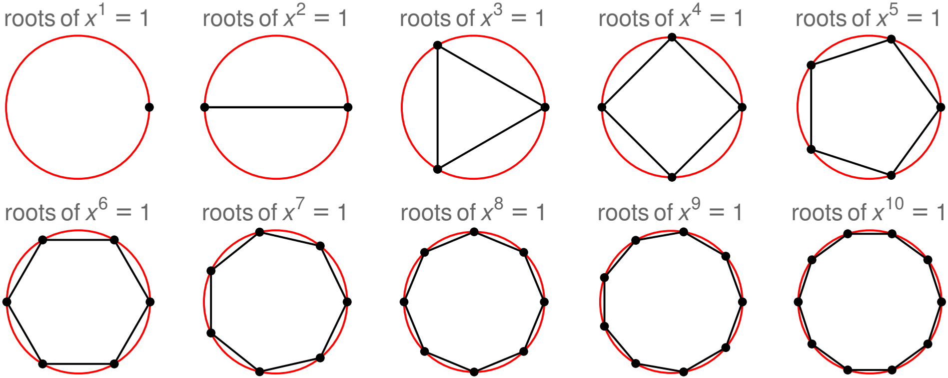
# What is...the Kronecker–Weber theorem?

Or: Field and Galois theory, application 2

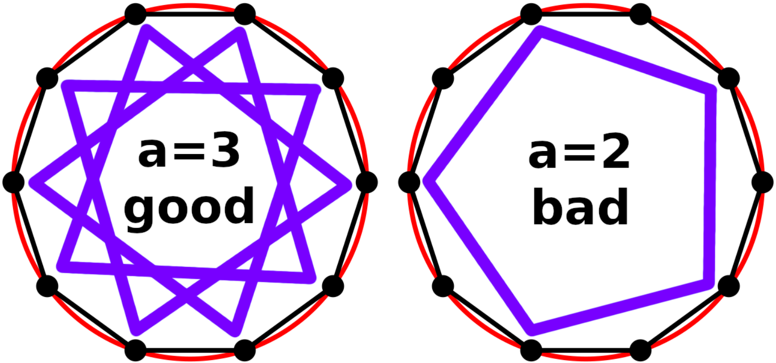
## Encircled polygons




- Complex $k$th roots of unity are of the form $e^{j \cdot 2\pi i/k}$ All
- The primitive ones are of the form $e^{j \cdot 2\pi i/k}$ for $\gcd(j, k) = 1$ Generators

# Galois and roots of unity

$k = 10$:




- $\mathbb{Q}(\zeta_k = e^{2\pi i/k})$ is the splitting field of $X^k - 1$
- $\mathbb{Q}(\zeta_k)$ is Galois over $\mathbb{Q}$ with

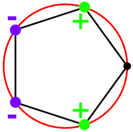
$$G(\mathbb{Q}(\zeta_k)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^* = \{a \text{ with } \gcd(a,k) = 1\}, (\zeta_k \mapsto \zeta_k^a) \leftrightsquigarrow a$$

- RoU $\Rightarrow$ abelian The Galois group of a roots of unity is abelian

## Weird coincidences

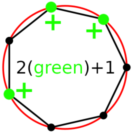
- $X^2 - 5$ has Galois group $G(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and

$$\sqrt{5} = e^{2\pi i/5} - e^{2\cdot 2\pi i/5} - e^{3\cdot 2\pi i/5} + e^{4\cdot 2\pi i/5}$$

$\sqrt{5} =$ 


- $X^2 + 7$ has Galois group $G(\mathbb{Q}(i, \sqrt{7})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and

$$\sqrt{-7} = 2 \cdot (e^{2\pi i/7} + e^{2\cdot 2\pi i/7} + e^{4\cdot 2\pi i/7}) + 1$$

$\sqrt{-7} =$ 


?RoU $\Leftarrow$ abelian? Abelian Galois group implies "being" a root of unity?

## For completeness: The formal statement

The union of all $\mathbb{Q}(\zeta_k)$ is the maximal abelian field extension of $\mathbb{Q}$, or equivalently

every finite Galois extension of $\mathbb{Q}$ with abelian $G(\mathbb{L}/\mathbb{Q})$ is contained in some $\mathbb{Q}(\zeta_k)$

RoU $\Leftrightarrow$ abelian

- Every $\sqrt{\pm n}$ is a linear combination of some $\zeta_k^a$
- *Ditto* for every algebraic integer with abelian Galois group
- Kronecker's Jugendtraum (a.k.a. Hilbert's twelfth problem): is this still true if $\mathbb{Q}$ is replaced ny any number field? Still open in 2021

*Es handelt sich um meinen liebsten Jugendtraum, nämlich um den Nachweis, dass die Abel'schen Gleichungen mit Quadratwurzeln rationaler Zahlen durch die Transformations-Gleichungen elliptischer Functionen mit singularen Moduln grade so erschöpft werden, wie die ganzzahligen Abel'schen Gleichungen durch die Kreisteilungsgleichungen.*

Kronecker in a letter to Dedekind in 1880 reproduced in volume V of his collected works, page 455

# This is also a no-go theorem

```
P< x >:=PolynomialAlgebra(Rationals());
f:=x^4-4*x^2+2;
G:=GaloisGroup(f);
print G;
```

Clear Submit

```
Permutation group G acting on a set of cardinality 4
Order = 4 = 2^2
    (1, 4)(2, 3)
    (1, 2, 4, 3)
```

Calculations are restricted to 120 seconds.
Input is limited to 50000 bytes.
Running Magma V2.26-4.
Seed: 4251428783; Total time: 0.110 seconds; Total memory usage: 32.09MB.

- The polynomial $X^4 - 4 \cdot X^2 + 2$ has splitting field $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ with

$$G(\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$$

  So $\sqrt{2+\sqrt{2}}$ "is" a root of unity See above

- The polynomial $X^4 - 5 \cdot X^2 + 2$ has splitting field $\mathbb{L}$ with

$$G(\mathbb{L}/\mathbb{Q}) \cong D_8$$

  So its roots, *e.g.* $\sqrt{\frac{1}{2}(5+\sqrt{17})}$, "are" not roots of unity Try yourself

**Thank you for your attention!**

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I hope that was of some help.