is to explain ideas and examples. Keep that in mind while preparing your talk.

General rule

The material in each chapter of the course book [EMTW20] are a bit too much for a talk. So skip anything unnecessary (proofs for example).

▷ 1^{th} talk "The classical theory I".

- **Speaker.** Daniel.
- **Date.** 21.Sep.2020, 13:00–14:45.
- **Topic.** Coxeter groups, the beginnings.

- **Plan.** Introduce Coxeter groups algebraically [EMTW20, Definition 1.1] and spent time to discuss the first two examples, type A [EMTW20, Section 1.1.2] and type B [EMTW20, Section 1.1.3]. The next example on the list is different in flavor, but also crucial: the dihedral groups [EMTW20, Section 1.1.5]. Then explain the geometric representation [EMTW20, Definition 1.27], with [EMTW20, Proposition 1.28] being the highlight of your talk. Take a short digression and explain the combinatorial notions presented in between [EMTW20, Section 1.2.1] and [EMTW20, Section 1.2.6], as they are crucial for understanding the combinatorics. (Don't miss the various facts regarding these, *e.g.* [EMTW20, Theorem 1.56].) Finish by mentioning the classification of finite Coxeter groups [EMTW20, Theorem 1.34].
- **Main goals.** Give a combinatorial introduction to the realm of Coxeter groups. Your talk is all about examples.
- **Note.** All the various combinatorial data (length function *etc.*) should be explained hand-in-hand with the type A and B examples. Try to solve [EMTW20, Exercise 1.58].
- **Literature.** [EMTW20, Chapter 1].

▷ 2^{th} talk "The classical theory II".

- **Speaker.** Rizacan.
- **Date.** 28.Sep.2020, 15:00–16:45.
- **Topic.** Coxeter groups and reflection groups.
- **Plan.** Start by explaining what reflection groups are and why and how they form Coxeter groups [EMTW20, Chapter 2 up to Theorem 2.11]. (Skip the various lemmas.) Then explain the geometric notion of a stroll with [EMTW20, Theorem 2.20] being then the main outcome. The crucial notion which you then need to explain carefully is the Coxeter complex [EMTW20, Section 2.6]. End with the surprising statement about the Coxeter complex [EMTW20, Theorem 2.30].
- **Main goals.** To explain the geometry behind Coxeter groups, *i.e.* reflections, strolls and the Coxeter complex.
- **Note.** In your talk pictures and examples are key, and the book gives several which you can use. (Note also additional material which you can find online.) Do [EMTW20, Exercise 2.25].
- **Literature.** [EMTW20, Chapter 2].

▷ 3^{th} talk "The classical theory III".

- **Speaker.** Daniel.

- **Date.** 05.Oct.2020, 13:00–14:45.
- **Topic.** Kazhdan–Lusztig theory.
- **Plan.** Start by explaining one of the main beasts of this seminar – the Hecke algebra [EMTW20, Definition 3.1] which is a deformation of its associated Coxeter group. The Hecke algebra has two crucial bases, the standard (a.k.a. easy, but not very useful) basis [EMTW20, Section 3.1.1] and the Kazhdan–Lusztig (a.k.a. hard, but interesting) basis [EMTW20, Section 3.2], and the my goal of your talk is to explain these. To explain why the Kazhdan–Lusztig exists, which is the crucial point but tricky, do the motivating example [EMTW20, Section 3.3.1] in all details and just state [EMTW20, Theorem 3.25]. Along the way explain notions such as the standard pairing, see *e.g.* [EMTW20, Theorem 3.21], and the various Kazhdan–Lusztig polynomials that appear.
- **Main goals.** Clearly your main objective is to explain [EMTW20, Theorem 3.25]. Do as many examples, as possible.
- **Note.** Example calculations are key to understand the combinatorics. For example [EMTW20, Exercise 3.24] is a good source of examples.
- **Literature.** [EMTW20, Chapter 3].

▷ 4^{th} talk "The classical theory IV".

- **Speaker.** Anna.
- **Date.** 12.Oct.2020, 13:00–14:45.
- **Topic.** Soergel bimodules, the beginnings.
- **Plan.** After some algebraic notions, take your time to explain the amazing Chevalley–Shephard–Todd theorem [EMTW20, Theorem 4.2], which is a vast generalization of the theory of symmetric polynomials. (Make this clear.) This theorem goes hand-in-hand with its associated examples, *e.g.* [EMTW20, Example 4.11]. Then go on with the Demazure operator and its properties [EMTW20, Lemma 4.15], and do some examples with it. Next, the key players, the Bott–Samelson bimodules [EMTW20, Section 4.5], which need to explained carefully. In particular, polynomial forcing [EMTW20, Exercise 4.28] is crucial. Then comes [EMTW20, Definition 4.29] which is the topic of the seminar. Finish by doing as many examples as possible [EMTW20, Section 4.7].
- **Main goals.** Introduce Soergel bimodules, making sure that it is clear that they arise via symmetric polynomials.
- **Note.** Soergel bimodules are explicit and combinatorial gadgets. This has to be made clear, so *e.g.* [EMTW20, Section 4.7] is crucial.
- **Literature.** [EMTW20, Chapter 4].

▷ 5^{th} talk "The classical theory V".

- **Speaker.** Najma.
- **Date.** 19.Oct.2020, 13:00–14:45.
- **Topic.** Soergel bimodules, the feast.
- **Plan.** Start by introducing standard bimodules [EMTW20, Definition 5.1], explain why they are "easy" [EMTW20, Exercise 5.3], and in what sense these categorify the standard basis [EMTW20, Section 5.2]. Now, if standard bimodules categorify the standard basis and Soergel bimodules the Kazhdan–Lusztig basis, then there should be a "change of basis matrix" between them, which is the content of [EMTW20, Section 5.3] (of particular importance are [EMTW20, Definition 5.11 and Exersice 5.12], and dually [EMTW20, Definition 5.15 and Exersice 5.16]). Inverting helps, which is the context of [EMTW20, Section 5.4], in particular, note [EMTW20, (5.24)]. The take your time and explain [EMTW20, Section 5.5] carefully, which in some sense is the main statement of the book.
- **Main goals.** Of course, explaining [EMTW20, Theorem 5.24].
- **Note.** Make sure to sell the philosophy that "standard=easy", "Kazhdan–Lusztig respectively Soergel=interesting" and how these two sides are related.¸"'
- **Literature.** [EMTW20, Chapter 4].

▷ 6^{th} talk "The diagrammatic theory I".

- **Speaker.** Katja.
- **Date.** 26.Oct.2020, 13:00–14:45.
- **Topic.** Drawing monoidal categories.
- **Plan.** Start by explaining string diagrams [EMTW20, Sections 2.1 and 2.2], which is Poincaré dual to the classical calculus of categories. Move on to [EMTW20, Example 7.11 and Section 7.4], which are prototypical examples of diagram categories, with [EMTW20, Remark 7.12] being a crucial remark. Finish by explaining how many categories come with a built-in topological interpretation [EMTW20, Section 7.5].
- **Main goals.** Explain the diagrammatic calculus of categories with the Temperley–Lieb category being the crucial example.
- **Note.** Of course, this talk lives from as many diagrams as possible. Also of interest are [EMTW20, Exercises 7.16 and 7.17].
- **Literature.** [EMTW20, Chapter 7].

▷ 7$^{\text{th}}$ talk "The diagrammatic theory II".

- **Speaker.** Aurelia.
- **Date.** 02.Nov.2020, 13:00–14:45.
- **Topic.** Frobenius extensions.
- **Plan.** The talk is about diagrammatics for Frobenius algebra objects, which starts right away after some basic definitions [EMTW20, Section 8.1.2]. The main statement is the [EMTW20, Proposition 8.6], which explains the diagrammatic nature of Frobenius algebras. Go an an play with isotopy [EMTW20, Section 8.1.3], explaining "non-sense pictures" and why this is the whole point of diagram calculus. Finally, after some intermediate section on Frobenius extensions [EMTW20, Section 8.1.4], come back to Soergel bimodules by explaining their diagrammatic calculus [EMTW20, Section 8.2].
- **Main goals.** Present the so-called one-color calculus carefully [EMTW20, Section 8.2], by also presenting some of the exercises (*e.g.* [EMTW20, Exercise 8.33] is fun).
- **Note.** Give a cobordism interpretation of the Frobenius relation, which explains why they are topological in nature.
- **Literature.** [EMTW20, Chapter 8].

▷ 8$^{\text{th}}$ talk "The diagrammatic theory III".

- **Speaker.** Daniel.
- **Date.** 09.Nov.2020, 13:00–14:45.
- **Topic.** The dihedral cathedral.
- **Plan.** The two-color Soergel calculus is the main topic of this talk, with [EMTW20, Theorems 9.5 and 9.6] being the first major statements. This was about the $m_{st}=\infty$ case, then start motivating the $m_{st}=m$ case [EMTW20, Remark 9.8]. Up next is the (slightly reformulated) Temperley–Lieb category again [EMTW20, Definition 9.11], which is the degree 0 part of the Soergel calculus [EMTW20, Proposition 9.13 and Theorem 9.22]. From there one can deduce the two-color relations which then gives [EMTW20, Theorems 9.31 and 9.32], the two main statements of this talk.
- **Main goals.** This talk is about the dihedral case, and the dihedral case is a subcase of the general situation, and should be presented as such.
- **Note.** There are some nice exercise which you might want to do and present, *e.g.* [EMTW20, Exercise 9.35].

- **Literature.** [EMTW20, Chapter 9].

▷ 9$^{\text{th}}$ talk "The diagrammatic theory IV".

- **Speaker.** Li.
- **Date.** 16.Nov.2020, 13:00–14:45.
- **Topic.** The diagrammatic Hecke category.
- **Plan.** After explaining the philosophy carefully again [EMTW20, Section 10.1] and a reminder on diagrams, move on to the main definition [EMTW20, Definition 10.6]. Although we have seen some of the relations in [EMTW20, Section 10.2], state them, where you should be too picky with the more complicated looking ones. More important are example [EMTW20, Section 10.2.3] and move on to the genuine presentation [EMTW20, Definition 10.18]. Afterwards one of the main statements of the whole book [EMTW20, Theorem 10.20], which you should explain carefully.
- **Main goals.** Explain the diagrammatic Hecke category and how to work with it.
- **Note.** Take your time explaining the philosophy [EMTW20, Section 10.1], which you can underpin using examples we have seen before.
- **Literature.** [EMTW20, Chapter 10].

▷ 10$^{\text{th}}$ talk "The diagrammatic theory V".

- **Speaker.** Hussein.
- **Date.** 23.Nov.2020, 13:00–14:45.
- **Topic.** Soergel's categorification theorem.
- **Plan.** Spent the first part of your talk explaining the philosophy and construction of light leaves [EMTW20, Section 10.4]. This is mostly explaining the algorithm [EMTW20, Section 10.4.2] which will lead you to [EMTW20, Corollary 10.43]. Then move on towards [EMTW20, Theorem 11.1], which you need to explain carefully. In order for this to make sense you need to explain a bit about [EMTW20, Section 11.2], but be as brief as possible.
- **Main goals.** Explain the light leaves basis and the main statement of the whole book a.k.a. the Soergel–Elias–Williamson categorification theorem.
- **Note.** Examples are key in the part about light leaves. Explain the algorithm on as many examples as you can.
- **Literature.** [EMTW20, Chapters 10 and 11].

References

[EMTW20] B. Elias, S. Makisumi, U. Thiel, G. Williamson. *Introduction to Soergel bimodules.* RSME Springer Series, volume 5. Springer International Publishing, 2020.

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