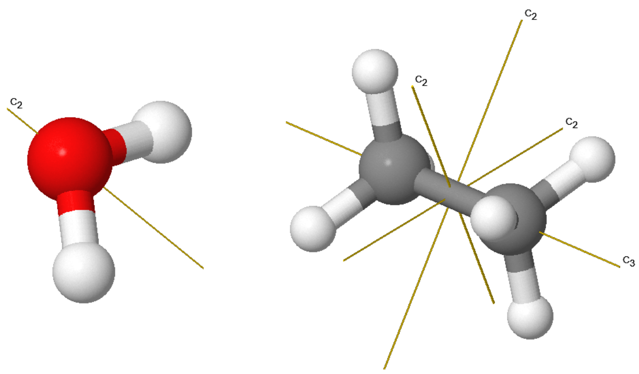
LECTURE: REPRESENTATION THEORY

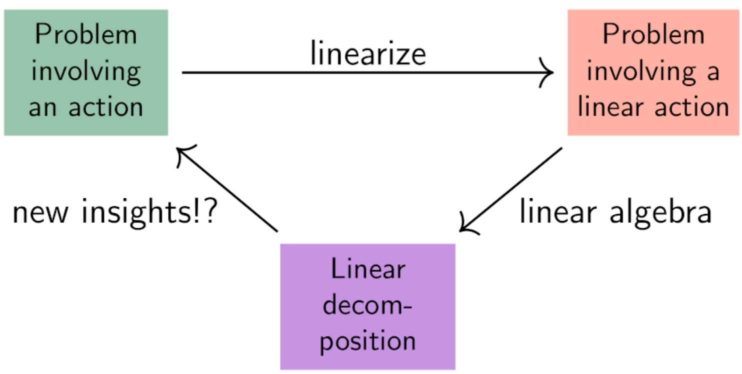
Disclaimer

Nobody is perfect, and I might have written or said something silly. If there is any doubt, then please check the references or contact me. All questions welcome!

What?
Symmetry is everywhere, and nature is designed symmetrically: Snails make their shells, spiders design their webs, and bees build hexagonal honeycombs, all based on the concept of symmetry. Indeed, symmetry is a general principle, which plays an important role in various areas of knowledge and perception, ranging from arts and aesthetics to natural sciences and mathematics. Representation theory is the systematic study of linear actions: in representation theory general symmetries are represented by matrices and one has now the whole power of linear algebra at hand to study symmetries.
For example, the symmetry of a molecule influence its chemical properties.



This picture displays a two-fold symmetry s_2 and a three-fold symmetry s_3 of two molecules. How to study symmetries of molecules systematically? You guessed it: (also) via representation theory! The symmetries s_2 and s_3 can be modeled by rotation matrices, and the molecules by vector spaces. The matrices now act on the vector space associated to the molecules (this is called a linear action), and this is the approach representation uses.
Representation theory, being the abstract study of the possible types of symmetry, is a fundamental area of algebra with applications throughout the sciences (not just chemistry as above): the methods of representation theory lead to conceptual and practical simplification of any problem in linear algebra where symmetry is present. The mnemonic is:



Let us zoom in a little bit. The study of group actions is of critical importance in mathematics and related fields such as physics and chemistry. Its significance can hardly be overestimated.

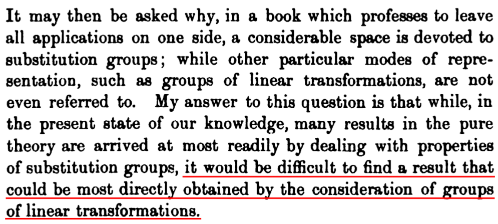
The approach of Frobenius ∼1895, Burnside ∼1900 and many others, nowadays called representation theory, is to linearly approximate such actions. For example, let G be a group or a ring or an algebra etc. Representation theory is the study of linear group actions $G \longrightarrow \mathrm{End}(V)$, $g \mapsto M(g)$ or $G \curvearrowright V$. That is, representation theory assigns to each group element a matrix M(g) acting on a vector space V – its linear shadow. The representation theory approach is that classifying linear G-actions has, in contrast to arbitrary group actions, a satisfactory answer for many groups.

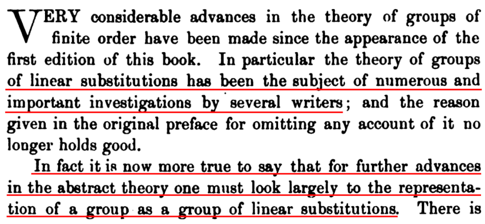
The basic building blocks V_i of such actions tell us a lot about the problem we started with. (The strategy of representation theorists is summarized below.) In fact, experience tells us that the collection of such linear shadows is an interesting structure in its own right and maybe even more worthwhile to study than G itself.

$$\begin{tikzcd} \begin{array}{c}\text{Problem involving}\\ \text{a group action}\\ G \curvearrowright \text{set}\end{array} \arrow[rr, dotted] & & \begin{array}{c}\text{Problem involving a}\\ \text{linear group action}\\ G \curvearrowright V\end{array} \arrow[dl, dotted] \\ & \begin{array}{c}\text{``Decomposition of}\\ \text{the problem''}\\ G \curvearrowright \bigoplus V_i\end{array} \arrow[ul, dotted] & \end{tikzcd}$$

Developing over the past century (and still in development), Frobenius and Burnside's theory is pervasive across many fields of mathematics. The success of representation theory has led to numerous generalizations and applications, e.g. in the aforementioned molecular chemistry or quantum physics, but also in engineering such as robotics. (How do you figure out how robots move before building them? Indeed, using representation theory.)

Still doubting the usefulness of representation theory? Well, Burnside did so as well. Here is a text from the introduction of Burnside's famous book "Theory of groups of finite order". The top picture is from the original edition and the second picture from the revision. Can you spot how Burnside changed their mind?


It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.


VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

(Linear substitutions is the old name for representations.)

Who?

Fourth semester students in Mathematics interested in a mixture of (linear) algebra, group theory and discrete mathematics, but everyone is welcome.

Where and when?

- The lecture.
  - Every Monday from 12:00–14:00.

  - ▷ Online.
- ▶ The tutorials.
  - ▷ Every Friday from 12:00–14:00.
  - ▷ Online.

Material for the lecture

- ▶ The lecture is a mix of various sources for group and monoid representations. The main source is [St12] for group representations and then [St16] for the monoid case, and the lecture follows the list of topics presented therein.

  The lecture sometimes takes a different perspective and potentially reading either of the classical references [CR62], [FH91] or [Se77] should be beneficial. [Be98] is a bit more abstract, but also a classic. Newer references are for example [Cr19], [E+11] (freely available), [Sa01] (for symmetric groups). These are also used for the lecture.
- ▶ Website www.dtubbenhauer.com/lecture-rt-2022.html
- ▶ Prerecorded lectures on the "What is...representation theory?" playlist here: www.youtube.com/c/VisualMath/playlists
- ▶ Exercise sheets are available on the course website.

Schedule and some details.

1. The beginnings – What is...representation theory?
   - ▶ **Speaker.** Daniel Tubbenhauer.
   - ▶ **Plan.** After some organizational preliminaries are addressed, this talk will be a collection of motivating example, none of which will be explained in a formal matter, but rather in an intuitive way. The talk also defines a representation and the equivalent notion of a module, as well as intertwiners.
   - ▶ **YouTube.** Videos 1, 2, 3 and 4 on the "What is...representation theory?" playlist.
   - ▶ **Literature.** This talk is not following any explicit literature, but is rather an overview of what will follow.

2. Simple and indecomposable representations I – The elements.
   - ▶ **Speaker.** Daniel Tubbenhauer.
   - ▶ **Plan.** This talk starts by introducing simple and indecomposable representations. The are the most basic representations and comparable to the elements in chemistry. We illustrate their similarities and differences in many examples.
   - ▶ **YouTube.** Videos 5 and 6 on the "What is...representation theory?" playlist.
   - ▶ **Literature.** A (restricted) collection of the concepts in [St12, Section 3.1], including some of the notions in earlier sections.

3. Simple and indecomposable representations II – More about elements.
   - ▶ **Speaker.** Daniel Tubbenhauer.
   - ▶ **Plan.** This talk will discuss the Jordan–Hölder theorem, explaining why simple representations are the elements of the theory. We also explain Maschke's theorem and the magical fact of semisimplicity over $\mathbb{C}$.
   - ▶ **YouTube.** Videos 7, 8 and 9 on the "What is...representation theory?" playlist.
   - ▶ **Literature.** The material in [St12, Section 3.2].

4. Characters I – The main players of representation theory!?
   - **Speaker.** Daniel Tubbenhauer.
   - **Plan.** The notion of characters is of fundamental importance and this lecture takes its time to introduce them. We also cover the character tables and discuss many examples.
   - **YouTube.** Videos 10, 11 and 12 on the "What is...representation theory?" playlist.
   - **Literature.** The material in [St12, Chapter 4] is broken up into this and the following two lectures.

5. Characters II – Schur's orthogonality relations.
   - **Speaker.** Daniel Tubbenhauer.
   - **Plan.** There are many magical numerical properties hidden in character tables and this lecture explains where they come from. We also, of course, motivate everything in many examples.
   - **YouTube.** Videos 13 and 14 on the "What is...representation theory?" playlist.
   - **Literature.** As in the previous lecture.

6. Characters III – Abelian groups and Fourier analysis.
   - **Speaker.** Daniel Tubbenhauer.
   - **Plan.** The character theory of finite abelian groups is very simple and we discuss it in this talk. It is also very applicable with finite Fourier analysis building almost completely on the characters of finite abelian groups.
   - **YouTube.** Videos 15 and 16 on the "What is...representation theory?" playlist.
   - **Literature.** As in the previous two lectures.

7. Burnside's theorem – An application.
   - **Speaker.** Daniel Tubbenhauer.
   - **Plan.** As a second main application of representation theory (after finite Fourier analysis) we explain Burnside's famous proof of the theorem with the same name.
   - **YouTube.** TBA.
   - **Literature.** This talk covers [St12, Chapter 6].

8. Induction and restriction – The classical adjoint pair.
   - **Speaker.** Daniel Tubbenhauer.
   - **Plan.** Adjoint functors are arguably one of the most important concepts in category theory. This talk is devoted to explain carefully where they (arguably) come from: Frobenius' theory of induction and restriction.
   - **YouTube.** TBA.
   - **Literature.** The topics of this talk are covered in [St12, Chapter 8].

9. Representations of symmetric groups – Young diagrams and co.
   - **Speaker.** Daniel Tubbenhauer.
   - **Plan.** Arguably the most important group is the symmetric group. So it is crucial to understand its representation theory, and the basics are covered in this talk.

- **YouTube.** TBA.
- **Literature.** [St12, Chapter 10] and also [Sa01].

10. Monoids I – Green's relations and friends.
    - **Speaker.** Daniel Tubbenhauer.
    - **Plan.** In order to explain the changes of representation theory of monoids compared to groups, we need to discuss a bit of background on monoids, most notably Green's relations.
    - **YouTube.** TBA.
    - **Literature.** We swap books and we are now covering [St16, Chapter 1].

11. Monoids II – The Clifford–Munn–Ponizovskiĭ theorem.
    - **Speaker.** Daniel Tubbenhauer.
    - **Plan.** A surprising fact about the representation theory of monoids is that it is almost entirely dictated by the representation theory of groups. This talk will explain the basics of that.
    - **YouTube.** TBA.
    - **Literature.** Without aiming for completeness, we explain [St16, Chapter 2].

12. Whats next? – Outlook.
    - **Speaker.** Daniel Tubbenhauer.
    - **Plan.** As a conclusion of the lecture, this talk will explain what potential further directions representation theory has to offer. For example, we explain a beautiful connection from representation theory to shuffling problems and probability.
    - **YouTube.** TBA.
    - **Literature.** Partially material beyond [St12] and [St16], but also [St16, Chapter 11].

References

[Be98] D.J. Benson. Representations and cohomology. I. Basic representation theory of finite groups and associative algebras. Second edition. Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, Cambridge, 1998. xii+246 pp.

[Cr19] D.A. Craven. Representation theory of finite groups: a guidebook. Universitext. Springer, Cham, 2019. viii+294 pp.

[CR62] C.W. Curtis, I. Reiner. Representation theory of finite groups and associative algebras. Reprint of the 1962 original. AMS Chelsea Publishing, Providence, RI, 2006. xiv+689 pp.

[E+11] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, E. Yudovina. Introduction to representation theory. With historical interludes by Slava Gerovitch. Student Mathematical Library, 59. American Mathematical Society, Providence, RI, 2011. viii+228 pp. https://math.mit.edu/~etingof/replect.pdf

[FH91] W. Fulton, J. Harris. Representation theory. A first course. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991. xvi+551 pp.

[Sa01] B.E. Sagan. The symmetric group. Representations, combinatorial algorithms, and symmetric functions. Second edition. Graduate Texts in Mathematics, 203. Springer-Verlag, New York, 2001. xvi+238 pp.

[Se77] J.P. Serre. Linear representations of finite groups. Translated from the second French edition by Leonard L. Scott. Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977. x+170 pp.

[St12] B. Steinberg. Representation theory of finite groups. An introductory approach. Universitext. Springer, New York, 2012. xiv+157 pp.

[St16] B. Steinberg. Representation theory of finite monoids. Universitext. Springer, Cham, 2016. xxiv+317 pp.

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