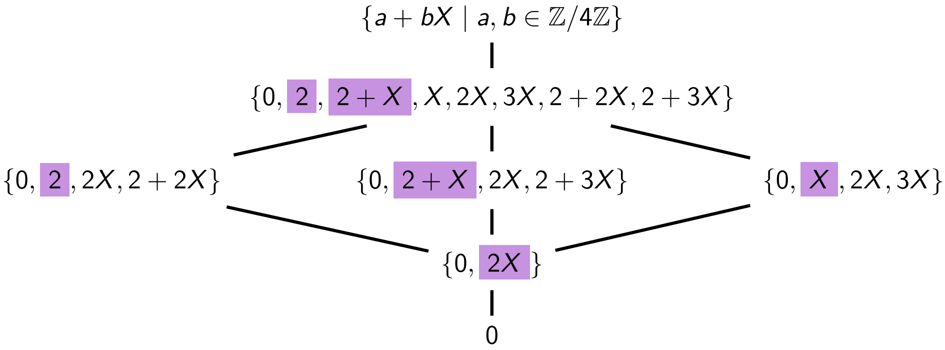
What is...the calculus of ideals?

Or: Ideals are like numbers

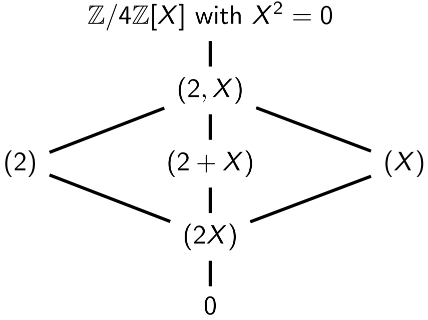
Example. $\mathbb{Z}/4\mathbb{Z}[X]$ with $X^2 = 0$



The elements outside of $\{0, 2, 2+X, X, 2X, 3X, 2+2X, 2+3X\}$ are all invertible:

- $1 \cdot 1 = 1$, $3 \cdot 3 = 9 \equiv 1 \bmod 4$
- $(a + bX)(c + dX) \equiv 1 \bmod 4 \Leftrightarrow (a = c \in \{1, 3\}$ and $b = -d)$
- Example. $(3 + 2X)(3 + 2X) = 9 + 12X + 4X^2 \equiv 1 \bmod 4$

A slick illustration

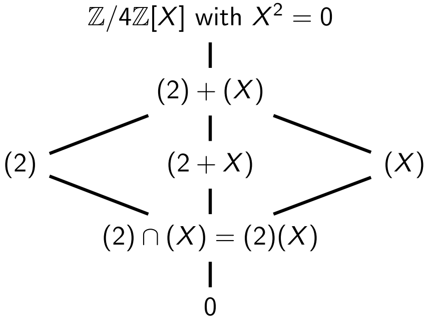


- $(2, X) \rightsquigarrow$ elements of the form $r \cdot 2 + r' \cdot X$
- $(2), (2 + X), (X) \rightsquigarrow$ elements of the form $r \cdot 2$, $r \cdot (2 + X)$, $r \cdot X$
- $(2X) \rightsquigarrow$ elements of the form $r \cdot 2 \cdot X$

(a) $Ra = \{r \cdot a \mid r \in R\}$, $aR = \{a \cdot r \mid r \in R\}$, $RaR = \{r \cdot a \cdot r' \mid r, r' \in R\}$

(b) $(a, b, ...) = \{r \cdot a + r' \cdot b + ...\}$ ideal generated by $a, b, ...$

Another slick illustration



- The sum $I + J$ of ideals gives an addition
- The product IJ of ideals gives a multiplication
- $I + J$ is a supremum, is $I \cap J$ an infimum

(a) $I + J = \{r \cdot i + r' \cdot j \mid r, r' \in R, i \in I, j \in J\}$, $IJ = \{\sum_{fin} i \cdot j \mid i \in I, j \in J\}$

(b) $I \cap J$ the set theoretic intersection **Warning.** $I \cup J$ is in general not an ideal

For completeness: The formal statement

Let R be a ring, then the collection of left ideal $\mathcal{I}$ forms a semiring and a lattice:

(a) $\mathcal{I}$ has an addition $+$, $\mathcal{I}$ has a multiplication $\cdot$ Two operations

(b) $(\mathcal{I}, +)$ is an abelian monoid

(c) $(\mathcal{I}, \cdot)$ is a monoid

(d) The two rules distribute over one another Compatibility

(e) $(\mathcal{I}, \subset)$ has $+$ and $\cap$ as a supremum respectively infimum Order

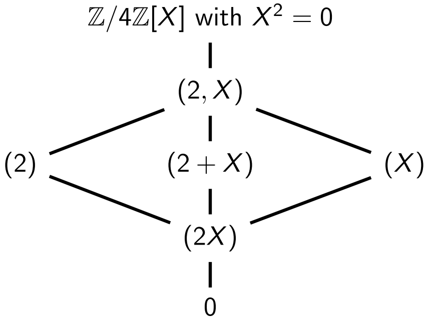
- The example of a semiring and a lattice is

$$\mathbb{N} = \{0, 1, 2, ...\}$$

Ideals behave like numbers!

- If R has a unit, then R is the unit of $\mathcal{I}$
- If R is commutative, then so is $\mathcal{I}$

Generalize properties of numbers



- $(2, X)$ is a maximal ideal
- $(2X)$ is a minimal ideal
- $(2, X)$ is a prime ideal
- (2), $(2 + X)$, (X) and $(2X)$ are principal ideals
- *etc.*

Thank you for your attention!

I hope that was of some help.