EXERCISES 10: LECTURE REPRESENTATION THEORY

Exercise 1. Given $i \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{Z}_{\geq 1}$ form the finite cyclic monoid $C_{i,p} = \langle a | a^{i+p} = a^i \rangle$ of cardinality $i + p$. What is the cell structure of $C_{i,p}$?

Hint:

$\mathcal{J}_t$	a^3, a^4	$\mathcal{H}(e) \cong \mathbb{Z}/2\mathbb{Z}$
$\mathcal{J}_{a^2}$	a^2	
$\mathcal{J}_a$	a	
$\mathcal{J}_b$	1	$\mathcal{H}(e) \cong 1$

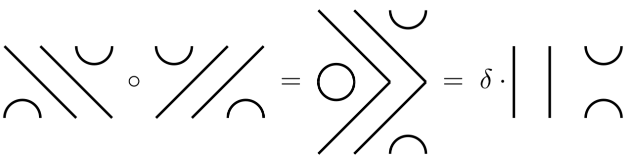
Exercise 2. The *transformation monoid* T_n on the set $\{1, \dots, n\}$ is $\mathrm{End}(\{1, \dots, n\})$. What is the cell structure of T_3?

Hint: With elements written in one-line notation, with (ijk) denoting the map $1 \mapsto i, 2 \mapsto j, 3 \mapsto k$, we get

$\mathcal{J}_t$		(111) (222) (333)		$\mathcal{H}(e) \cong S_1$
$\mathcal{J}_m$	(122), (211) (133), (311) (233), (322)	(121), (212) (313), (131) (323), (232)	(221), (112) (113), (331) (223), (332)	$\mathcal{H}(e) \cong S_2$
$\mathcal{J}_b$		(123), (213), (132) (231), (312), (321)		$\mathcal{H}(e) \cong S_3$

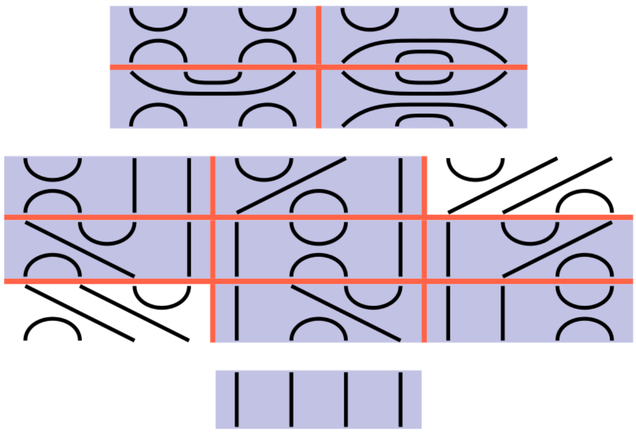
Exercise 3. The Temperley–Lieb monoid TL_n is the monoid of isotopy classes of diagrams of matchings of $2n$ points in the strip $\mathbb{R} \times [0, 1]$, with n points at the bottom and n points at the top line of the strip, called crossingless matchings. The relations on them are such that two diagrams represent the same element if and only if they represent the same crossingless matching.

Composition $\circ$ of crossingless matchings is given by vertical gluing (and rescaling), using the convention to glue $a\colon n \to n$ on top of $b\colon n \to n$, which is denoted using the operator notation $a \circ b$. This will give another crossingless matching, but with potentially internal circles. To get rid of this ambiguity, we remove such internal circles, *e.g.*:



Show that $TL_n = (TL_n, \circ)$ is a monoid, but not a group.

Exercise 4. What is the cell structure of TL_n (as in Exercise 3)?
Hint:



$\mathcal{J}_0$ $\qquad \mathcal{H}(e) \cong 1$

$\mathcal{J}_2$ $\qquad \mathcal{H}(e) \cong 1$

$\mathcal{J}_4$ $\qquad \mathcal{H}(e) \cong 1$

- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-rt-2022.html.
- Slogan: “Everything that could be finite is finite, unless stated otherwise.”. For example, groups are finite and representations are on finite dimensional vector spaces.
- There might be typos on the exercise sheets, my bad, so be prepared.