MAT558: SEMINAR ON TENSOR CATEGORIES – PLAN

DANIEL TUBBENHAUER

Disclaimer

Please do not hesitate to contact me in case of questions. (The contact data can be found below.) Never forget to give many examples during your talk!

What?

The philosophy is: Categories generalize (categorify) sets; categories with additional structure should categorify sets with additional structure. In particular, the notion of a tensor category and their categorical representations can be seen as a categorification of rings and algebras and their representations, making the rich story of rings and algebras even more interesting. The purpose of this seminar is to understand how this works precisely.

The seminar follows the book [EGNO15].

Who?

Master students and upwards interested in a mixture of category theory, algebra and representation theory.

Where and when?

- ▶ Time and date.
  - Every Monday from 15:15–17:00.
  - Room Y27H28, University Zurich, Institute of Mathematics.
  - First meeting: 19.Feb.2018.
- ▶ Preliminary meeting: 05.Feb.2018, 15:15–17:00, room Y27H28.
- ▶ Website http://www.dtubbenhauer.com/seminar-tensor-2018.html

Schedule and some details.

Note: You can always find more references at the end of each chapter in [EGNO15], called bibliographical notes.

- ▷ 1th talk "Abelian categories I+II".
  - **Speaker:** Andres
  - **Date:** 19.Feb.2018, 15:15–17:00.
  - **Topic:** Abelian categories, basic notions, properties and examples & functors between abelian categories.

- **Plan:** Define additive categories, define abelian categories, notion of exact sequences. State the Jordan–Hölder theorem [EGNO15, Theorem 1.5.4], the Krull–Schmidt theorem [EGNO15, Theorem 1.5.7], and introduce the Grothendieck group & introduce projective and injective objects and resolutions by these, and finite abelian categories. Discuss (representable) functors between abelian categories and the two notions of Grothendieck groups.
- **Main goal:** Explain the content of Mitchell's theorem [EGNO15, Theorem 1.3.8]. Then discuss that Schur's Lemma [EGNO15, Lemma 1.5.2] and the theorems of Jordan–Hölder and Krull–Schmidt still hold in the abelian setting & the proposition about hom-spaces between simples [EGNO15, Proposition 1.8.4], the representability of functors [EGNO15, Proposition 1.8.10] and the Cartan matrix of an abelian category..
- **Note:** The book [EGNO15] is a bit sketchy in the first chapter. (Details can be found in [Bor94, Chapter 1].) Try to give many examples to illustrate the claims and statements.
- **Literature:** [EGNO15, Sections 1.1 to 1.5], see also [Bor94, Chapter 1], [ML98, Chapter VIII] & [EGNO15, Sections 1.6 to 1.8], see also [Bor94, Chapter 1].

▷ 2^{th} talk "Monoidal categories I"

- **Speaker:** Mariya
- **Date:** 26.Feb.2018, 15:15–17:00.
- **Topic:** Monoidal categories and functors, basic properties and examples.
- **Plan:** Introduce monoidal categories, their functors and some first basic properties. Give many examples, e.g. the example/proposition [EGNO15, Proposition 2.5.4] and the category of tangles [EGNO15, Example 2.3.14].
- **Main goal:** The only possible way to understand (versions of) category theory is to have many examples in mind. The category of vector spaces, the category of representations of a group and the category of endofunctors are further key examples. (These are [EGNO15, Examples 2.3.3, 2.3.4 and 2.3.12].)
- **Note:** There are a lot of diagrams to draw. Focus on some of them and only refer to the others. Moreover, please note [EGNO15, Remark 2.2.9] and [EGNO15, Sentence after Definition 2.4.5] – two possible confusing points.
- **Literature:** [EGNO15, Sections 2.1 to 2.6], see also [ML98, Chapter XI] or [Lei98].

▷ 3^{th} talk "Monoidal categories II"

- **Speaker:** Mariya
- **Date:** 05.Mar.2018, 15:15–17:00.
- **Topic:** Coherence and monoidal categories.
- **Plan:** Explain Mac Lane's coherence theorem and the strictification of monoidal categories. Explain [EGNO15, Remark 2.8.6] (in case the example therein was not explained in previous talks, explain it as well).

- **Main goal:** Carefully prove [EGNO15, Theorem 2.8.5] and [EGNO15, Theorem 2.9.2].
- **Note:** As for the previous talk, there are a lot of diagrams to draw. Focus on some of them and only refer to the others. (The point of your talk is that we want to "forget" these diagrams.)
- **Literature:** [EGNO15, Sections 2.8 to 2.9], see also [ML98, Chapter XI] or [Lei98].

▷ 4^{th} talk "Monoidal categories III"

- **Speaker:** Mariya
- **Date:** 12.Mar.2018, 15:15–17:00.
- **Topic:** Categorical groups and categorical actions.
- **Plan:** Explain the notion of rigidity, uniqueness of duals [EGNO15, Proposition 2.10.5], the tensor-hom adjunction [EGNO15, Proposition 2.10.8]. Explain the notion of a group acting on a category [EGNO15, Section 2.7] and the notion of a categorical group [EGNO15, Section 2.11].
- **Main goal:** Rigid monoidal categories play an important role and should be carefully explained. Next, categorical actions and categorical groups give a first hint about the "categorification which lies ahead".
- **Note:** As for the previous two talks, there are a lot of diagrams to draw. Focus on some of them and only refer to the others. (The point of your talk is that we want to "forget" these diagrams.)
- **Literature:** [EGNO15, Sections 2.10, 2.7, 2.11] (in this order), see also [Kas95, Chapter XIV].

▷ 5^{th} talk "$\mathbb{Z}_+$-rings I"

- **Speaker:** Jon
- **Date:** 19.Mar.2018, 15:15–17:00.
- **Topic:** $\mathbb{Z}_+$-rings and their Perron–Frobenius dimensions.
- **Plan:** Define $\mathbb{Z}_+$-rings, discuss their basic properties and explain [EGNO15, Example 3.1.9]. Recall and prove the Perron–Frobenius theorem [EGNO15, Theorem 3.2.1] and define the Perron–Frobenius dimension [EGNO15, Definition 3.3.3]. Explain [EGNO15, Example 4.10.6] from the viewpoint of $\mathbb{Z}_+$-rings.
- **Main goal:** The Perron–Frobenius theorem [EGNO15, Theorem 3.2.1] is a very classical and useful statement. State it and prove the first basic properties which follow for $\mathbb{Z}_+$-rings [EGNO15, Proposition 3.3.4].
- **Note:** Beware that the standard name is Perron–Frobenius theory. Moreover, morally speaking, the $\mathbb{Z}_+$-rings should be called $\mathbb{Z}_{\geq 0}$-rings.
- **Literature:** [EGNO15, Sections 3.1 to 3.3], see also [CE08, Section 4].

▷ 02.Apr.2018 Eastern

▷ 6^{th} talk “$\mathbb{Z}_+$-rings II”

- **Speaker:** Jon
- **Date:** 26.Mar.2018, 15:15–17:00.
- **Topic:** $\mathbb{Z}_+$-rings and their representations.
- **Plan:** Continue after [EGNO15, Proposition 3.3.4] with some more properties of the Perron–Frobenius dimension. Explain [EGNO15, Proposition 3.3.15] carefully. Introduce $\mathbb{Z}_+$-modules with [EGNO15, Proposition 3.4.6] being the most important statement here. Do not spent to much time on the later sections, but rather explain [EGNO15, Example 4.10.6] and [EK95, Section 3.3] from the viewpoint of $\mathbb{Z}_+$-rings.
- **Main goal:** $\mathbb{Z}_+$-rings have an $\mathbb{Z}_{\geq 0}$-structure. So their representation should reflect this structure, where the Perron–Frobenius theory kicks in. Explain this in examples.
- **Note:** As for the previous talk, beware that the standard name is Perron–Frobenius theory. Moreover, morally speaking, the $\mathbb{Z}_+$-rings should be called $\mathbb{Z}_{\geq 0}$-rings.
- **Literature:** [EGNO15, Sections 3.3 to 3.8] and [EK95, Section 3.3], see also [CE08, Section 4].

▷ 7^{th} talk “Tensor categories I”

- **Speaker:** Nino
- **Date:** 09.Apr.2018, 15:15–17:00.
- **Topic:** Tensor categories and tensor functors.
- **Plan:** Define tensor categories and give the first examples, i.e. [EGNO15, Examples 4.1.2 and 4.1.3]. Define tensor functors and state their basic properties, e.g. left duals imply right duals [EGNO15, Proposition 4.2.10]. Discuss semisimplicity of the unit object [EGNO15, Section 4.3], be brief in [EGNO15, Section 4.4].
- **Main goal:** The purpose of this talk is to slowly discuss the notion of tensor categories and their functors. The first crucial result is [EGNO15, Theorem 4.3.8], showing that the unit plays an important role.
- **Note:** Beware that some people understand something different under the name tensor category, e.g. some authors do not assume tensor categories to be abelian.
- **Literature:** [EGNO15, Sections 4.1 to 4.4].

▷ 8^{th} talk “Tensor categories II”

- **Speaker:** Nino
- **Date:** 16.Apr.2018, 15:15–17:00.
- **Topic:** Traces and tensor subcategories.
- **Plan:** Discuss the notion of a categorical trace and prove its basic properties [EGNO15, Proposition 4.7.3]. Explain pivotality and prove [EGNO15, Proposition

4.7.12]. Discuss the relationship between left and right traces [EGNO15, Theorem 4.7.15], as well as [EGNO15, Propositions 4.8.1 and 4.8.4]. Finish with the notion of a tensor subcategory [EGNO15, Section 4.11].

- **Main goal:** The notion of traces is a very important tool to study tensor categories. Explain the concept carefully with examples.
- **Note:** As in the previous talk, beware that some people understand something different under the name tensor category, e.g. some authors do not assume tensor categories to be abelian.
- **Literature:** [EGNO15, Sections 4.8 4.11].

▷ 9^{th} talk "Tensor categories III"

- **Speaker:** Felix
- **Date:** 23.Apr.2018, 15:15–17:00.
- **Topic:** Grothendieck rings and Perron–Frobenius dimensions of tensor categories.
- **Plan:** Discuss the Grothendieck groups of tensor categories [EGNO15, Sections 4.4 and 4.9]. Hereby, [EGNO15, Propositions 4.5.4 and 4.9.1]. Also important are [EGNO15, Remark 4.5.6 and Proposition 4.5.7]. Explain what categorification of $\mathbb{Z}_+$-rings means. Give many examples along the way, e.g. [EGNO15, Example 4.10.6].
- **Main goal:** Explain how tensor categories categorify $\mathbb{Z}_+$-rings, i.e. discuss [EGNO15, Section 4.10] carefully.
- **Note:** As in the previous talks, beware that some people understand something different under the name tensor category, e.g. some authors do not assume tensor categories to be abelian.
- **Literature:** [EGNO15, Sections 4.4, 4.9 and 4.10] and see what you need from the sections between [EGNO15, Section 4.4] and [EGNO15, Section 4.10].

▷ 10^{th} talk "Finite tensor categories"

- **Speaker:** Felix
- **Date:** 30.Apr.2018, 15:15–17:00.
- **Topic:** Finite tensor categories and Perron–Frobenius revisited.
- **Plan:** Discuss projective and injective objects of finite tensor categories, and what kind of functors preserve projective [EGNO15, Theorem 6.1.16] or simples [EGNO15, Proposition 6.3.1]. Then do the comparison of Perron–Frobenius dimensions of finite tensor categories [EGNO15, Corollary 6.2.2, Proposition 6.3.3 and Corollary 6.3.5]. End with [EGNO15, Theorem 6.6.1].
- **Main goal:** The notion of Perron-Frobenius dimension is very important for finite tensor categories. That is, [EGNO15, Corollary 6.2.2, Proposition 6.3.3 and Corollary 6.3.5] are categorical analogs of injective/surjective linear maps between between vector spaces.
- **Note:** We skip the chapter on Hopf algebras. Throughout, avoid all notions to Hopf algebras. (You can use them as a black-box if necessary.)

- **Literature:** [EGNO15, Sections 6.1 to 6.3 and 6.6].

▷ 11^{th} talk "Module categories I"

- **Speaker:** Raul
- **Date:** 07.May.2018, 15:15–17:00.
- **Topic:** Module categories – basic constructions.
- **Plan:** Introduce module categories in two different ways, i.e. [EGNO15, Definition 7.1.1] and [EGNO15, Proposition 7.1.3]. Explain the notion of module functors and equivalence of module categories [EGNO15, Section 7.2]. Same for multitensor categories [EGNO15, Section 7.3]. End with the examples in [EGNO15, Section 7.4]. State the classification of module categories related to [EGNO15, Example 4.10.6] and [EK95, Section 3.3], see [Os03, Theorem 6.1] (and also [KO02]).
- **Main goal:** Explain the first examples of module categories [EGNO15, Section 7.4].
- **Note:** Beware that many statements are formulated in remarks or exercises.
- **Literature:** [EGNO15, Sections 7.1 to 7.4] and [Os03, Theorem 6.1], see also and also [KO02].

▷ 12^{th} talk "Module categories II"

- **Speaker:** Raul
- **Date:** 14.May.2018, 15:15–17:00.
- **Topic:** Module categories and $\mathbb{Z}_+$-modules.
- **Plan:** Discuss module categories over finite tensor categories [EGNO15, Section 7.5]. Prove the reducibility proposition [EGNO15, Proposition 7.6.7]. Next, explain [EGNO15, Proposition 7.7.2]. If time suffices, say more about [Os03, Theorem 6.1]. (See previous talk.)
- **Main goal:** State and explain [EGNO15, Proposition 7.7.2] which is the categorification of representation theory of $\mathbb{Z}_+$-rings.
- **Note:** As in the previous talk, beware that many statements are formulated in Remarks or exercises.
- **Literature:** [EGNO15, Sections 7.5 to 7.7] and [Os03, Theorem 6.1], see also and also [KO02].

References

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D.T.: Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, Campus Irchel, Office Y27J32, CH-8057 Zürich, Switzerland, www.dtubbenhauer.com

E-mail address: daniel.tubbenhauer@math.uzh.ch