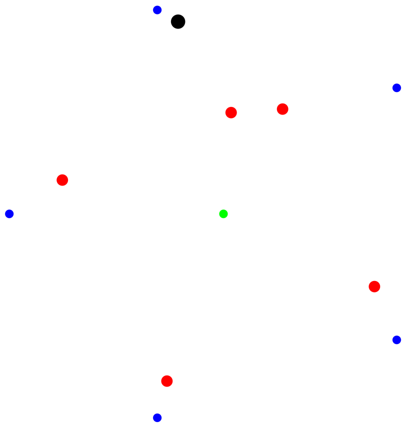
What are...normal and separable extensions?

Or: Linear factors matter

Groups actions on roots



Groups act on roots of polynomials

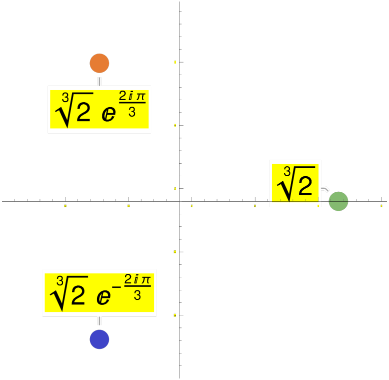
How is this reflected in field extensions?

An ill-behaved example

Roots of $X^3 - 2$

$\{\sqrt[3]{2}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2}\}$:

$\zeta = e^{2\pi i/3}$



- $\mathbb{Z}/2\mathbb{Z}$ acts on the roots of $X^3 - 2$:

$$\zeta \leftrightarrows \zeta^2$$

- This information is lost in $\mathbb{Q}(\sqrt[3]{2})$
- $X^3 - 2$ does not split in $\mathbb{Q}(\sqrt[3]{2})$

A well-behaved example

Roots of $f = X^6 + 3X^5 + 6X^4 + 3X^3 + 9X + 9$

$$\{\zeta^a + \zeta^b \sqrt[3]{2} \mid a = 1, 2, b = 0, 1, 2\}$$

$$\zeta = e^{2\pi i/3}$$

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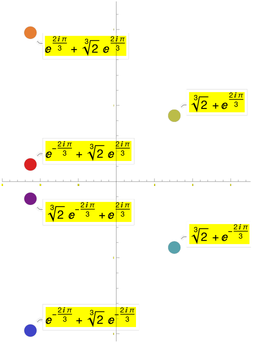
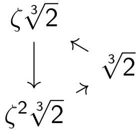


- The symmetric group S_3 permutes the roots of f:

$\zeta \leftrightarrows \zeta^2$,



- This information is present in $\mathbb{Q}(\zeta + \sqrt[3]{2}) = \mathbb{Q}(\zeta, \sqrt[3]{2})$
- f does split in $\mathbb{Q}(\zeta + \sqrt[3]{2})$

For completeness: The formal definitions

An algebraic field extension $\mathbb{K} \subset \mathbb{L}$ is normal over $\mathbb{K}$ if

every irreducible $f \in \mathbb{K}[X]$ with a root in $\mathbb{L}$ splits

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An algebraic field extension $\mathbb{K} \subset \mathbb{L}$ is separable over $\mathbb{K}$ if for all $x \in \mathbb{L}$

the minimal polynomials $m_x \in \mathbb{K}[X]$ have $\partial_X m_x \neq 0$

- Minimal normal extensions exist and are unique Existence and uniqueness
- Every algebraic extension of a field of characteristic zero or a finite field is separable Morally, separable is a non-condition

No multiple roots

For $f = a_n X^n + ... + a_0$ define $\partial_X f = n a_n X^{n-1} + ... + a_0$

- ∂_X satisfies linearity:

$$\partial_X(a \cdot f + b \cdot g) = a \cdot \partial_X f + b \cdot \partial_X g$$

- ∂_X satisfies the product rule:

$$\partial_X(fg) = \partial_X(f)g + f\partial_X(g)$$

- For irreducible $f \in \mathbb{K}[X]$ we have

$$(\partial_X f \neq 0) \Leftrightarrow f \text{ has no multiple roots}$$

Thank you for your attention!

I hope that was of some help.